278. \[ \sqrt{16x + 1} + \sqrt{5x} = \frac{4}{5} \]
\[ \sqrt{16x + 1} - \sqrt{5x} = \frac{1}{5} \quad x = 5 \]

279. \[ (1 - x)^{\frac{1}{3}} = 2 - (1 + x)^{\frac{1}{3}} \quad x = 1 \]

280. \[ \sqrt{36x + 1} + \sqrt{36x} = 9 \{ \sqrt{36x + 1} - \sqrt{36x} \} \quad x = \frac{9}{8} \]

281. \[ \frac{1 + x + x^2}{1 + x} = \frac{62}{63}, \quad 1 - \frac{x^2 + x}{1 - x} \quad x = \frac{1}{3} \]

282. \[ \left( \frac{m + x}{m - x} \right)^{\frac{1}{3}} - 1 = \frac{m + x}{mn} \quad x = m \left( 1 + 2 \sqrt{\frac{n}{p}} \right) \]

283. \[ \frac{1 + x^2}{1 + x^2} = m - \frac{1 - x^2}{(1 - x)^2} \quad x = \sqrt{\left( \frac{m - 2}{m + 4} \right)} \]

284. \[ \frac{-1}{(1 + x)^{\frac{1}{3}} + 1} \quad \frac{1}{(1 + x)^{\frac{1}{3}} - 1} \quad x = \frac{1}{2} \sqrt{3} \]

285. \[ \frac{x + 1}{3 + x + (1 + x^2)^{\frac{1}{3}}} = 4 + \frac{x - 1}{1 - x + (1 + x^2)^{\frac{1}{3}}} \quad x = \sqrt{3} \]

286. \[ (1 + p^2)^{\frac{1}{3}} \left( \frac{1 + x}{1 - x} \right)^{\frac{1}{3}} + (1 - p^2)^{\frac{1}{3}} \left( \frac{1 - x}{1 + x} \right)^{\frac{1}{3}} = 2(1 - p^2)^{\frac{1}{3}} \quad x = -p^2 \]

287. \[ \frac{m + x + \sqrt{(2mx + x^2)}}{m + x - \sqrt{(2mx + x^2)}} = n^2 \quad x = \frac{m}{2n}(n - 1)^2 \]

288. \[ \sqrt{(m + x)} + \sqrt{(m - x)} = \sqrt{(m^2 + x^2)} + \sqrt{(m^2 - x^2)} \quad x = \frac{3}{5} \quad \sqrt{5} \]

289. \[ \sqrt{\frac{m + x}{n + \sqrt{(m + x)}} + \frac{m - x}{n + \sqrt{(m - x)}}} = m \quad x = \frac{m + \sqrt{3}}{2} \]

290. \[ \sqrt{(y^4 - 1)} = y^2 - y^4 \quad \sqrt{(y^8 - 1)} \quad y = \left( \frac{1 + \sqrt{2}}{2} \right)^{\frac{1}{4}} \]

291. \[ \sqrt{9(w^2 + 39x + 374)^{\frac{1}{2}}} = \left( \frac{w + 22}{w + 17} \right)^{\frac{1}{8}} + \sqrt{w^2 + 2w + 51} \quad x = 78 \]
ALGEBRAICAL EXERCISES.

292. \[3(mn^2)^{\frac{1}{3}} + \omega = n - m\]
\[\omega = (n^{\frac{1}{3}} - m^{\frac{1}{3}})^2\]

293. \[\omega = \sqrt{a^2 + \omega\sqrt{(\delta^2 + \omega^2)}} - \alpha\]
\[\omega = \frac{\delta^2 - 4\alpha\omega}{4\alpha}\]

294. \[\frac{m^4 + (m^4 - \omega^2)^{\frac{1}{2}}}{m^4 - (m^4 - \omega^2)^{\frac{1}{2}}} = n^2 \quad \frac{(m^2 + \omega)^{\frac{1}{2}} - (m^2 - \omega)^{\frac{1}{2}}}{(m^2 + \omega)^{\frac{1}{2}} + (m^2 - \omega)^{\frac{1}{2}}} = \frac{2m^2p}{1 + p^2}\]

295. \[\left(\frac{x + \sqrt{(x^2 - 9)}}{x - \sqrt{(x^2 - 9)}}\right)^2 = 243\left(\frac{\sqrt{(x + 3)} - \sqrt{(x - 3)}}{\sqrt{(x + 3)} + \sqrt{(x - 3)}}\right)\]
\[\omega = 5\]

PROBLEMS.

1. Find a number which being multiplied by 8, and having 18 added to the product, the sum shall be 90. 
   Ans. 9.

2. What number is that whose seventh part added to its fifth part will make 24? 
   Ans. 70.

3. Twenty five is subtracted from a certain number and \(\frac{3}{5}\) of the remainder is 50; What is the number? 
   Ans. 100.

4. The twelfth part of a certain number exceeds the fifteenth part by 1; find the number. 
   Ans. 60.

5. Find a number from which if 8 be taken, and the remainder multiplied by 3 and then 65 added to the product, this sum divided by 4 shall give the required number. 
   Ans. 41.

6. Find two numbers whose sum is 70 and whose difference is 28. 
   Ans. 21, 49.

7. A father had two sons; the age of the eldest is \(\frac{1}{4}\) of the age of the father, and the age of the youngest is \(\frac{1}{4}\) of that of his brother, the sum of their ages is 84; What is the age of each? 
   Ans. 49, 28, 7.

8. Sham is thrice as old as Ram. Fifteen years ago he was 6 times as old as Ram. What is Sham's age? 
   Ans. 75.
9. Two numbers when multiplied together give 96 for the result, but when the greater is decreased by 1, the product is decreased by 8. What are the numbers? Ans 8, and 12

10. The cube root of a certain number is \( \frac{1}{3} \) of the square root; find the number.

11. Divide Rs. 14 among 30 persons, men and boys, giving 10 as each to the men and 5 as each to the boys. how many were there of each? Ans 16 men, 14 boys.

12. Forty-two years hence a boy will be seven times as old as he was six years since; how old is he? Ans 14

13. Find a number such that if 10, 40 and 25 be successively subtracted from it, the sum of the third, fourth and fifth parts of the respective results shall be equal to 60. Ans 100

14. A boy's age was thrice the number indicated by the clock, and his father's age was 3 times that of the boy. The sum of their ages was 96. What o'clock was it? Ans 8

15. After A has received 12 Rs. from B, he finds that he has 47 Rs. more than B; they have 147 Rs. between them; what sum had each at first? Ans 85 and 62

16. A and B entered into a transaction with equal sums of money, B first gave 24 Rs. to A, but afterwards received from A \( \frac{1}{4} \) of the money he then had; A then found, that he had 14 Rs. less than what B had; what money had each at first? Ans 100

17. Divide the number 100 into two such parts that if \( \frac{1}{3} \) of the smaller part be added to the greater the result is 89. 33 and 67

18. A person has 10 times as many eight anna pieces as two-anna pieces and together they amount to 328 Rs. How many has he of each. Ans 640 eight anna pieces and 64 two-anna pieces.

19. Divide 360 Rs. among 3 persons, A, B, C, so that B shall have twice as much as A, and C three times as much as B. Ans A's 40; B's 80; C's 240.
20. A debt of £3, 14s was paid in florins and half crowns and one more of the former was paid than of the latter. How many florins were paid? 

Ans 17

21. Divide 48 into four such parts that the first increased by 3, the second diminished by 3, the third multiplied by 3 and the fourth divided by 3, shall all give the same result. 

Ans 6, 12, 3, 27

22. A and B have equal sums of money. A gains 180 Rs and B loses 140 Rs and then A has thrice as much as B. How much had each at first? 

Ans 800 Rs

23. Find a number such that its third is as much above 80 as its fifth part is below 80. 

Ans 300

24. Two baskets were full of equal number of ripe mangoes; after 75 mangoes had been taken from one basket and 50 from the other there remained just one and a half times as many in one basket as in the other. What did each basket contain when full? 

Ans 25

25. Divide 240 into two parts such that \( \frac{2}{3} \) of one part together with \( \frac{2}{3} \) of the other part may be equal to 100. 

Ans 100, 140

26. A gentleman starts on a hunting excursion and has just 10 hours at his disposal: how far may he ride in a gharry which travels 12 miles an hour, so as to return home in time driving at the rate of 6 miles an hour. 

Ans 40 miles

27. A number is divided into both \( p \) and \( p + 1 \) equal parts; the product of the \( p \) parts is \( p \) times the product of the \( p + 1 \) parts find the number. 

Ans \( \left( \frac{p+1}{p} \right)^{p+1} \)

28. A waterman finds by experience that he can with the advantage of a common tide row down a river from A to B, which is 18 miles, in an hour and a half, and that to return from B to A against an equal tide, though he rows back along the shore, where the stream is only \( \frac{3}{2} \) as strong as in the middle, takes him just two
PROBLEMS.

hours and a quarter. At what rate per hour the tide runs in the middle, where it is strongest. Ans 2\frac{1}{2} miles per hour

29. Suppose two hands of a clock (a) and (b) were together on Sunday noon at 12 o'clock, and that the motion of each was such that (a) moved round the horary circle in one hour and (b) in 1\frac{3}{4} hour, when will they be together again for the 1st time.

Ans 61 hours.

30. The difference of two numbers is 54, and the quotient of the greater by the less is 4; what are the numbers.

Ans 72, 18

31. Divide 1500 Rs among A, B, C and D in such a manner that B shall get 3 times as much as A, C shall get twice as much as A and B together and D shall get \frac{1}{4} as much as A, B, and C together.

Ans A, 's = 100 ; B's = 300 ; C's = 800 and D's = 300

32 A person bought a pony chaise at a certain price and paid the same price for the horse; if the horse had cost 50 Rs less and the ghary 150 Rs more, the price of the pony would have been only half that of the chaise, find the cost of the chaise.

Ans 250

33. A carpenter is engaged for 45 days, on condition that he receives 5 as for every day he works and pays 4 as fine for every day he is idle; at the end of the time he receives 5 Rs 10 as. How many days did he work and how many was he idle.

Ans He worked 30 days, and was idle 15 days

34. Between 7 and 8 when will the two hands of a clock be (1) together (2) at right angles to each other (3) exactly opposite.

Ans Together at 33\frac{2}{11} ; right angles at 21\frac{9}{11} ; and opposite at 5\frac{9}{11}.

35. After m o'clock the hour and minute hands of a clock are distant d of the minute divisions from each other find the time.

Ans \frac{12}{11} \left(5m + \frac{d}{11}\right)

36. A draper becoming insolvent found that the sum of money remaining unreasied from his customers was equal in amount to his liabilities, and further he finds that on 4000 Rs of the unrecovered money he can only get 12 as in the rupee, and the expences
of the bankruptcy are 5 per cent on his liabilities; if he pay 13½ as. in the rupee what is the amount of his liabilities? Ans 8000 Rs.

37. A man at a party at cards betted 3 shillings to two upon every deal. After 20 deals he won 5 shillings. How many deals did he win? Ans 13

38. Hemadrish and Kristo were playing together, the former first won 12 Rs. from the latter and then both had equal sums; but Kristo on winning back his own money and 6 Rs. more, had at last 2½ times as much money as Hemadrish: what money had each at first. Ans 36, 60

39. A bill of 3 Rs. 10 as. was paid in 4 as. pieces and 2 as. pieces and the whole number of coins was 18; how many coins were there of each kind. Ans 11 and 7

40. A confectioner buys 1½ mds of sugar at 7 as. a seer and chana (curdled milk) at 6 as. a seer. What quantity of the latter must he add to the former so as to sell the mixture (sodesh) at 6⅔ as. a seer. Ans A maund

41. A regiment was drawn up in a solid square; and on a certain emergency 1500 soldiers were taken away; it was again drawn up in a solid square and then it was found that there were 10 men fewer in a side: what was the original number of men in the regiment. Ans 6400

42. Divide 80 into 2 parts such that the first divided by 3 shall be equal to the second multiplied 3. Ans 8 and 72

43. A person sitting at a tavern distributed 3 pice each to those present there and had 7 pice left; had he 3 as. more with him he could have given them at the rate of 4 pice each; how many persons were there? Ans 19

44. Sham and Ram begin to travel in the same direction from Howrah by the grand Trunk Road. Sham travels 15 miles per day and after 18 days turns and goes back as far as Ram has travelled during the 9 days; he then turns again and pursuing Ram overtakes
him at the end of 36 days from the commencement of their journey. How many miles Ram travelled in a day? Ans 10 miles

45. A confectioner buys sugar at 6 as. a seer and chana at 8 as. a seer: in what proportion must he mix the two to prepare his sondesh so as to gain 20 per cent by selling it at 9 as. a seer? Ans 1 : 3

46. A person started at the rate of 3 miles an hour to walk to the Howrah railway station in order to catch a train, but after he had walked one third of the distance he was detained 15 minutes and was obliged in consequence to walk the rest of the way at the rate of 4 miles an hour. How far off was the station? Ans 4 1/2 miles

47. A grocer has two sorts of sugar, one sort worth 8 as. a seer, and the other worth 10 as. a seer; from these he wants to make a mixture of 2 maunds worth 9 1/2 as a seer. How many seers must be take from each sort. Ans 32 seers and 48 seers

48. Soodeen and Hurry start in different directions from a certain place in the circumference of a circular island. Soodeen travels 5 miles an hour and Hurry 7 miles per hour always keeping in the circumference. Soodeen wants 5 miles of being half way when he meets Hurry. Required the circumference of the island. Ans 60 miles.

49. A locomotive engine Alexander sets out from Howrah to go to Jamalpore, at the same time that another engine Napoleon sets out from Jamalpore to go to Howrah; Alexander arrives at Jamalpore 4 hours and Napoleon at Howrah 9 hours after they met on the way; how long did each take to perform the journey. Ans. Alexander 10 hours; Napoleon 15 hours.

50. Shoshee and Khetter play at marbles, Shoshee wins half of Khetter’s and then loses 5 when he finds that he has as many as Khetter had at the commencement of the play; how many had each at first if they had 50 between them? Ans Shoshee 20, Khetter 30.

51. The marjee (helmsman) of a boat of 8 oars weighs 1 ind. 32 Srs. The average weight of the crew exclusive of the marjee
is greater than their average weight inclusive of him by 2 srs. What is their whole weight? Ans 19 mds 32 Srs.

52. Saroda, who travels at the rate of 8 miles an hour, starts from Ooloobaria to go to Ghatal a distance of 50 miles while another Barada starts at the same time from Ghatal to come to Ooloobaria, his rate of travelling is 6 miles an hour. Saroda when went a certain distance from the starting point recollected to have left his bag in a shop at Ooloobaria and therefore turned back and on arriving at the place from which he started, again continued his journey; they arrived both at the same time to their destination. At what distance from Ooloobaria did Saroda at first travel? Ans $8\frac{3}{4}$ miles.

53. A steamer was seen to pass by Dacca when the minute-hand of a clock was at right angles to the hour-hand for the first time after 10 A. M.; when it arrived at Nariangunge, the minute-hand was opposite to the hour-hand after 11 A. M. The rate of the steamer is eleven miles an hour; find the distance of Nariangunge from Dacca by water. Ans 15 miles.

54. A man had a certain number of oranges, he sells $\frac{1}{4}$ of the number and 2 more to one person; $\frac{1}{2}$ the remainder and one more to a second person; $\frac{2}{3}$ of the remainder and 3 more to a 3rd person, after which he had 3 with him; how many had he altogether. Ans 50.

55. A ship’s crew sailed with provisions for 30 days; after being at sea 20 days they encountered a storm in which they lost 8 men and 2 days after the storm, they took on board 18 persons who had been wrecked and were without provisions; they then found that to make their provisions last as long as was intended, each person’s daily allowance must be reduced to $\frac{4}{5}$ of what it had been before; how many persons were on board when the ship set sail? Ans 51.

56. A fish was caught in the river Tista whose tail weighed 9 seers, his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail together; find the weight of the fish. Ans 1 md. 32 seers.
57. A ship sails with a supply of biscuit for 60 days at a daily allowance of one seer a head; after being at sea for 20 days she encounters a storm in which 5 men were washed overboard and damage suffered that will cause a delay of 24 days and it is found that each man's allowance must be reduced to \( \frac{5}{7} \) of a seer. Find the original number of the crew.  

Ans 40

58. An officer in drawing up his battalion in the form of a solid square finds that he has 6 men too many, and that he would want 19 men, to increase the side of the square by one man; how many men were there in the army?  

Ans 150

59. An officer can form the men of his regiment into a hollow square 8 deep. The number of men in the regiment is 1024. Find the no of men in the front of the hollow square.  

Ans 40

60. A colonel wished to form a solid square of his men. The first time he had 39 men over; the second time he increased the side of the square by one man, and then he found he wanted 50 men to complete it. Of how many men did the regiment consist?  

Ans 1975

61. A person walked out from the Oriental Seminary to the Howrah Station at the rate of 3½ miles an hour and then walked along side of the railway to Bally at the same rate and on reaching the Bally station had to wait 6 minutes for the train which was then 3 miles off, and was proceeding towards Howrah. On arriving at the seminary which was a mile from the Howrah Station he found that he had been out 2 hours 35½ minutes. Find the distance of Bally from Howrah.  

Ans 6 miles.

62. A stone is thrown into a well, and it is observed that 1½" elapse before the sound of its striking the bottom is heard; neglecting the time occupied by the transmission of the sound, find the depth of the well. Having given that \( s = 16 \cdot 1 \times t^2 \), where \( s \) = space in feet described by falling bodies from rest, on the surface of the earth, and \( t \) = time in seconds occupied for the descent.  

Ans 36.225 ft.
63. In passing over 180 ft. of the Chitpore Road the forewheel of a barouche had to make 3 turns more than the hind wheel and the periphery of the first is 3 ft. less than that of the 2nd; find that of each.
   Ans 12 ft., 15 ft.

64. Two workmen A and B were employed together for 50 days, at 5 shillings per day each. A spent 6d. a day less than B did, and at the end of the 50 days he found he had saved twice as much as B, and the expense of 2 days over. What did each spend per day?
   Ans A 50d, B 56d a day.

65. A hare 50 of her leaps before a greyhound, takes 4 leaps to the greyhounds three; but two of the greyhounds' leaps are as much as 3 of the hare's. How many leaps must the greyhound take to catch the hare?
   Ans 300 leaps.

66. Divide the no 116 into 4 such parts that if the 1st be increased by 5, the second diminished by 4, the 3rd multiplied by 3, and the 4th divided by 2, the result in each case shall be the same.
   Ans 22, 31, 9 and 54.

67. Kadar and Boykanto began to trade with equal sums of money. In the 1st year Kadar gained 40 Rs. and Boykanto lost 40; but in the 2nd Kadar lost \( \frac{1}{3} \) of what he then had and Boykanto gained a sum less by 40 Rs, than twice the sum that Kadar had lost; when it appeared that Boykanto had twice as much money as Kadar. What money did each begin with?
   Ans 320 Rs.

68. Sham and Ram being at play severally cut packs of cards so as to take off more than they left. Now it happened that Sham cut off twice as many as Ram left, and Ram cut off seven times as many as Sham left. How were the cards cut by each?
   Ans Sham 48, Ram 28.

69. A person at play won twice as much as he began with and then lost 16 Rs. After this he lost \( \frac{1}{3} \) of what remained, and then won as much as he began with and counting his money, found he had 80 Rs. What sum did he begin with?
   Ans 52 Rs.
70. A man wished to inclose a garden with palisades, and found that if he put them a foot apart he should have too few by 150, but if he set them a yd asunder, he should have too many by 70. How many had he? Ans 180

71. A constable in pursuing a thief, who was \( \frac{3}{5} \) of a mile from him runs at the rate of 5 miles an hour; the thief runs at the rate of 4\( \frac{1}{2} \) miles an hour. What distance will the thief run before he is caught. Ans 9\( \frac{1}{2} \) miles

72. Divide the number 75 into four parts such that the first increased by 2, the second diminished by 3, the third multiplied by 4\( \frac{1}{2} \) and the 4th divided by 6 may all be equal Ans 7, 12, 2, 54

73. A ryot made an agreement with his zemindar that his annual rent would be 35 Rs and the value of a certain number of maunds of rice; when rice is 2 Rs a maund the whole rent becomes 10 per cent lower than when it is 3 Rs a maund. Find the number of maunds of rice which are stipulated as part of the rent. Ans 5 maunds

74. The interest on a certain sum of rupees at a certain rate of interest in one year is 405 Rs, but if the rate were increased by 1\( \frac{1}{2} \) per cent, the interest would amount to 607 Rs 8 as. Find the principal and the rate per cent. Ans Principal 13500 Rs, rate 3 per cent

75. A train 88 yds in length overtook Naran walking along the line in the same direction at the rate of 4 miles an hour and passed him in 10 seconds; 20 minutes afterwards the train overtook Ram and passed him in 9 seconds; when will Naran overtake Ram. Ans 3\( \frac{1}{3} \) hrs.

76. A market woman being asked how many mangoes she had replied ‘If I had as many more, half as many more and one egg and a half I should have 10\( \frac{1}{2} \) eggs; how many had she?’ Ans 41

77. A man buys a certain no of oranges at two a penny, four times as many at 5d a dozen, five times as many at 8d a score, and sells them at 3s. 8d a hundred, gaining by the transaction 3s. 6d. How many oranges did he buy? Ans 1800
78. A gentleman walked from Calcutta to Ootterparah at the rate of 3 miles an hour and returned to Calcutta by the same route at the rate of 2 miles an hour; it took him 5 hours to perform the journey; find the distance of Ootterparah from Calcutta. Ans 6 miles

79. A gentleman walks from a village to a railway station at the rate of 3 miles an hour, runs part of the way back at the rate of \(8\frac{1}{2}\) miles an hour, and then walks the remainder in 1 hr 5 min. If he is out 2 hrs 44 min, find the distance to the railway station. Ans \(4\frac{1}{2}\) miles

Exercise 22.

SIMPLE EQUATIONS OF TWO UNKNOWN QUANTITIES.

Find the values of \(x\) and \(y\) in the following equations.

1. \(4x + 5y = 23\) \(\begin{cases} x = 2 \\ x + 2y = 8 \end{cases}\) \(\begin{cases} y = 3 \\ 2w + 3y = 25 \end{cases}\)

2. \(3x + 2y = 30\) \(\begin{cases} x = 3 \\ 2w + 3y = 25 \end{cases}\)

3. \(6x + 8y = 74 = 0\) \(\begin{cases} x = 3 \\ 10w - 3y = 51 = 0 \end{cases}\) \(\begin{cases} y = 7 \\ 5x + 2y = 9 \end{cases}\)

4. \(8x - 3y = 2\) \(\begin{cases} x = 1 \\ 5x + 2y = 9 \end{cases}\)

5. \(3x + 2y = 39\) \(\begin{cases} x = 11 \\ 5x - 6y = 37 \end{cases}\) \(\begin{cases} y = 3 \\ 8x - y = -2 \end{cases}\)

6. \(5x + 6y = 12\) \(\begin{cases} x = 0 \\ 8x - y = -2 \end{cases}\)

7. \(10x - 3y = 54\) \(\begin{cases} x = 6 \\ 2w - 3y = 6 \end{cases}\) \(\begin{cases} y = 2 \\ 3y - 2x = 21 \end{cases}\)

8. \(4x - y = 3\) \(\begin{cases} x = 3 \\ 4w - y = 3 \end{cases}\)

9. \(4x - 3y = 20\) \(\begin{cases} x = 2 \\ 8x + 3y = 4 \end{cases}\) \(\begin{cases} y = -4 \\ 10x - 2y = -28 \end{cases}\)

10. \(11y - 3x = -2\) \(\begin{cases} x = -3 \\ 10x - 2y = -28 \end{cases}\)

11. \(4x - 2y = 50 \end{cases}\)

12. \(\frac{3x}{x} + 8y = 102\) \(\begin{cases} x = 8 \\ 6x + \frac{2y}{3} = 56 \end{cases}\)

13. \(ax + by = c \end{cases}\)

14. \(7x + 2y = 40\) \(\begin{cases} x = 4 \\ 5wy = (x + 1)(5y - 6) \end{cases}\)

\(\begin{cases} y = \frac{cp - bd}{ap - bm} \\ mx + py = q \end{cases}\)

\(\begin{cases} y = \frac{cm - ad}{bm - ap} \end{cases}\)
15. \((x + 5)(y + 7) = (x + 1)(y - 9) + 112\) \(\Rightarrow\) \(x = 3\)  
\[2x + 10 = 3y + 1\] \(\Rightarrow\) \(y = 5\)

16. \((a + c)x - by = bc\) \(\Rightarrow\) \(a = b\)  
\[\begin{align*} \frac{c}{a} + \frac{c}{b} = \frac{a}{b} \end{align*}\] \(y = a\)

17. \(a + y = a + b\) \(\Rightarrow\) \(x = a\)  
\(b + x + ay = 2ab\) \(\Rightarrow\) \(y = b\)

18. \(\frac{x}{2} + \frac{y}{3} = 8\) \(\Rightarrow\) \(x = 8\)  
\(\frac{y}{5} + \frac{x}{5} = \frac{5}{2}\) \(\Rightarrow\) \(y = 12\)

19. \(\begin{align*} \frac{x}{2} + \frac{2y}{7} &= 4 \\ \frac{x}{1} + \frac{y}{2} &= 5 \end{align*}\) \(\Rightarrow\) \(x = 6\)  
\(\frac{5x}{3} - y = 24\) \(\Rightarrow\) \(y = 15\)

20. \(\frac{x + y}{3} + a = 15\) \(\Rightarrow\) \(x = 10\)  
\(\frac{x - y}{5} + y = 6\) \(\Rightarrow\) \(y = 5\)

21. \(\begin{align*} \frac{5x}{6} + \frac{2y}{3} &= 24 \Rightarrow x = 20 \\ \frac{3x}{5} - y &= 0 \Rightarrow y = 12 \end{align*}\)

22. \(\frac{a + y}{8} - \frac{a - y}{6} = 5\) \(\Rightarrow\) \(x = 20\)  
\(\frac{b + y}{5} + \frac{a - y}{11} = 8\) \(\Rightarrow\) \(y = 20\)

23. \(\begin{align*} \frac{2 - 3x}{6} + \frac{6y - 1}{12} &= \frac{1}{3} \\ \frac{2x + 3y}{2} + 3y &= 2 \end{align*}\) \(\Rightarrow\) \(y = \frac{1}{3}\)

24. \(0.08x + 0.21y = 0.035\) \(\Rightarrow\) \(a = 1\)  
\(0.12x + 0.7y = 3.54\) \(\Rightarrow\) \(y = \) 

25. \(\begin{align*} \frac{16}{x} - \frac{12}{y} &= 1 \\ \frac{64}{x} + \frac{36}{y} &= 11 \end{align*}\) \(\Rightarrow\) \(y = 12\)  
\(\frac{x}{2y} + 9 = \frac{7x - 4y}{y} \Rightarrow y = 5\)

26. \(\begin{align*} x + 2y - 1 &= 10 \\ y - 1 - y &= 4 \\ x - y &= 2 \end{align*}\) \(\Rightarrow\) \(y = 2\)

27. \(\begin{align*} 3x - 2y &= 42 \\ \frac{y}{2x} &= \frac{5x - y}{2} \end{align*}\)

28. \(\begin{align*} 2(x - 3) - \frac{1}{3}(y - 3) &= 3 \Rightarrow a = 5 \\ 3(y - 5) + \frac{1}{3}(x - 2) &= 10 \Rightarrow y = 8 \end{align*}\)

29. \(\begin{align*} \frac{x - 1}{3} &= \frac{y + 1}{4} \\ 2x - 3 &= 13 - 2y \end{align*}\) \(\Rightarrow\) \(y = 3\)

30. \(\begin{align*} \frac{x - 3}{y} &= \frac{1}{4} \Rightarrow a = 4 \\ \frac{2x - 3}{13 - 2y} &= \frac{3}{4} \Rightarrow y = 3 \end{align*}\)
31. \( a(x + y) + b(x - y) = 1 \quad x = \frac{1}{a + b} \)  
\( a(x - y) + b(x + y) = 1 \quad y = 0 \)

32. \( (m + n)x = 4mn + (m - n)y, (m - n)x = 2(m + n)(m - n) - (m + n)y \)  
\( x = m + n \quad y = m - n \)

33. \( (a + m)x + (b - m)y = c \quad x = \frac{c}{a + b} \)  
\( (b + n)x + (a - n)y = c \quad y = \frac{c}{a + b} \)

34. \( x - y = 3, 3\left(\frac{1}{y} + \frac{1}{x}\right) - 11\left(\frac{1}{y} - \frac{1}{x}\right) \)  
\( x = 7 \quad y = 4 \)

35. \( 2(x - y) = 3(x - 4y) \) \( x = 20 \)  
\( 14(x + y) = 11(x + 8y) \) \( y = 2 \)

36. \( 16x + 6y - 1 = \frac{128x^2 - 18y^2 + 217}{8x - 3y + 2} \)  
\( 10x + 10y - 35 = 5 - \frac{54}{3x + 2y - 1} \)  
\( x = 6 \quad y = 5 \)

37. \( 2x - \frac{y - 3}{5} = 4 \) \( 3y + \frac{x - 2}{3} = 9 \)  
\( x = 2, y = 3 \)

38. \( \frac{1}{3}(4x + 2y) = 6 - \frac{1}{3}(5y - 3x) \) \( x = 3 \)  
\( \frac{1}{3}(8y - 10) = \frac{1}{3}(5x + 3y) + 5 \) \( y = 5 \)

39. \( y^{\frac{1}{2}} - (x - x)^{\frac{1}{2}} = (y - x)^{\frac{1}{2}} \) \( x = \frac{4}{5}\alpha \)  
\( 2(y - x)^{\frac{1}{2}} - 3(x - x)^{\frac{1}{2}} \) \( y = \frac{5}{4}\alpha \)

40. \( \frac{4x - 8y + 1}{2} = \frac{10x^2 - 12y^2 - 14xy + 2w}{5x + 3y + 3} \)  
\( 2\sqrt{(6 + w)} = 3\sqrt{(6 - y)} \)  
\( x = 3 \quad y = 2 \)

41. \( (x^w)^u = c^u \) \( v = (c^u)^w \)  
\( w = c^\frac{1}{2u} \quad y = 1 \)
42. \( x^2 P - y^2 P = m^2 \)
\[
\frac{a^2}{2n} + \frac{y^2}{2n} = n
\]
\[
a = \left( \frac{m^2 + n^2}{2n} \right)^2
\]
\[
y = \left( \frac{n^2 - m^2}{2n} \right)^2
\]

43. \( x + y = 5 \)
\( xy = 6 \)
\( a = 3 \)
\( y = 2 \)

44. \((a^2 - b^2)(3a + 5y) = (4a - b)2ab\)
\[
a^2 \frac{b}{a + b} + (a + b + c)by = b^2 x + (a + 2b)ab
\]
\[
a = \frac{ab}{a - b} , y = \frac{ab}{a + b}
\]

45. \( \sqrt{(a - 3)} = \frac{1}{2} \sqrt{(a + y)} \)
\( 25y = 5a \)
\( a = -3 \frac{3}{4} \)
\( y = -\frac{3}{8} \)

46. \( a(x^2 + y^2) - b(x^2 - y^2) = 2a \)
\( (a^2 - b^2)(x^2 - y^2) = 4ab \)
\( a = \sqrt{\left( \frac{a + b}{a - b} \right)} \)
\( a = \sqrt{\left( \frac{a - b}{a + b} \right)} \)

47. \( \sqrt{y} - \sqrt{(20 - a)} = \sqrt{(y - a)} \)
\( 2\sqrt{(y - a)} + 2\sqrt{(20 - a)} = 5\sqrt{(20 - a)} \)
\( a = 16 \)
\( y = 25 \)

48. \( x^2 + xy = 45 \)
\( y^2 + xy = 36 \)
\( a = 5 \)
\( y = 4 \)

49. \( x + y = 5 \)
\( x^2 - y^2 = 5 \)
\( x = 3 \)
\( y = 2 \)

50. \( 2x + 2y = 10 \)
\( x - y = 1 \)
\( x = 3 \)
\( y = 2 \)

51. \( x^2 + y^2 = 13 \)
\( x + y = 5 \)
\( x = 2 \)
\( y = 3 \)

52. \( x^2 + y^2 = 5 \)
\( x - y = 1 \)
\( x = 2 \)
\( y = 1 \)

53. \( x^2 - xy = 8 \)
\( xy - y^2 = 4 \)
\( x = 4 \)
\( y = 2 \)

54. \( x^2 + y^2 = 35 \)
\( x + y = 5 \)
\( x = 2 \)
\( y = 3 \)
ALGEBRAICAL EXERCISES.

55.  \( x^2 + xy = 36 \) \( x = 4 \)
     \( xy + y^2 = 45 \) \( y = 5 \)

56.  \( (x^2 - xy + y^2)(w^2 + y^2) = 15 \) \( w = 2 \)
     \( (x^2 - xy + y^2) = \frac{21}{w^2 + xy + y^2} \) \( y = 1 \)

57.  \[ \frac{5(w + z)^{\frac{1}{2}}}{w} \cdot \frac{5(w + z)^{\frac{1}{2}}}{z} = 6 \frac{2}{3} \]
     \[ \frac{3(w - z)^{\frac{1}{2}}}{z} \cdot \frac{3(w - z)^{\frac{1}{2}}}{w} = 3 \frac{3}{4} \]
     \( w = 5 \)
     \( z = 4 \)

Exercise 23.

SIMPLE EQUATIONS OF MORE THAN TWO UNKNOWNS.

1. \[
\begin{align*}
5x + 2y + z &= 12 \\
2x + 3y + 4z &= 20 \\
3x + 4y + 5z &= 26
\end{align*}
\]
\( \begin{array}{c} w = 1 \\ y = 2 \\ z = 3 \end{array} \)

2. \[
\begin{align*}
5x + 8y - 3z &= 59 \\
2x - 3y + 11z &= 23 \\
3x + 2y - z &= 21
\end{align*}
\]
\( \begin{array}{c} w = 4 \\ y = 6 \\ z = 3 \end{array} \)

3. \[
\begin{align*}
4x + 2y - z &= 20 \\
6x - 7y + 3z &= -32 \\
7x + y - 2z &= 14
\end{align*}
\]
\( \begin{array}{c} w = 2 \\ y = 8 \\ z = 3 \end{array} \)

4. \[
\begin{align*}
7x + 8y - 4z &= 31 \\
3x + 2y + z &= 17 \\
w + 3y + 4z &= 32
\end{align*}
\]
\( \begin{array}{c} w = 1 \\ y = 5 \\ z = 4 \end{array} \)

5. \[
\begin{align*}
2y + 3y - z &= -2 \\
3x - 3y + 2z &= -4 \\
5x + 2y - 3z &= -18
\end{align*}
\]
\( \begin{array}{c} w = -2 \\ y = 2 \\ z = 4 \end{array} \)

6. \[
\begin{align*}
\omega + 2y + 3z &= 6 \\
2x + 4y + 2z &= 8 \\
3x + 2y + 8z &= 101
\end{align*}
\]
\( \begin{array}{c} \omega = 45 \\ y = -21 \\ z = 1 \end{array} \)

7. \[
\begin{align*}
3x + 4z &= 57 \\
5x + 3y &= 65 \\
2y - z &= 11
\end{align*}
\]
\( \begin{array}{c} w = 7 \\ y = 10 \\ z = 9 \end{array} \)
8. \[ \begin{align*}
2x - 3y - 4z &= 21 \\
3x - 6y + 3z &= 0 \\
4x + 6y - z &= -22
\end{align*} \] \[ \begin{align*}
x &= -2 \\
y &= -3 \\
z &= -4
\end{align*} \]

9. \[ \begin{align*}
w + y &= 5 \\
w + z &= 6 \\
y + z &= 7
\end{align*} \] \[ \begin{align*}
w &= 2 \\
y &= 3 \\
z &= 4
\end{align*} \]

10. \[ \begin{align*}
\frac{3x - y}{17 - z} &= 1, & \frac{5x - 2z}{10 - 3y} &= 1, & \frac{7x + 4y}{5z + 3} &= 1
\end{align*} \] \[ x = 4, y = 0, z = 5 \]

11. \[ \begin{align*}
\frac{w + 2y}{7} &= \frac{3y + 4z}{8} = \frac{5x + 6z}{9}, & w + y - z &= 126, & w = 51, & y = 76, & z = 1
\end{align*} \]

12. \[ \begin{align*}
\frac{1}{x} + \frac{1}{y} &= \frac{3}{5}, & \frac{1}{y} + \frac{1}{z} &= \frac{5}{12}, & \frac{1}{x} + \frac{1}{z} &= \frac{7}{14},
\end{align*} \] \[ x = 2, y = 4, z = 6 \]

13. \[ w + y + z = m^2 + n^2 + p^2, \quad w + m^2 = y + n^2 = z + p^2 \]

14. \[ \frac{1}{x} - \frac{1}{y} = \frac{1}{2}, \quad \frac{1}{y} + \frac{1}{z} = \frac{3}{2}, \quad \frac{y}{z} + \frac{7}{y} = 13 \frac{3}{4}, \quad w = 3, \quad y = 4, \quad z = 5 \]

15. \[ w + y + z = 5, \quad w - y + z = 11, \quad y + z - x = 1 \]

16. \[ \begin{align*}
\frac{x}{2} + \frac{y}{3} + \frac{z}{4} &= 1 \\
x + \frac{y}{2} + \frac{z}{3} &= 1 \\
w + \frac{y}{3} + \frac{z}{4} &= 1
\end{align*} \] \[ w = y = z = 1 \frac{2}{3} \]

17. \[ \begin{align*}
\frac{xy}{x + y} &= 1 \frac{1}{3} \\
\frac{ax}{ax} &= 1 \frac{1}{3} \\
\frac{yz}{yz} &= 1 \frac{1}{7}
\end{align*} \] \[ w = 2, \quad y = 3, \quad z = 4 \]

18. \[ wx + 3, \quad yz = 6 \]

19. \[ wxyz = 2, \quad wxw = 6, \quad yzw = 3 \]

m
20. \( w^2yz = 6, \ xy^2z = 12, \ wyz^2 = 18 \) \( \omega = 1, \ y = 2, \ z = 3 \)

21. \( y^2z = 36; \ z^2w = 32, \ w^2y = 12 \) \( \omega = 2, \ y = 3, \ z = 4 \)

22. \( w^2y^2z = 1\frac{1}{2}, \ \omega^{-1}yz^2 = 18, \ w^2y^2z^2 = 108 \) \( \omega = 1, y = 2, z = 3 \)

23. \( (x + y)(y + z) = 35, \ (x + z)(y + x) = 42, \ (w + y)(x + z) = 30 \) \( \omega = 2, \ y = 3, \ z = 4 \)

24. \( \omega y = 1, \ (yz)^{\frac{1}{3}} = 24x, \ (wz)^{\frac{1}{3}} = 4y \) \( \omega = \frac{1}{\sqrt[3]{6}}, \ y = \frac{1}{\sqrt[3]{6}}, \ z = 96 \)

25. \( 3w + 4y = 19, \ 2y - z = 5, \ 7x + 2z = 13 \) \( \omega = 1, y = 4, z = 3 \)

26. \( 2(\omega - y) = 3z - 2, \ w - 3z = 3y - 1, 2\omega + 3z = 4(1 - y) \) \( \omega = 2, \ y = -3, z = 4 \)

27. \( 9xy = 20(x + y), \ 7xz = 12(x + z), \ 8yz = 15(y + z) \) \( \omega = 4, \ y = 5, z = 3 \)

28. \( \omega y + 20(\omega - y) = 0, \ yz + 30(y - z) = 0, \ 3\omega - 2z = 0 \) \( \omega = 4, y = 5, z = 6 \)

29. 
\begin{align*}
\omega + w + y + z &= 10 \quad \omega = 2 \\
2w + w + y + z &= 11 \quad y = 3 \\
w + y + z &= 8 \quad z = 4 \\
w + 2y + z &= 11 \quad w = 1
\end{align*}

30. \( \omega + y + z = 14, \ \omega^2 + y^2 + z^2 = 84, \ \omega z = y^2 \) \( \omega = 8, \ y = 4, \ z = 2 \)

31. \( \omega^2 + y^2 + z^2 = 3xyz \) and \( 3m - \omega + z = 3n - y + \omega = 3p - z + y \)
\[ \omega = m - n, \ y = n - p, \ z = p - m \]

32. \( \omega^2 + y^2 + z^2 = 29, \ \omega y + wz + yz = 26, \ x = y - z = 5 \)
\[ \omega = 4, \ y = 3, \ z = 2 \]

33. \( xyz = \frac{2}{3} (y^2 + z^2) = \frac{6}{5} (z^2 + \omega^2) = \frac{9}{4} (\omega^2 + y^2) \)
\[ \omega = 2, \ y = 3, \ z = 4 \]

34. \( x + y + z = 0, \ (m + n)x + (m + p)y + (n + p)z = 0, \ mnx + mpy \)
\[ + npx = 0 \quad \omega = \frac{1}{(m - p)(n - p)}, \ y = \frac{1}{(m - n)(n - p)}, \ z = \frac{1}{(m - n)(m - p)} \]
35. \( x - m^2 y + m^4 z = m^6, \quad a - n^2 y + n^4 z = n^6, \quad a - p^2 y + p^4 z = p^6 \)

\( x = m^3 n^2 p^2, \quad y = m^2 n^2 + m^2 p^2 + n^2 p^2, \quad z = m^2 + n^2 + p^2 \)

36. \( ax = a(x + y), \quad az = b(x + z), \quad yz = c(y + z) \)

\[ a = \frac{2abc}{ac + bc - ab}, \quad y = \frac{2abc}{ab + bc - ac}, \quad z = \frac{2abc}{ab + ac - bc} \]

37. \( a(x + y + z) = 18, \quad y(x + y + z) = 27, \quad z(x + y + z) = 36 \)

\[ x = 2, \quad y = 3, \quad z = 4 \]

38. \( ax = p(x + y + z), \quad yz = m(x + y + z), \quad az = n(x + y + z) \)

\[ a = \frac{mn + mp + np}{m}, \quad y = \frac{mn + mp + np}{n}, \quad z = \frac{mn + mp + np}{n} \]

39. \( ax = m(yz - ax - ax) = n(az - ay - yz) = p(ax - yz - az) \)

\[ x = \frac{2np}{n + p}, \quad y = \frac{2mp}{m + p}, \quad z = \frac{2mn}{m + n} \]

40. \( (x - m) + (y - n) + (z - p) = 0, \quad (n - p)x + (p - m)y + (m - n)z = 0, \quad p(x + y + z) = m^2 + n^2 + p^2 \)

41. \( a(x + y) = 1, \quad b(x + z) = 1, \quad c(y + z) = 1 \)

\[ x = \frac{1}{2a} + \frac{1}{2b} - \frac{1}{2c}, \quad y = \frac{1}{2a} - \frac{1}{2b} + \frac{1}{2c}, \quad z = \frac{1}{2b} + \frac{1}{2c} - \frac{1}{2a} \]

42. \( ax + y + z = 0, \quad m^2 x + n^2 y + p^2 z = 0 \) and \( n^2 p^2 x + m^2 p^2 y + m^2 n^2 z \)

\( = (n^2 - m^2)(n^2 - p^2)(p^2 - m^2), \quad a = n^2 - p^2, \quad y = p^2 - m^2, \quad z = m^2 - n^2 \)

43. \( ax + y + z = 70, \quad ay + z = 10, \quad ax + z = 20 \)

\[ a = 5, \quad y = 1, \quad z = 2 \]

44. \( x^2 + x^3 + z^3 = 3xyz, \quad x - y = y - n = z - p. \quad x = m - \frac{1}{3}(m + n + p) \)

\[ z = p - \frac{1}{3}(m + n + p), \quad y = n - \frac{1}{3}(m + n + p) \]

45. \( 3x^2 + 4y = 2xy, \quad 4y + 3z = 3yz, \quad 6z + 5x = 4xz \)

\[ a = 4, \quad y = 3, \quad z = 2 \]

46. \( \frac{1}{x} + \frac{1}{y} = c, \quad \frac{1}{y} + \frac{1}{z} = a, \quad \frac{1}{x} + \frac{1}{z} = b \)

\[ a = \frac{2}{b + c - a}, \quad y = \frac{2}{a - b + c}, \quad z = \frac{2}{a + b - c} \]
47. \( \frac{1}{xz} + \frac{1}{xy} = \frac{5}{3} \), \( \frac{1}{yz} + \frac{1}{xy} = \frac{7}{3} \), and \( \frac{1}{yz} + \frac{1}{xz} = \frac{1}{3} \)  
\( x = 1, y = 2, z = 3 \)

48. \( \omega(y + z) = 5, y(w + z) = 8, z(w + y) = 9 \)  
\( w = 2, y = 3, z = 5 \)

49. \( \frac{x}{w^2} = \frac{1}{2}, \quad \frac{y^2}{w} = \frac{75}{2}, \quad \omega y^2 z^2 = 2250 \)  
\( x = 2, y = 3, z = 5 \)

50. \( x^2(y - z) = 8, y^2(z - w) = -9, z^2(w - x) = -1 \)  
\( x = 2, y = 3, z = 1 \)

51. \( x^2(y + z) = 5, y^2(x + z) = 16, \omega yz = 6 \)  
\( x = 1, y = 2, z = 3 \)

52. \( \omega yz = \frac{3}{2}(y + z) = \frac{3}{2}(x + z) = \frac{3}{2}(w + y) \)  
\( x = 1, y = 2, z = 4 \)

53. \( x^2 y^2 z^2 = 324, \omega x^3 y^2 z^2 = 72, \omega yx^2 z^2 = 432 \)  
\( x = 2, y = 1, z = 3 \)

54. \( x^2 + y^2 + z^2 = 14, \omega + y + z = 6, \omega(y + z) = 5 \)  
\( x = 1, y = 3, z = 2 \)

55. \( \omega yz = 24, \omega yw = 12, yzw = 6, \omega zw = 8 \)  
\( x = 4, y = 3, z = 2, \omega = 1 \)

56. \( x + 2y + 3z + 4u = 27, 3x + 5y + 7z + u = 48, 5x + 8y + 10z - 2u = 65 \)  
\( 7x + 6y + 5z + 4u = 53 \)  
\( x = 1, \omega = 2, \upsilon = 3, z = 4 \)

57. \( \frac{1}{x} - \frac{1}{y} = \frac{5}{3}, \frac{1}{z} + \frac{1}{z} = \frac{1}{2} \)  
\( x = 12, y = -24, z = 36 \)

58. \( (m - p)x + (p - m)y + (n - m)z = 0, (m - n)x + (p - m)y + (n - p)z = 0 \)  
\( m(x - m) + p(y - p) + n(z - n) + 3mn\upsilon = 0 \)  
\( x = y = z = m^2 + n^2 + p^2 - mn - mp - np \)

59. \( (x - m) + (y - n) + (z - p) = 0 \)  
\( m(x - n) + n(y - p) + p(z - m) = 0 \)  
\( (n - p)x + (p - m)y = (n - m)z \)  
\( x = \frac{n + p}{2}, \quad y = \frac{m + p}{2}, \quad z = \frac{m + n}{2} \)

60. \( (x - m) + (y - n) + (z - p) = 0 \)  
\( (n - p)x + (p - m)y + (m - n)z = 0 \)  
\( px + qy + rz = mn + np + mp \)  
\( x = m, y = n, z = p \)
EXERCISE 24.

PROBLEMS.

1. What fraction is that which becomes equal to \( \frac{3}{5} \) if 1 be added to the numerator and equal to \( \frac{4}{7} \) if 2 be added to the denominator. Ans \( \frac{8}{9} \).

2. If the numerator of a certain fraction be increased by 1 and the denominator be diminished by 2 the value will be 1; if the numerator be increased by the denominator and the denominator diminished by the numerator the value will be \( \frac{3}{2} \); find the fraction. Ans \( \frac{7}{4} \).

3. If the length and breadth of a floor be increased by 3 ft. and 2 ft. respectively, the area would be increased by 40 sq. ft; but if the length be diminished by 2 ft. and the breadth increased by 4 ft. the area would be increased by 12 ft: find the length and breadth of the floor.

Ans 8 ft, 6 ft.

4. A certain number of two digits is equal to 3 times the sum of its digits, and if 45 be added to the number the digits are reversed: find the number.

Ans 27.

5. A and B can perform a piece of work together in 6 days; B and C in 7 days; and A, B and C in 4 days; how long would A and C take to do it.

Ans 5\( \frac{1}{4} \) days.

6. A boat sailing from Chandernagore with a fair wind, arrives in Calcutta in 2 hours; and on its return the wind being contrary, it proceeds 6 miles an hour slower than it went; now when it is half way over, the wind changing, it sails two miles an hour faster and reaches Chandernagore sooner than it would have done had the wind not changed in the proportion of 6 : 7. Required the rates of sailing and the distance between Chandernagore and Calcutta.

Ans Distance 22 miles and in returning it sails 5 and 7 miles an hour.

7. If A and B work together, they can earn 9 as in 2 days; if A and C work together, they can earn 1\( \text{Rs.} \) 4 as. in 4 days, and if B and C work together, they can earn 2\( \text{Rs.} \) 12 as. in 8 days; find what each will earn in a day.

Ans A 2 as; B 2\( \frac{1}{2} \) as; C 3 as.
8. Two trains, one 80 ft long and the other 50 ft, are observed to run on parallel rails; when they move in opposite directions they are found to pass each other in $1\frac{3}{7}$ seconds and when they move in the same direction they are found to pass each other in 9 seconds. Find the speeds of the trains.

Ans The faster runs 40 miles and the other 30 miles an hour.

9. A person sells 4 sheep and 5 goats for 30 Rs, and 9 sheep and 7 goats for 59 Rs: find the price of each. Ans Sheep 5Rs; Goats 2Rs

10. A cistern is filled in 24 minutes by 3 pipes, one of which conveys 8 gallons more, and another 7 gallons less than the third, every 3 minutes. The cistern holds 1050 gallons. How much flows through each pipe in a minute?

Ans $17\frac{5}{6}$, $14\frac{7}{2}$, $12\frac{5}{6}$

11. Two shepherds A and B while driving their cattle on a pasture were talking in the following way; A said to B “if 5 of your sheep come to my flock, the two flocks would be equal in number whereas if I send 10 sheep to yours, my flock would have only half the no of your flock” ; find the number of sheep in each flock.

Ans 40 and 50

12. A certain number of two digits is equal to 6 times the sum of the digits increased by 7, and if 27 he subtracted from the number the digits are reversed; find the number. Ans 85

13. A, B, C, D, E, play together on the condition that he who loses shall give to all the rest as much as they have already. First A loses, then B, then C, then D, then E; all lose in turn, and yet at the end of the fifth game they all have the same sum viz 32Rs. How much had each before they began to play. Ans A 81 Rs, B 41 Rs, C 21 Rs, D 11 Rs, E 6 Rs

14. A and B together can perform a piece of work in a days, A and C together the same in b days, and B and C together in c days; find the time in which each can perform it separately.

Ans A in $\frac{2abc}{ac+bc-ab}$, B in $\frac{2abc}{ab+bc-ac}$, C in $\frac{2abc}{ab+ac-bc}$ days
15. To complete a certain work A requires $m$ times as long a time as B and C together; B requires $n$ times as long as A and C together and C requires $p$ times as long as A and B together. Compare the times in which each would do it, and prove that \[ \frac{1}{m+1} + \frac{1}{n+1} = \frac{p}{p+1} \]

16. A person rowed 12 miles down a river and back again in 8 hours and found that it took thrice as long a time to row against the tide as to row with the stream. Find the rate of the stream, and of the boat in still water. Ans stream 2, boat 4 miles an hour.

17. A person rowed down the river Hoogly from Serampore to Calcutta a distance of 12 miles. When he came down 4 miles of the way there occurred the flood tide by which the progress of the boat was retarded. His rate of pulling in still water is 6 miles per hour and he observed that the time it occupied to row with the ebb tide was to the time it occupied to struggle against the flood tide as 3 : 16 also that the rate of the ebb tide is \( \frac{2}{3} \) of that of the flood tide. Find the rates of the ebb tide and flood tide. Ans Ebb tide 2 miles, flood tide 3 miles per hour.

18. Two vessels contain mixtures of sugar and water: in one there is twice as much sugar as water, in the other 3 times as much water as sugar. How much must be drawn off from each to fill a tumbler, in order that its contents may be half sugar and half water? It being known that the 3 vessels have the same capacity.

Ans \( \frac{2}{3} \) of the tumbler, from the 1st and \( \frac{2}{3} \) from the 2nd

19. Two casks A and B contain mixtures of wine and water; in A the quantity of wine is to the water as 4 : 3; in B the like ratio is 2 : 3. If A contains 70 gallons, what must B contain so that when the two are put together, the new mixture may be half wine and half water.

Ans 20 gallons of wine and 30 of water

20. There were two scholarships to be competed by 5 boys in a certain seminary, and after examination it was found that Nogender obtained \( \frac{2}{5} \)ths of the whole number of marks given, Jotish twice as
many as Nogender got more than Hjem, who obtained 3 times as many as Jotish got more than Surrut; that Surrut obtained half as many as Nogender, Jotish and Hjem together, and Preo one third more than the excess of the sum of Nogender, Jotish and Hjem's marks together over Surrut. Determine the successful boys.  

Ans Jotish and Preo.

21. A and B run a race round a two mile course. In the first heat B reaches the winning post 2 minutes before A. In the second heat A increases his speed 2 miles an hour, and B diminishes his by the same quantity; and A then reaches the winning post 2 minutes before B. Find at what rate each ran in the first heat.

Ans A 10 miles an hour and B 12 miles.

22. A person rows from Monghyr to Bhagulpore a distance of 20 miles and back again in 10 hours, the stream flowing uniformly in the same direction all the time and he finds that he can row two miles against the stream in the same time that he rows 3 miles with it. Find the time of his going and returning.

Ans, 4 hours, 6 hrs.

23. A, B, C, sit down to play; in the first game, A loses to each of B and C as much as each of them has, in the second B loses similarly to each of A and C, and in the third C loses similarly to each of A and B; and now they have each 24 Rs. What had they each at first.

Ans 30 Rs. 21 Rs. 12 Rs.

24. There is a number consisting of two digits, the 2nd of which is greater than the first; and if the number he divided by the sum of its digits the quotient is 4; but if the digits be inverted, and that number divided by a number greater by 2 than the difference of the digits, the quotient becomes 14. Required the number.

Ans 48.

25. A number consisting of 2 digits when divided by 4, gives a certain quotient and a remainder 3; when divided by 9 gives another quotient and remainder 8. Now the value of the digits on the left hand is equal to the quotient which was got when the number was divided by 9 and the other digit is equal to \( \frac{1}{17} \) th of the quotient got when the number was divided by 4, Required the number.

Ans 71.
26. If 19 lbs. of gold weigh 18 lbs. in water and 10 lbs. of silver weigh 9 lbs. in water; find the quantity of gold and silver weighing 106 lbs. in air and 99 lbs. in water. Ans 76, 30.

27. A and B severally cut packs of cards, so as to cut off less than they left; now the number of cards left by A added to the number cut off by B make 50; also the number of cards left by both exceed the number cut off, by 64. How many did each cut off?

Ans A 11; B 9.

28. A and B playing at bowls, says A to B, "if you will give me a guinea I will bet you ½ a crown to 18d on each game, and will play 36 games together." B won his guinea back again and £1.17s besides. How many games did each win?

Ans A 8, B 28 games.

29. One of the digits of a number is greater by 5 than the other. When the digits are inverted, the number becomes ⅔ of the original number. Find the digits.

Ans 72.

30. The sum of the digits is 9. Nine times one of the numbers they form is equal to twice the other number: find the digits.

Ans 8 and 1.

31. There is a number consisting of 3 digits, the right hand one being 0. If the left hand and middle digits be interchanged the number is increased by 180. If the left hand digits be halved and the middle and right hand digits be interchanged the number is diminished by 136. Find the number.

Ans 240.

32. Henry challenged Roberts to ride a bicycle of 1760 yds. Henry at the first heat gave Roberts a start of 60 ft. and beat him by ½ a min. At the second heat Henry gave Roberts a start of 32 seconds and beat him by 10½ yds. How many miles per hour did Henry run.

Ans 12 miles.
Exercise 25.

EXPONENTIAL EQUATIONS.

Solve the following equations,

1. $5^x = 125$ \hspace{1cm} \text{Ans} \ x = 3

2. $2^x = 256$ \hspace{1cm} \text{Ans} \ x = 8

3. $(1\frac{1}{2})^x = 5\frac{1}{10}$ \hspace{1cm} \text{Ans} \ x = 4

4. $3^{x+3} = 213$ \hspace{1cm} \text{Ans} \ x = 1

5. $25^x = 5$ \hspace{1cm} \text{Ans} \ x = \frac{1}{2}

6. $4^x + 2^{x+1} = 24$ \hspace{1cm} \text{Ans} \ x = 3

7. $2^{x+1} + 4^x = 80$ \hspace{1cm} \text{Ans} \ x = 3

8. $\alpha^{-x}(\alpha^x + b^{-x}) = \frac{a^2 b^2 + 1}{a^x b^x}$ \hspace{1cm} \text{Ans} \ x = 2

9. $4^{x+1} + \frac{1}{16} = 320$ \hspace{1cm} \text{Ans} \ x = 3

10. $(2^x)^2 + (4^x)^3 = 16 + 16 \times 2^{10}$ \hspace{1cm} \text{Ans} \ x = 2

11. $7^{x+y} = 2401$ and $6^{x+y} = 1296$ \hspace{1cm} \text{Ans} \ x = 4, \ y = 6

12. $x^3 = y^x$, and $x^4 = y^2$ \hspace{1cm} \text{Ans} \ x = 2, \ y = 4

13. $6^x = 54y$, and $4^x = 16y$ \hspace{1cm} \text{Ans} \ x = 3, \ y = 1

14. $2^y + 3^y = 11$, and $9^x - 4^y = 77$ \hspace{1cm} \text{Ans} \ x = 2, \ y = 1

15. $2^{x+y} = 32$, and $3^x - 9^y = 3$ \hspace{1cm} \text{Ans} \ x = 3, \ y = 1

16. $\sqrt{2} + 1 - \frac{1}{2} = 0$ \hspace{1cm} \text{Ans} \ x = 2

17. $\sqrt{a^x}, \sqrt{a^y} = a^3$ ; $(b^x)^3 = b^6. (b^y)^3$ \hspace{1cm} \text{Ans} \ x = 4, \ y = 2

18. $a^x \cdot a^{3y+1} = a^{3m+5}$ \hspace{1cm} \text{Ans} \ x = 2

$a^y \cdot (a^x + 3) = a^4m+7$ \hspace{1cm} \text{Ans} \ y = m
EXponential Equations.

19. \(5^{x+1} + \frac{1}{25^x} = 150\) \(x = 2\)

20. \(x^y = y^x, x^m = y^n\) \(x = \left(\frac{m}{n}\right)^{\frac{n}{m-n}}, y = \left(\frac{m}{n}\right)^{\frac{m}{n}}\)

21. \(4^x = 2^{x+y} \times 8\) and \(x = 3\), \(y = \frac{3}{2}\)

22. \(\frac{64^x}{8} = \frac{1}{4096^x}\) \(x = \frac{1}{6}\)

23. \((6.25) \times (4.25)^x = 1\) \(x = 2\)

24. \(m^2 - x^2 = m^2 - x^2\) \(x = 1\)

25. \(m^2(x-1), m^{x(x+2)} = m^{6x}\) \(x = 2\)

26. \(3^x \cdot 9^y = 27\) and \(4^x \cdot 8^y = 32\) \(x = 1, y = 1\)

Exercise 26.

RATIO.

1. Find the ratio of 12 as to 3 Rs. \(\text{Ans} \ 1 : 4\).

2. Arrange the following ratios in the order of magnitude

   \(5 : 6, 6 : 9, 8 : 10, 5 : 8\) \(\text{Ans} \frac{5}{8}, \frac{6}{9}, \frac{8}{10}, \frac{5}{8}\).

3. Find the ratio compounded of 2 : 13 and 26 : 30 \(\text{Ans} \frac{2}{5}\).

4. Find the ratio compounded of \(p : n, x : y\) and \(y : p\) \(\text{Ans} \frac{x}{y}\).

5. What is the proportion deducible from the equation \(x^2 + y^2 = 2ax\) \(\text{Ans} \ x : y : y : 2a - x\)

6. Four given numbers are represented by \(a, b, x, y\) required the quantity which added to each will make them proportionals.

   \(\text{Ans} \ \frac{ay - bx}{b + x - a - y}\)
7. Shew that the ratio \( a^2 - x^2 : a^2 + x^2 \) is greater than the ratio \( a - x : a + x \)

8. Prove that \( x^2 + y^2 : x^2 + y^2 \) is greater than \( x^2 + y^2 : x + y \)

9. Shew that the ratio \( x^2 + 9x + 20 : x^3 + 7x^2 + 14x + 8 \) is equal to the ratio \( x + 5 : x^2 + 3x + 2 \)

10. What quantity must be added to each of the terms of the ratio \( a : b \) that it may become equal to \( m : n \)

\[ \text{Ans} \quad \frac{am - bm}{m - n} \]

11. What is the ratio resulting from the composition of \( a^2 + b^2 : a^2 - b^2 \) and \( (a + b)^2 : (a - b)^2 \)

\[ \text{Ans} \quad \frac{(a + b)(a^2 + b^2)}{(a - b)^2} \]

12. If \( x \) be to \( y \) in the duplicate ratio of \( a : b \) and \( a \) be to \( b \) in the subduplicate ratio of \( a + x : a - y \) then will \( 2x : a = x - y : y \)

12 (a) Which is the greater, the ratio of \( 2x + 3 : \frac{2}{3}x + 3 \) or that of \( 2x + 6 : \frac{2}{3}x + 4 \)

\[ \text{Ans} \quad 2x + 6 : \frac{2}{3}x + 3 > 2x + 3 : \frac{2}{3}x + 3 \]

13. Compound the ratios of \( 41 : 6, 6 : 11 \) and \( 11 : 4 \) and then reduce the resulting ratio to its lowest terms

\[ \text{Ans} \quad 41 : 4 \]

14. Compound the subduplicate ratio of \( x^2 : y^2 \) with the triplicate ratio of \( x^2 \cdot y^2 \)

\[ \text{Ans} \quad x^2 : y^2 \]

15. Which is the greater \( x^2 + y^2 : x - y \) or \( x^2 - y^2 : x + y \),

\[ \text{Ans} \quad x^2 + y^2 : x - y \]

16. Find a mean proportional to \( \frac{a + b}{a - b} \) and \( \frac{a - b}{a + b} \)

\[ \text{Ans} \quad 1 \]

17. How is the ratio \( a : a - 2b \) affected by adding \( d \) to both terms

18. Find the ratio compounded of \( 16 : 5 \), the subtriplicate ratio of \( 8 : 27 \) and the sub-duplicate ratio of \( 16 : 25 \).

\[ \text{Ans} \quad 128 : 75 \]

19. If the ratios \( a - x : a + x \) and \( a + x : 2 + y \) be compounded together; shew that the resulting ratio is a ratio of equality.

20. Two numbers are in the ratio of \( 4 : 7 \) and if \( 4 \) be added to each, the ratio is that of \( 2 : 3 \); find the numbers.

\[ \text{Ans} \quad 8 : 14 \]
21. Two numbers are in the ratio of 9:11 and if 6 be taken from each the ratio is that of 3:4; find the numbers. Ans 18, 22

22. Two numbers are in the ratio of 2:3; if 6 be added to the less number and 3 taken from the greater number the ratio is that of 10 to 11; find the numbers. Ans 24:36

23. Find the number which added to each term of the ratio 9:7 makes it 4 of what it would have been if the same number had been taken from each term. Ans 3

24. If the ratios of 3a+2:6a+1 and of 2a+3:a+2 be compounded together, is the resulting ratio a ratio of greater or less inequality. Ans A ratio of greater inequality

25. Divide 18 into 3 parts such that the ratio of the first two shall be 1:3 and that of the last two 3:5. Ans 2, 6, 10

26. What is the proportion deducible from the equation \( ax = x^3 + a^3 \)
   Ans \( x = x + a : x^2 - ax + a^2 : a \)

27. Prove that \( a^3 + b^3 : a^2 + b^2 \) is greater than \( a^2 - b^2 : a + b \) unless \( a = b \)

28. Prove that if \( a:b \) is a greater ratio than \( c:d \), \( a + c:b + d \) is a less ratio than \( a:b \) but a greater than \( c:d \).

29. Find two numbers in the ratio of 4:5 such that their difference has to the difference of their squares the ratio of 1:18. Ans 8, 10

30. Find two numbers in the ratio of 5:7 such that their sum has to the sum of their squares the ratio of 3:37. Ans 10:14

31. Find two numbers in the ratio of 3:4 such that their sum has to the difference of their squares the ratio of 1:3
   Ans 9:12

32. Find \( x \) so that the ratio \( x:3 \) may be the duplicate ratio of \( 3:x \). Ans 3
33. Find \( x \) so that the ratio \( 6 - x : 12 - x \) may be the duplicate of the ratio \( 6 : 12. \) Ans 4

34. If \( \frac{x}{a - b} = \frac{y}{b - c} = \frac{z}{c - a} \) then \( a + y + z = 0 \)

35. Find the ratio compounded of \( x^6 - y^6 : x^4 + 2x^2y^2 + y^4, \) \( x^2 + y^2 : ax + by + cy, \) and \( x + y : x^3 - y^3. \) Ans \( (x + y)^2 : x^2 + y^2 \)

36. The ratio of the sum to the difference of two numbers is that of 7:3. Shew that if half the less be added to the greater, and half the greater to the less, the ratio of the numbers so formed will be that of 4:3.

37. If \( \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \) then each of these ratios

\[
= \sqrt{\left( \frac{pa^2 + qc^2 + re^2}{pf^2 + qd^2 + rf^2} \right)} \quad \text{or} \quad \left( \frac{pa^3 + qc^3 + re^3}{pf^3 + qd^3 + rf^3} \right)^{\frac{1}{3}}
\]

\[
= \left( \frac{pa^n + qc^n + re^n}{pf^n + qd^n + rf^n} \right)^{\frac{1}{n}}
\]

38. If \( \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \) prove that each of these ratios \( \frac{a + c + e}{b + d + f} \)

39. Prove that a ratio of greater inequality is diminished and of less inequality increased, by adding the same quantity to each of its terms.

40. Prove that a ratio of greater inequality is increased, and a ratio of less inequality is diminished by taking from both terms of the ratio any number which is less than each of those terms.

41. If \( \frac{a - m}{m - n} = \frac{b - p}{p - q} = \frac{c - r}{r - s} \) then each of these ratios

\[
= \frac{am - mb}{mq - mu} = \frac{br - pc}{ps - uq} = \frac{cm - ar}{wr - ms} = \frac{a + b + c - (m + p + r)}{m + p + r - (n + q + s)}
\]

42. There are two vessels A and B each containing a mixture of
water and sugar, A in the ratio of 3 : 4, B on the ratio of 2 : 3.
What quantity must be taken from each in order to form a third mixture which shall contain 7 seers of water and 10 seers of sugar.

\[\text{Ans } 1\text{st } 7 \text{ seers, } 2\text{nd } 10 \text{ seers.}\]

**Exercise 27.**

**PROPORTION.**

Find the value of \(x\) in each of the following proportions

1. \(12 : 16 :: 24 : x\) \[\text{Ans } x = 32\]
2. \(6 : 8 :: x : 12\) \[x = 9\]
3. \(3 : x :: x : 27\) \[x = 9\]
4. \(x : 4 :: 36 : x\) \[x = 12\]
5. \(x + 4 : x + 2 :: x + 8 : x + 5\) \[x = 4\]
6. \(4x - 4 : 2x :: 3x + 7 : 2x + 6\) \[x = 3\]
7. \(x^2 + x + 1 :: 62(x + 1) :: x^2 - x + 1 : 63(x - 1)\) \[x = 5\]
8. If \(xy = mn\) and \(yz = np\) then \(x : m :: z : p\)
8a. Find the 2nd proportional to the nos 32, 4, 1 \[\text{Ans } 8\]
8b. Find a fourth proportional to 2, 3, and 8 \[\text{Ans } 12\]
8c. Find the mean proportional to the nos 3 and 27 \[\text{Ans } 9\]
8d. Find the mean proportional between \(\frac{x + y}{x - y}\) and \(\frac{x^2 - y^2}{x^2 y^2}\)
\[\text{Ans } \frac{x + y}{xy}\]

If \(a : b :: c : d\) shew that

9a. \(a + b : b :: c + d : d\) (compare)
9b. \( a - b : b :: c - d : d \) (Dividendo)
9c. \( a : a - b :: c : c - d \) (Convertendo)
9d. \( a : c :: b : d \) (Alternendo)
9e. \( b : a :: d : c \) (Invertendo)
9f. \( a + b : a - b :: c + d : c - d \) (Compenendo and Dividendo)
10. \( a + c : c :: b + d : d \)
11. \( a + b : a :: c + d : c \)
12. \( a^2 : b^2 :: a^2 + c^2 : b^2 + d^2 \)
13. \( ma + nb : b :: mc + nd : d \)
14. \( \frac{a + c}{m} : a :: \frac{b + d}{m} : b \)
15. \( a^2 + b^2 : a^2 - b^2 :: c^2 + d^2 : c^2 - d^2 \)
16. \( ma + nc : pa + qc :: mb + nd : pb + qd \)
17. \( a + mc : b + md :: a : b \)
18. \( ma + b : mb + a :: mc + d : md + c \)
19. \( a + b + c + d : b + d :: c + d : d \)
20. \( (a \pm b)^2 : ab :: (c \pm d)^2 : cd \)
21. \( a(a + c) : c^2 :: b(b + d) : d^2 \)
22. \( a : c :: (a^3 + b^3) : (c^3 + d^3) \)
23. \( (a + b + c + d)(a + d - b - c) = (a + c - b - d)(a + b - c - d) \)

If \( a : y :: y : z \) Shew that
24. \( ax = y^2 ; x^2 - y^2 : x :: y^2 - z^2 : z \)
\( a^2 : z :: x^2 : y^2 \)
25. \( 7x + 9y : 7y + 9z :: 7x - 9y \cdot 7y - 9z \)
26. \( a + 2y + z : a^2 :: (y + z)^2 : z \)
27. \( a^4 + a^2 + z^2 = (a^2 - y^2 + z^2)(a^2 + y^2 + z^2) \)
28. \( y^2(x^2 - y^2 + z^2) = x^2 - y^2 + z^2; y^2(x + y + z) = x^2 + y^2 + z^2; \)
\( y^2(x + y + z) = x^2 + y^2 + z^2; y^2(x^2 + y^2 + z^2) = x^2 + y^2 + z^2 \)
\( y^2(x + y + z) = x^2 + y^2 + z^2; y^2(x^2 + y^2 + z^2) = x^2 + y^2 + z^2 \)

29. \( \frac{a^2 + b^2}{xy + yz} = \frac{xy + yz}{y^2 + z^2} \)

30. If \( x : y : z : w \) and \( m : n = p : q \) show that \( mn : ny : px : qw \)

31. \( n : m \) is : \( qz : pw \)

32. If \( w : y = z : q \) and \( py = qw \), show that \( w : q : z : p \)

33. If \( \frac{a}{b} = \frac{c}{d} \) prove that \( (a + b)^2 : (c + d)^2 : (c - d)^2 : d^2 : a^2 + b^2 : c^2 + d^2 \)

If \( x : y : z : w = r : s \) then

34. \( a^2 : y^2 : rs : sw \)

35. \( z^2 : w^2 = x^2 + z^2 + r^2 : y^2 + w^2 + s^2 \)

36. \( z^2 + w^2 : r^2 + s^2 : wz : rs \)

37. \( \alpha = mx + r : y - nw + s : nx - nz + qr : mw + nw : x : z \)

38. If \( m : n : a^2 : b^2 \) and \( n^2 + a^2 : a^2 - y^2 \) prove that \( r^2(a - b) = ab(a + b) \)

39. \( a_1, a_2, a_3, a_4, a_5 \), be in continued proportion then \( a_1 : a_5 = a_1 : a_2 \)

40. If \( 2a + 3b : 4a + 5b :: 2a + 3y : 4a + 5y \) then \( a : b = a : y \)

41. There are 3 numbers in continued proportion, the middle number is 13 and the sum of the others is 20; find the numbers.

Anss. 4, 8, 16

42. If \( x : y = z : w \) and \( w \) the greatest of the four, then \( w + w > y + w \) and \( w^2 + w^2 > y^2 + z^2 \)

43. If \( w : y = z : w \) show that \( (x - mw)^2 : (y - mw)^2 :: x^2 + z^2 : y^2 + w^2 \)
44. If \( \frac{a+c}{b} = \frac{c}{a} = \frac{a}{c-b} \) determine the ratios \( a : b : c \)

\[ \text{Ans. } 2:3:4 \]

45. If \( \omega : y : z : w \) prove that

\[ \omega(a - z - y + w) = (\omega - y)(\omega - z); \ (x + y) - (z + w) = \frac{(x + y)(y - w)}{y} \]

46. \( \sqrt{\frac{\omega^2 - \omega y^2 + \omega z^2 + y^2}{\omega^2 + y^2}} = \frac{\omega^2 - 5\omega z^2 - 5\omega w^2 + \omega^2}{\omega^2 + w^2} \)

47. \( \frac{x + y}{w + z} = \frac{1}{y} + \frac{1}{x} + \frac{(\omega - y)(\omega - z)}{xyz} \)

48. \( m\omega = n\omega \pm q\omega : p\omega = q\omega : m\omega + n\omega : p\omega + q\omega \)

49. \( \sqrt{\frac{(\omega^2 - x^2)(y^2 - z^2)}{\omega^2 - y^2}} = \frac{z}{y}; \ (x + y + z + w) = (x + y)(x + z) \)

50. \( \frac{\omega^2 - xy + y^2}{\omega^2} = \frac{z^2 - wz + w^2}{z^2} \)

51. \( \text{If } 4(x + y)(z + w) = yw \left( \frac{x + y}{y} + \frac{z + w}{w} \right)^2 \)

52. \( \frac{1}{mx} + \frac{1}{ny} = \frac{1}{yz} \left( \frac{a}{q} + \frac{y}{p} + \frac{z}{n} + \frac{w}{m} \right) = \frac{1}{pz} - \frac{1}{qw} \)

53. If \( \frac{\omega}{a} = \frac{y}{b} = \frac{z}{c} \) prove that each of these ratios = \( \frac{a^2 + b^2 + c^2}{\omega z^2 + 2 + m^2} \)

54. If \( x : y : z : w : m : n \) show that \( (aw + mz + nw)(y^2 + w^2 + n^2) = (yw + nw + ny)(aw^2 + az^2 + m^2) \)

If \( x : y = m : n \) prove that.

55. \( (\omega + m)(\omega^2 + m^2)(y - n)(y^2 - n^2) = (\omega - m)(\omega^2 - m^2)(y + n)(y^2 + n^2) \)

56. \( (px^2 + qxy + ry^2)(lm^2 + smn + tn^2) = (pm^2 + qmn + rny)(aw^2 + swy + ty^2) \)

57. \( nx \left( \frac{1}{x} - \frac{1}{2y} - \frac{1}{3m} + \frac{1}{4n} \right), \quad \frac{x^2 - y^2 - z^2 + n^2}{4} \)
58. \( \sqrt[3]{w^3 + m^3 w : y^3 n + y n^3 :: (w + m)^3 : (y + n)^3} \)

59. \( \sqrt[3]{x + y : m + n :: \frac{y - x}{m} : \frac{n - m}{m}} \)

60. If \( \frac{b}{a + b} = \frac{a + c - b}{b + c - a} = \frac{a + b + c}{2a + b + 2c} \) find the ratio between \( a \), \( b \) and \( c \). \( \text{Ans} \ 2, 3, 4 \)

61. If \( \frac{a}{b} = \frac{c}{f} \) prove that \( \frac{a^3 - c^3}{b^3 - d^3} = \frac{a^2 c^2 e^2 - (a^2 - c^2 + e^2)^3}{b^2 d^2 f^2 - (b^2 - d^2 + f^2)^3} \)

62. Find 3 numbers in continued proportion such that their sum may be 14 and the sum of their squares 84. \( \text{Ans} \ 2, 4, 8 \)

63. If \( a : b = b : c \) show that \( a - 2b + c = \frac{(a - b)^2}{a} = \frac{(b - c)^2}{c} \)

64. If \( a : b = b : c \) show that \( \frac{a + b + c}{a - b + c} = \frac{(a + b + c)^2}{a^2 + b^2 + c^2} \)

65. \( 2a : b :: b : 2c \) show \( a : c :: 4a^2 : b^2 \)

66. If \( a + \omega : a - \omega :: 11 : 7 \) find the value of \( a : \omega \). \( \text{Ans} \ 9 : 2 \)

67. Find two numbers in the ratio of 2 : 3 that their sum : their product :: 5 : 12. \( \text{Ans} \ 4 \text{ and } 6 \)

68. If \( a : b :: b : c \) and \( b : c :: c : d \) show that \( a + b : b + c :: b + c : c + d \)

69. Shew that \( a : b \) is duplicate of the ratio \( a + c : b + c \) if \( c \) be a mean proportional between \( a \) and \( b \)

70. Divide the number \( n \) into 2 parts so that one shall be to the other in the ratio of \( n : 1 \). \( \text{Ans} \ \frac{n^2}{n + 1}, \frac{n}{n + 1} \)

71. Find the number to which if one and three be successively added the resulting numbers are in the proportion of 2 : 7. \( \text{Ans} \ -1 \).
72. If 4 quantities are proportionals and the 2nd is a mean proportion between the 3rd and 4th, the 3rd will be a mean proportion between the 1st and 2nd.

If \( a : b = c : d = e : g \) shew that

\[ a - c : b - g : : e : d \; ; \; a : b = \sqrt[p]{p^a + q^a + r^a + s^a} = \sqrt[q]{q^b + r^b + s^b + t^b} \]

73. \( ma + nc + re = mb + nd + rg = pa + qa + le = pb + qd + lg \)

74. \( \frac{p^a + q^a}{a^2 - b^2} : (c + md)^2 = a^2 - b^2 : c^2 - d^2 \)

75. \( \frac{a^2 + c^2}{ab + cd} = \frac{ab + cd}{b^2 + d^2} \) prove that \( a : b = c : d \)

76. If \( \frac{a}{b} = \frac{c}{d} \) prove that \( \frac{a^2}{d^2} = \frac{b^2}{c^2} \)

77. If \( m : n : p : q \) : prove that \( \frac{(m - n)(m - p)}{m} = m - n + q - p \)

78. If the duplicate ratio of \( a + c \) to \( b + c \) be as \( a \) to \( b \) and if \( a \) and \( b \) are unequal prove that \( c^2 = ab \).

79. If \( x^2 : y^2 = m : n \) prove that \( \frac{x^4 + y^4}{m + n} = \frac{x^6}{x^2 + y^2(m^2 + n^2)} \)

80. If \( \frac{7x - 6y}{8a + 9b} = \frac{7y - 6z}{8b + 9c} = \frac{7z - 6x}{8c + 9a} \) prove that \( (172x + 337b + 239c)(w + y + z) = 17(a + b + c)(95x + 59y - 110z) \)

81. If \( \frac{3x + 3y - z}{5a + b - c} = \frac{3y + 3z - x}{5c + a - b} = \frac{3z + 3x - y}{5b + c - a} \) prove that \( \frac{x + y + z}{a + b + c} = \frac{6x + 10y + 14z}{4a + 14b + 12c} \)

82. If \( x + z = 2y ; x + y = 2z ; y + z = 2x \) prove that \( x = y = z \)

83. If \( \frac{2x - y}{2m + p} = \frac{2y - z}{2n + p} = \frac{2z - x}{2p + m} \) shew that \( \frac{21(x + 2y + 3z)}{x + y + z} = \frac{41m + 38n + 47p}{m + n + p} \)
85. If \( \frac{1 + aw}{x} = \frac{1 + wz}{w - z} \) prove that \( \frac{x - w}{y - z} = \frac{1 + wx}{1 + yz} \).

86. If \( \frac{x + 2y}{2a + b} = \frac{2y + z}{2b + c} = \frac{x + z}{2c + a} \) prove that \( 2(x + 2y + z)(9a + 16b + 17c) = 3(a + b + c)(7w + 18y + 12z) \).

If \( a : b = c : d = e : f \) shew that

87. \( a^2 - ab + b^2 : ab + ad - bc :: c^2 - cd + d^2 : cd - ad + bc \).

88. \( \frac{m - ne}{mb - nf} = \frac{a - c - e}{b - d - f} = \frac{ma - nc - pe}{mb - nd - pf} = \frac{m(a + e) - nc}{m(b + f) - nd} \).

89. \( a^3 - \sqrt{ace} + e^3 : b^3 - \sqrt{bdf} + f^3 :: (\sqrt{a} + \sqrt{c} + \sqrt{e})^3 = (\sqrt{b} + \sqrt{c} + \sqrt{f})^3 \).

90. \( a + b : c + d :: a^2(c - d) : c^2(a - b) \).

91. If \( \frac{a}{b} = \frac{c}{d} = \frac{m}{n} \) prove that \( \sqrt{(ab)} + \sqrt{(cd)} + \sqrt{(mn)} \).

\[ = \sqrt{(a + c + m)(b + d + n)} \quad \text{and} \quad (a^2 + c^2 + m^2)(b^2 + d^2 + n^2) = (ab + cd + mn)^2 \]

92. If \( \frac{ax + by}{cz} = \frac{cz + ax}{by} = \frac{by + cz}{ax} = w + y + z \) find \( x, y, z \).

\[ x = \frac{2bc}{bc + ac + ab}, \quad y = \frac{2ac}{bc + ac + ab}, \quad z = \frac{2ab}{bc + ac + ab} \]

93. Before noon, a clock which is too fast, and points to afternoon time, is put back 5 hrs 40 min ; and it is observed that the time before shewn is to the true time as 29 : 105. Required the true time.

Ans 8 hrs 45 min.

94. A gentleman sitting in a railway saloon observes that another train running on a parallel line in the opposite direction occupies two seconds in passing him, but if the two trains had been proceeding in the same direction, it would have taken 30 seconds to pass him; if the
rate of the faster train be 24 miles an hour, find the rate of the other and the length of the quicker train.

Ans 21 miles an hour and length 132 ft.

95. There are four towns situated in the order of the four letters A, B, C, D. The distance from A to D is 34 miles, the distance from A to B : distance from C to D :: 2 : 3, and \( \frac{1}{4} \) of the distance from A to B added to half the distance from C to D is three times the distance from B to C; what are the respective distances.

Ans \( AB = 12 \), \( BC = 4 \), \( CD = 18 \)

96. A was following B, and after a time B turned and without changing his pace walked in the opposite direction and in consequence A approached B 3 times as fast as before. Compare the rates of A and B.

Ans 2 : 1,

27. Two locomotive engines A and B set out to meet each other, A leaving the station Howrah at the same time that B left Ranneegunge, and on meeting, it appeared that A had run 18 miles more than B; and that A could have run B's distance in \( 15\frac{3}{4} \) hours, but B would have been 28 hours in performing A's journey. What was the distance between Howrah and Ranneegunge and the distance run by each engine.

Ans distance 126 miles; A 72, B 54 miles.

98. Two pieces of long-cloth of equal goodness but of different lengths, were bought, the one for 5 Rs, the other for 6 Rs 8 as; now if the lengths of both pieces were increased by 10 yds, the numbers resulting would be in the proportion of 5 to 6. How long was each piece and how much did they cost a yd.

Ans 20 and 26 yds; 4 as
Exercise 28.

Identities.

Shew that

1. \((x^2 + y^2)(a^2 + b^2) = (ax + by)^2 + (bx - ay)^2\)
2. \((x + y)^2 + (y + z)^2 + (x + z)^2 = (x + y + z)^2 + x^2 + y^2 + z^2\)
3. \(x(1 - y^2)(1 - z^2) + y(1 - z^2)(1 - w^2) + z(1 - w^2)(1 + y^2) = 4xyz\)
   \(\cdot (x + y + z - xyz)(1 - xy - yz - wz)\)
4. \((x + y + z)^3 = x^3 + y^3 + z^3 + 3(x + y)(x + z)(y + z) = x^3 + y^3 + z^3\)
   \(+ 3x^2(y + z) + 3y^2(x + z) + 3z^2(x + y) + 6xyz\)
5. \(\frac{x^4(y^2 - x^2) + y^4(z^2 - x^2) + z^4(x^2 - y^2)}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)} = (x + y)(y + z)(x + z)\)
   \(\cdot (L. C. E. 1881)\)
6. \((x - b)(x - a)(x - c) + (b - c)(x - b)(x - c) + (c - a)(x - c)(x - a)\)
   \(\cdot = (a - b)(a - c)(b - c)\)
7. Shew that the difference of the squares of any two consecutive numbers is equal to the sum of the numbers.
8. \((1 + \frac{x}{y})(1 + \frac{y}{z})(1 + \frac{z}{x}) + 1 = (x + y + z)(\frac{1}{x} + \frac{1}{y} + \frac{1}{z})\)
9. \(\frac{x^2 \left( \frac{1}{y} - \frac{x}{z} \right) + y^2 \left( \frac{1}{z} - \frac{1}{x} \right) + z^2 \left( \frac{1}{x} - \frac{1}{y} \right)}{x \left( \frac{1}{y} - \frac{1}{z} \right) + y \left( \frac{1}{z} - \frac{1}{x} \right) + z \left( \frac{1}{x} - \frac{1}{y} \right)} = x + y + z\)
   \(\cdot (F. A. 1881)\)
10. \((a + b + c)^2 + (a + b + c)(a - b - c) - 2ab - 2ac = 2a^2\)
11. \((a + b + c)^2 - (b + c - a)^2 - (a - b - c)^2 - (a + b - c)^2 = 2a^2b^2c^2\)
12. \((x + y + z)^2 + (x + y - z)^2 + (x + z - y)^2 + (y + z - x)^2\)
   \(\cdot = 4(x^2 + y^2 + z^2)\)
13. \((x - y)^2 + (y - z)^2 + (z - x)^2 = 3(x - y)(y - z)(z - x)\)
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14. \((m^2 - np)^2 + (n^2 - mp)^2 + (p^2 - mn)^2 = (m^2 + n^2 + p^2 - 3mn)\)
   \[= 3(m^2 - np)(n^2 - mp)(p^2 - mn)\]

15. \((x + y)^2 + (x + z)^2 + (x + w)^2 + (y + z)^2 + (y + m)^2 + (z + w)^2 = (x + y + z + w)^2 + 2(x^2 + y^2 + z^2 + w^2)\]

16. \({(4a + 3b)^2 + (6a - 2y)^2\}}\{(4a + 3b)^2 -(6a - 2y)^2\}\]
   \[= 18(4a^2 + y^2)(5y^2 - 20a^2 + 48ay)\]

17. \((a + b)^2(a + c)^2 - (a + c + b)^2 - (b + d)^2 = 2(a - d)(a - b + c + d)\)

18. Shew that the continued product of any four consecutive numbers together with unity is a square number.

20. \((m + n + p)^2 + m^2 + n^2 + p^2 = 12(mn + mp + np) + (m + n + p)^2 + (m + p)^2\)

21. \(\left(\frac{a}{b} + \frac{b}{a}\right)^2 + \left(\frac{a}{c} + \frac{c}{a}\right)^2 + \left(\frac{b}{c} + \frac{c}{b}\right)^2 = 4 + \left(\frac{a}{b} + \frac{b}{a}\right)\left(\frac{a}{c} + \frac{c}{a}\right)\left(\frac{b}{c} + \frac{c}{b}\right)\)

22. \(\frac{x^2 - y^2}{x^2 - z^2} + \frac{y^2 - z^2}{y^2 - x^2} + \frac{z^2 - x^2}{z^2 - y^2} = (x + y)(y + z)(x + z)\)

23. \(x - y)^2 + (x - z)^2 + (y - z)^2 = 2\{(x - y)^2(y - z)^2 + (z - w)^2\} + (z - w)^2(z - w)^2\}

24. \(\{(m - n)^2 + (n - p)^2 + (p - m)^2\} = 2\{(m - n)^2 + (n - p)^2 + (p - m)^2\}\)

25. \((x + y)^7 - (x^7 + y^7) = 7xy(x + y)(x^2 + xy + y^2)\)

26. \(\left(1 - \frac{x}{y}\right)^{-1} \left(1 - \frac{y}{x}\right)^{-1} + \left(1 - \frac{z}{y}\right)^{-1} \left(1 - \frac{y}{z}\right)^{-1} \left(1 - \frac{x}{z}\right)^{-1} = 1\)

27. Shew that the product of any four consecutive integers increased by 16 is a perfect square.

28. \(\frac{2^{13} + 2 \times 8^x - 16}{4^{13} + 32^{3x} + 2^{3x}} = 1\)

29. \((x + y + z)(a^3 + b^3 + c^3 + xyz) = a^4 + b^4 + c^4 + (xy + yz + wz)(a^3 + b^3 + c^3)\)
30. \((mx - nx)^2 + (nx - mz)^2 + (nz - py)^2 + (mx + ny + pz)^2\)
\[= (m^2 + n^2 + p^2)(x^2 + y^2 + z^2)\]

31. \((y - z)^7 + (z - x)^7 + (x - y)^7\) = \(((y - z)^{-1} + (z - x)^{-1} + (x - y)^{-1})^7\)

32. \[\frac{25}{21}(y - z)^7 + (z - x)^7 + (x - y)^7\]
\[= \frac{((y - z)^{\frac{5}{3}} + (z - x)^{\frac{5}{3}} + (x - y)^{\frac{5}{3}})^3}{(y - z)^3 + (z - x)^3 + (x - y)^3}\]

33. \((1 - a)(1 + a)(1 + a^2)(1 + a^4)\) & c. to \(n + 1\) factors = \(1 - a^{2n}\)

34. \(((mp + nq)a + (mq - np)y)^2 = (mp + nq)a - (mq - np)y)^2\)
\[= (m^2 + n^2)(p^2 + q^2)(a^2 + y^2)\]

35. \(x^4 + y^4 + z^4 - (x^2 - z^2)^2 + (y^2 - z^2)^2 + (x^2 - y^2)^2\) = \((x + y + z)^4\)
\[= (x + y + z)(x - y + z)(y + z - x)\]

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**Exercise 29.**

**CONDITIONAL IDENTITIES.**

If \(x + y + z = 0\) prove that

1. \(x^3 + y^3 + z^3 = 3xyz\)

2. \(\frac{x^3}{yz} + \frac{y^3}{xz} + \frac{z^3}{xy} = 3\)

3. \(\frac{y^3 - z^3}{y - z} - \frac{z^3 - x^3}{z - x} - \frac{x^3 - y^3}{x - y} = 0\)

4. \(x(x^2 - yz) + y(y^2 - wz) + z(z^2 - wx) = 0\)

5. \(x^2 - yz = y^2 - wz = z^2 - wx\)

6. If \(x^3 + y^3 + z^3 = 0\) prove that \((x + y + z)^3 = 27xyz\)

7. If \(x(y + z)^{-1} + z(x + y)^{-1} = 2y(x + z)^{-1}\) prove that \(\frac{1}{3}(x^2 + z^2) = y^2\) and \(x + y = -z\)
8. If \( a = 2c - b \) prove that \( \frac{a^2(b + c) + b^2(a + c) + c^2(a + b)}{c^2} = 6 \)

9. If \( x + \frac{1}{x} = p + \frac{1}{p} \) prove that \( x^3 + \frac{1}{x^3} = p^3 + \frac{1}{p^3} \)

10. If \( \omega(y^2 - x^2) \neq \frac{y - x}{x^2} \), shew that \( x^2 = (y + x)(y - x) \)

11. If \( \frac{y - x}{x - z} + \frac{x^2}{y - z} = \frac{(xy)^{-1}}{(x - z)(y - z)} \), prove that \((x + y)z = xy = x^2 + y^2 \)

12. If \( \frac{2x}{y - x} + \frac{2z}{y - z} = 0 \) prove that \( x^{-1} + z^{-1} = 2y^{-1} \)

13. If \( \frac{a + 2a}{w - b} + (2(a + b) - w)(w - a)^{-1} = 2 \) prove that \( a = 2(a + b) \)

14. If \( x^2 + z^2 = 2y^2 \) prove that \((y + z)^{-1} + (x + y)^{-1} = 2(a + z)^{-1} \)

15. If \( w^y y = 1 \) shew that \( \frac{w}{y} = y^{-(x - 1)} \)

16. If \( w^y + w^z = w^2 + w^2 = 1 \) shew that \( \frac{w^y + w^z}{w + z} = \frac{w - z}{w^y - w^z} \)

and \((w^y - w^z)^2 + (w^y + w^z)^2 = 1 \)

17. If \( 2b = a + c \) prove that \( \frac{1}{2} (a + b + c)^3 = a^2(b + c) + b^2(a + c) + c^2(a + b) \)

18. If \( 2wz = \sqrt{(a + y + z)(a + y + z)(x + z - y)(y + z - x)} \) shew that \( z^2 = w^2 + y^2 \)

19. If \( c = a(b + c)^{2/3} + b (1 - a^2)^{1/3} \), shew that \( (a + b + c)(a + b - c) \)

\( (a - b + c)(b + c - a) = 4a^2b^2c^2 \)

20. If \( a^2 - b^2 = w^2 \), \( b^2 - ac = y^2 \), and \( c^2 - ab = z^2 \), shew that \( aw^2 + by^2 + cz^2 = (a + b + c)(w^2 + y^2 + z^2) \),

21. If \( n^2p^2x = m^2(y - z) \), \( m^2p^2y = n^2(z - x) \) and \( m^2n^2z = p^2(x - y) \),

shew that \( n^4p^4 + m^4p^4 + m^4n^4 + m^4n^4 + p^4 = 0 \)
22. If \( \frac{ny + na}{m} = \frac{ny - na}{n} = 1 \) shew that \( x^2 + y^2 = 1 \).

23. If \( x^2(x^2 - 2m^2) = (y^2 + z^2)^2 \), \( y^2(y^2 - 2m^2) = (z^2 + x^2)^2 \) prove that
\[
x^2 + y^2 + z^2 = m^2
\]

24. If \( (m^2 - np)(n^2 - mp)(p^2 - mn) = 0 \) prove that
\[
m^{-2} + n^{-2} + p^{-2} = (m^2 + n^2 + p^2)(mpn)^{-2}
\]

If \( x = a + b + c \) prove that

25. \( x(x - 2b)(x - 2c) + x(x - 2c)(x - 2a) + x(x - 2a)(x - 2b) \)
\[
= (x - 2a)(x - 2b)(x - 2c) + 8abc
\]

26. \( a^2 - \left( \frac{a^2 + b^2 - c^2}{2b} \right)^2 = \frac{x(x - 2c)(x - 2b)(x - 2a)}{4b^2} \)

27. \( x(x - a)(x - b) + c(x + a)(x + b) + a(x - a)(x - c) + x(x + a)(x - c) \)
\[
= (x + a)(x + b)(x + c)
\]

28. \( (x - a)^2 + (x - b)^2 + (x - c)^2 - 3(x - a)(x - b)(x - c) \)
\[
= 2(a^2 + b^2 + c^2 - 3abc)
\]

29. \( x^4 = a^4 + b^4 + c^4 + 3a^2b^2 + 3b^2c^2 + 3c^2a^2 + 6abc \)

If \( 2x = a + b + c \) shew that

30. \( (x - a)^2 + (x - b)^2 + (x - c)^2 + x^2 = a^2 + b^2 + c^2 \)

31. \( (x - a)(x - b)(x - c) = x^3 - \frac{3}{2}(a^2 + b^2 + c^2) - abc \)

32. \( (x - a)^{-1} + (x - b)^{-1} + (x - c)^{-1} = x^{-1} \)
\[
= abc x^{-1} - (x - a)^{-1} - (x - b)^{-1} - (x - c)^{-1}
\]

33. \( 2(x - a)(x - b) + 2(x - b)(x - c) + 2(x - c)(x - a) + a^2 + b^2 + c^2 \)
\[
= 2x^2
\]

34. \( 2(x - a)(x - b)(x - c) + a(x - b)(x - c) + b(x - a)(x - c) + c(x - a)(x - b) \)
\[
+ a(x - a)(x - b) - abc = 0
\]

35. \( 16x(x - a)(x - b)(x - c) = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^2 - b^2 - c^2 \)

36. \( (2x - b)(x - c) + (2x - c)(x - a) + (2x - a)(x - b) = ab + ac + bc \)
37. If \( a = b + c + \ldots \) to \( r \) terms shew that \( \frac{x - a}{x} + \frac{x - b}{x} + \frac{x - c}{x} + \ldots = r - 1 \)

38. If \( p_1 + p_2 + p_3 + \ldots + p_r = \frac{r^2}{3} \), \( p_1^2 + p_2^2 + p_3^2 + \ldots p_r^2 = \frac{r^3}{3} \)

and \( p_1^3 + p_2^3 + p_3^3 + \ldots p_r^3 = 0 \)

prove that \( (x - p_1)^3 + (x - p_2)^3 + (x - p_3)^3 + \ldots (x - p_r)^3 = r \)

39. If \( 2x = a + b + c \) and \( 2x^2 = a^2 + b^2 + c^2 \) shew that \( x \)

\[
(s^2 - a^2)(s^2 - b^2) + (s^2 - b^2)(s^2 - c^2) + (s^2 - c^2)(s^2 - a^2)
= 4s(x - a)(x - b)(x - c)
\]

40. If \( x^{-1} + y^{-1} + z^{-1} = (x + y + z)^{-1} \) shew that \( (x^{-1} + y^{-1} + z^{-1})^3 = (x^3 + y^3 + z^3)^{-1} \)
and generally \( (x^{-1} + y^{-1} + z^{-1})^{2r+1} = (x^{2r+1} + y^{2r+1} + z^{2r+1})^{-1} \)

41. If \( x^2 + y^2 = 1 \) shew that \( 2(x^6 + y^6) - 3(x^4 + y^4) + 1 = 0 \)

42. If \( m = ax + cy + bz, n = bx + ay + cz, p = bx + ay + cz \) shew that \( m^3 + n^3 + p^3 - 3mnp = (a^3 + b^3 + c^3 - 3abc)(a^3 + y^3 + z^3 - 3xyz) \)

43. If \( \frac{a + b - c}{a + b} = \frac{b + c - a}{b + c} = \frac{c + a - b}{c + a} \) and that \( a + b + c \) is not \( = 0 \)

show that, \( a = b = c \)

44. If \( a^m = a^m, b = y^b, y = z^p \) shew that \( mnp = 0 \).

45. If \( \frac{a^2 + b^2}{a^3} + \frac{a^2 + c^2}{b^3} + \frac{b^2 + c^2}{c^3} = a - b - c + a^2 + b^2 + c^2 \)

that \( \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = 1 \), if \( a^2 + b^2 + c^2 \) is not \( = 0 \)

46. If \( \frac{a + b + c - abc}{2b} = \frac{1 - bc - ac - ab}{1 - b^2} \) prove that \( a + c - b \)

(1 - ac)

47. If \( ab + ac + bc = 1 \) prove that \( \frac{a + b}{1 - ab} + \frac{b + c}{1 - bc} + \frac{a + c}{1 - ac} \)

\[
\frac{(a + b)(b + c)(a + c)}{(1 - ab)(1 - bc)(1 - ac)}
\]
48. If \( \frac{a-b}{c} + \frac{b-c}{a} + \frac{a+c}{b} = 1 \) and \( a-b+c \) is not = 0

\[ \frac{1}{a} = \frac{1}{b} + \frac{1}{c} \]

49. If \( \omega + y + z = 0 \) show that \( \frac{\omega^2}{2\omega^2 + yz} + \frac{y^2}{2y^2 + wz} + \frac{z^2}{2z^2 + xy} = 1 \)

50. If \( \frac{a}{b+c} = \omega, \frac{b}{a+c} = y, \frac{c}{a+b} = z \) show that

\[ \frac{a^2}{x(1-xy)} \quad \frac{b^2}{y(1-zx)} \quad \frac{c^2}{z(1-xy)} \]

and \( xy + yz + xz + 2xyz = 1 \)

51. If \( \omega = ab + ac + bc \) show that \( (x-a)(x-b)(x-c) + b(x-a) \)

\( (x-c) + c(x-a)(x-b) - 2abc = 0 \)

52. If \( 2x = \omega + b + c \) show that \( (x-c)(2\omega - c) + (\omega - b)(2x - b) \)

\( + (x-a)(2x-a) = a^2 + b^2 + c^2 \)

53. If \( \omega^2 + yz)(y^2 + wz) (z^2 + xy = 0 \) prove that

\[ \frac{1}{\omega^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{\omega^2 + y^2 + z^2}{\omega^2 y^2 z^2} \]

54. If \( 2s = a + b + c \) show that \( s^2 - (s-a)^2 - (s-b)^2 - (s-c)^2 = 3abc \)

55. If \( \omega = cy + bx, y = az + cx, z = bx + ay \) show that

\[ \frac{\omega^2}{1-\omega^2} = \frac{y^2}{1-y^2} = \frac{z^2}{1-z^2} \]

56. If \( \omega^2 (y + z) = m^2, y^2 (x + z) = n^2, z^2 (x + y) = p^2, \omega y z = mnp \)

show that \( m^2 + n^2 + p^2 + mnp = 0 \)

57. If \( \omega (y + z)^2 = 1 - \frac{y^2}{1 - \omega^2} \) prove that \( \omega^2 + y^2 + z^2 + 2xyz = 1 \)

(Cambridge examination papers.)

58. If \( a + ac + bc = 1 \) show that

\[ \frac{4abc}{(1-a^2)(1-b^2)(1-c^2)} \]
59. If \( x = ny + pz + qu, y = mx + px + qu, z = mx + ny + qu, \)
\[ u = mx + ny + pz \text{ shew that } \frac{m}{1 + m} + \frac{n}{1 + n} + \frac{p}{1 + p} + \frac{q}{1 + q} = 1 \]

60. If \( xy + yz + zx = 1 \) prove that \( \frac{1 + x^2}{(x + y)(x + z)} + \frac{1 + y^2}{(y + z)(y + x)} + \frac{1 + z^2}{(x + z)(y + x)} = 3 \)

61. If \( x + y + z = xyz \) prove that \( \frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2} = \frac{(3x - x^3)(3y - y^3)(3z - z^3)}{(1 - 3x^2)(1 - 3y^2)(1 - 3z^2)} \)

62. If \( c = \frac{2ab}{a + b} \) shew that \( \frac{1}{c - a} + \frac{1}{c - b} = \frac{1}{a} + \frac{1}{b} \)

63. If \( (a^2 + bc)^2 (b^2 + ac)^2 (c^2 + ab)^2 = (a^2 - bc)^2 (b^2 - ac)^2 (c^2 - ab)^2 \) prove that either \( a^2 + b^2 + c^2 + abc = 0 \) or \( a^{-2} + b^{-2} + c^{-2} + a^{-1}b^{-1}c^{-1} = 0 \)

64. If \( c^2 = a^2 + b^2 \) shew that \( (a + b + c)(a + b - c)(a + c - b) = (b + c - a) = 4a^2b^2 \)

65. \( 2y + b^{-x} = b^x \) prove that \( b^{-x} = y + \sqrt{1 + y^2} \)

66. If \( x + y + z = 2a, \) and \( x^2 + xy + y^2 + z^2 = 2a(x + y) \) shew that \( (x - a)^2 + (y - a)^2 + (z - a)^2 = a^2 \)

67. \( \sqrt{x} + \sqrt{y} + \sqrt{z} = 0 \) Shew that
\[ \{x^2 + y^2 + z^2 - 2(yz + xz + xy)\}^2 = 128 \, xyz(x + y + z) \]

68. If \( \frac{x}{b + c - a} = \frac{y}{c + a - b} = \frac{z}{a + b - c} \) shew that

(1) \( (b - c)x + (c - a)y + (a - b)z = 0 \)

(2) \( (a + b + c)(ax + by + cz) = 2(x + y + z)(ax + by + cx) \)

69. If \( b^2 = ac \) prove that \( a^2b^2c^2(q^{-b} + b^{-a} + c^{-a}) = a^2 + b^2 + c^2 \)
70. If \( \frac{a}{b} = \frac{y}{c} = \frac{z}{d} \) shew that
\[
\frac{x^2 + a^2}{x + a} + \frac{y^2 + b^2}{y + b} + \frac{z^2 + c^2}{z + c} = \frac{(x + y + z)^2 + (x + b + c)^2}{(x + y + z) + (x + b + c)}
\]
(Cambridge papers.)

71. If \( \frac{a + b}{a - b} = \frac{b + c}{2(b - c)} = \frac{a + c}{3(c - a)} \) shew that
\[
8a + 9b + 5c = 0
\]

72. If \( \frac{a + y}{3a - b} = \frac{y + z}{3b - c} = \frac{x + z}{3c - a} \) shew that
\[
(x + y + z)(a^2 + b^2 + c^2) = (ax + by + cz)(a + b + c)
\]

73. If \( 2x = a^2 - b^2 - c^2 + d^2 \) prove that \( (ad + bc)^2 - a^2 \)
\[
= \frac{1}{4}(a + b + c - d)(a + b + d - c)(a + c + d - b)(b + c + d - a)
\]

74. If \( x + y + z = \frac{14x}{3} = \frac{7y}{2} \) prove that \( x + y = z \)

75. If \( x - y = 7z \) and \( x - z = 4y \) shew that \( x = 9(y - z) \)

76. If \( x^2 + y^2 = 123z \) and \( x^2 - y^2 = 27z \) shew that \( xy = 60z \)

77. If \( ab = \frac{1}{2}(a + b)(p + q) - pq \) and \( cd = \frac{1}{2}(c + d)(p + q) - pq \)
shew that
\[
\left( \frac{p - q}{2} \right) = \frac{(a - c)(a - d)(b - c)(b - d)}{(a + b - c - d)^2}
\]

78. If \( ax^2 = by^2 = cz^2 \) and \( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{d} \) shew that \( ax^2 + by^2 + cz^2 \)
\[
= \left( \frac{a^3}{d} + b^3 + c^3 \right) \]

79. If \( 2x = a + b + c \) shew that \( (x - a)^3 + (x - b)^3 + (x - c)^3 \)
\[
- 3(x - a)(x - b)(x - c) = \frac{1}{4}(a^2 + b^2 + c^2 - 3abc)
\]

80. If \( x + y = 2z \), \( x + z = 2y \) and \( y + z = 2x \) prove that \( x = y = z \)

81. If \( x + y + z = xyz \) prove that \( \frac{2x}{1 - x^2} + \frac{2y}{1 - y^2} + \frac{2z}{1 - z^2} \)
\[
= \frac{2x}{1 - x^2} \cdot \frac{2y}{1 - y^2} \cdot \frac{2z}{1 - z^2}
\]
82. If \( \frac{w^2 - yz}{w - xyz} = \frac{y^2 - xz}{y - xyz} \) prove that each of these ratios
\[
\frac{z^2 - xy}{z - xyz} = \frac{w + y + z}{w} = 1 + \frac{1}{y} + \frac{1}{z}
\]
(Cambridge papers.)

83. If \( a^2 + bc = a^2, \ y^2 + ac = b^2, \ z^2 + ab = c^2 \) show that
\[
(a + b + c)(a^2 + b^2 + c^2) = a(a^2 + b^2 + c^2)
\]

84. If \( \frac{cy + bz}{b - c} = \frac{cz + ax}{c - a} = \frac{bx + ay}{a - b} \) show that
\[
(a + b + c)(a + y + z) = ab + by + cx.
\]

85. If \( a + b + c = 0 \) show that \( c(a^2 + b^2 - c^2) = b(a^2 + c^2 - b^2) \)
\[
= 2(a^2 + b^2 - c^2)
\]

86. If \( x = \frac{2ab}{b^2 + 1} \) show that \( \frac{\sqrt{(a + x)} + \sqrt{(a - x)}}{\sqrt{(a + x)} - \sqrt{(a - x)}} = b \)

87. If \( a + b + c = 0 \) show that \( (a^2 + b^2 + c^2)^2 = 4(a^4 + b^4 + c^4) \)
\[
= 4(a^2 + b^2 + c^2)^2
\]

88. If \( a = b^2, \ b = c^3 \) and \( c = a^2 \) show that \( xyz = 1 \)

89. If \( a = yz, \ y = wz \) and \( z = w^2 \) prove that \( xyz = 1 \)

90. If \( \frac{b^2z^2 + a^2yz}{a} = \frac{a^2 + b^2z}{b} \) show that \( b^2 + a^2 = ab \) or \( a^2 + b^2 = ab \)

Exercise 30.

INEQUALITIES.

1. If \( a \) and \( y \) are any two positive integers prove that \( x^2 + y^2 > 2xy \).

2. If \( a, y, \) and \( z \) be such that any two of them are greater than the third, prove that \( 2xy + 2xz + 2yz - a^2 - y^2 - z^2 > 0 \).

3. Also \( 2a(y + z) + 2y(a + z) + 2z(a + y) > (a + y + z)^2 \).
4. Prove that \((a + b + c)^3 > 27abc\) unless \(a = b = c\).

5. Shew that \(a^2 + b^2 + c^2 > ab + ac + bc\) unless \(a = b = c\).

6. Shew that \(9(a^2 + b^2 + c^2) > (a + b + c)^2\).

7. Shew that \(x > x + 1\) for all values of \(x\) not less than 3.

8. Prove that \(x^2 + y^2 + z^2 > xyz(x + y + z)\).

9. Prove that \(a^3 + b^3 > a^2b + a^2c + b^2a + b^2c + c^2a + c^2b\).

10. Given that \(\frac{a + 2}{4} + \frac{a + 2}{3} < \frac{x + 2}{2} = \frac{3x + 5}{6}\) find a whole number for the value of \(x\) Answer \(x = 5\).

11. Which is greater \(2(y + 1)^2\) or \(y + 2\).

12. Shew that \(abc > (a + b - c) (a + c - b) (b + c - a)\) unless \(a = b = c\).

13. What is the integral value of \(x\) when \(6x - 7 < 4x + 2\) and \(3x + 1 > 13 - x\) Answer \(x = 4\).

14. If \(x^2 = a^2 + b^2\), and \(y^2 = c^2 + d^2\), shew that \(xy > ac + bd\) unless \(ad = bc\).

15. Prove that \(n^2 + 1 > n^2 + n\) unless \(n = 1\).

16. Given that \(\frac{a + 2}{4} + \frac{a + 2}{3} < \frac{x + 4}{2} + 3\), and \(\frac{x + 1}{2} + \frac{1}{3}\), find \(x\) Answer \(x = 5\).

17. Prove that \(xyz(y + z) + xz(x + z) > 6xyz\).

18. Shew that \(\frac{x^2 - y^2}{x^2 + y^2 + 1}\) lies between 3 and \(\frac{1}{3}\) for all real values of \(x\).

19. If \(x\) be greater than \(y\) prove that \(x^n - y^n < n\cdot x^{n-1}(x - y)\), and \(ny^{n-1}(x - y)\).

20. Shew that \((x^4 + y^4 + 1)(x^2 + y^2 + 1) > (x^2x^2 + y^2x^2 + 1)^2\).

21. Prove that \(a^6 + a^4y^2 + b^6y^2 + y^6 > (a^2 + y^3)^2\).

Q
22. Which is greater \(\left(\frac{1}{2}\right)^{\frac{3}{2}}\) or \(\left(\frac{3}{2}\right)^{\frac{3}{2}}\)? \(\text{Ans} \left(\frac{3}{2}\right)^{\frac{3}{2}}\)

23. Which is greater \(a - b\) or \((\sqrt{a} - \sqrt{b})^2\) if \(a > b\) \(\text{Ans} \ a - b\)

24. If \(x > 3\) prove that \(x^{\frac{1}{2}} > (x + 1)^{\frac{1}{2}}\).

25. The double of a certain number increased by 7 is not greater than 19, and its triple diminished by 5 is not less than 13. What is the number? \(\text{Ans} \ 6\)

26. Shew that \(2(x + a^2 + 1) > 3a(1 + a^2)\) unless \(x = 1\)

27. Shew that \(a + b + c + d > 4\sqrt[4]{abcd}\) unless \(a = b = c = d\).

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Exercise 31.

Eliminations.

1. Eliminate \(a\) and \(y\) from the equations \(x + y = m\), \(a^2 + y^2 = n\). \(a^2 + y^2 = p\). \(\text{Ans} \ 2p = 3mn - m^2\)

2. Eliminate \(m\) and \(n\) from the equations \(a + m + n = x\), \(am + an + mn = y\), \(amn = z\). \(\text{Ans} \ a^2 - a^3 = 2a^2\)

3. Eliminate \(x\) and \(y\) from the equations \((ax + by)^2 = c^2x^2 + d^2y^2\) and \((ay - bx)^2 = c^2y^2 + d^2x^2\). \(\text{Ans} \ a^2 + b^2 = c^2 + d^2\)

4. Eliminate \(x\) and \(y\) from the equations \(x - y = a\), \(x^2 - y^2 = b\). \(3xy = a^2\) \(\text{Ans} \ b = 2a^2\)

5. Eliminate \(x\) and \(y\) from the equations \(x^2 + y^2 = \frac{c}{d}\), \((x + y)^3 = m\), \(3xy = n\). \(\text{Ans} \ m^2 = 8n^2\)

6. Eliminate \(x\) and \(y\) from the equations \(y^2 - ay = ax - bx\), \(4xy = ax + by\), \(x^2 + y^2 = 1\). \(\text{Ans} \ (a + b)^{\frac{3}{2}} = \frac{a + b}{3}\)

7. Eliminate \(x\), \(y\) and \(z\) from the equations \(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = m\), \(\left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right) = p\) \(\text{Ans} \ mn = 1 + p\)
8. Eliminate \( x \) and \( y \) from the equations \( x^2 + xy + y^2 = a, \) \( x + y = b, \) \( xy = a^2 \)

\[ \text{Ans } \frac{a^2 + a - b^2}{2} \]

9. Eliminate \( a, b, c \) from the equations \( a + b + c = m, \) \( a^2 + b^2 + c^2 = n, \) \( ab + ac + bc = p \)

\[ \text{Ans } n + 2p = m^2 \]

10. Eliminate \( x \) and \( y \) from the equations

\[ \frac{a^2 - a^2}{y^2 - b^2} = \frac{2a + 3b}{3a + 2b}, \]

\[ x^2 - y^2 = (a - b)^2, \]

\[ a^2 + y^2 = c^2 \]

\[ \text{Ans } \frac{a^2 + b^2}{c^2} \]

11. Eliminate \( x, y, z \) from the equations

\[ (w - y)(y - z)(z - w) = a^3 \]

\[ (x + y)(y + z)(z + w) = b^3 \]

\[ (x^2 + y^2)(y^2 + z^2)(z^2 + w^2) = \frac{a^4}{2} \]

\[ (x^4 + y^4)(y^4 + z^4)(z^4 + w^4) = \frac{a^6b^6}{2} \]

\[ \text{Ans } 2a^3 = b^3 \]

12. Eliminate \( m, n, p, q \) from the equations

\[ \frac{x - p}{m} + \frac{y + q}{n} = \frac{pm}{a} + \frac{qn}{b} = \frac{m^2}{a^2} - \frac{n^2}{b^2} = \frac{p^2}{a^2} + \frac{q^2}{b^2} - 1 = 0 \]

\[ \text{Ans } \frac{x}{a} + \frac{y}{b} = \sqrt{2} \]

13. Eliminate \( b \) and \( c \) from the equations

\[ a + b + c + p = 0, \]

\[ a(b + c) = q - bc, \]

\[ abc + r = 0 \]

\[ \text{Ans } a^3 + pa^2 + qa + r = 0 \]

14. Eliminate \( m, n, p \) from the equations

\[ \left( \frac{x}{m} \right)^a + \left( \frac{y}{n} \right)^a + \left( \frac{z}{p} \right)^a = 1 = \left( \frac{m}{p} \right)^b + \left( \frac{n}{q} \right)^b + \left( \frac{p}{q} \right)^b \]

\[ \text{and} \]

\[ \frac{a^a}{m + b} = \frac{b^a}{n + s} = \frac{c^a}{p + a} \]

\[ \text{Ans } \frac{a^b}{a + b} + \frac{b^a}{b + c} + \frac{c^b}{c + a} = \frac{a^b}{a + b} \]
Exercise 32.

APPLICATION OF ALGEBRA TO GEOMETRY.

1. One side of a right angled triangle is 8 and the hypotenuse is 10: find the other side. Ans 6

2. One side of a right angled triangle is 5 and the difference between the hypotenuse and the other side = 1; find the hypotenuse and the other side. Ans 13, 12.

3. One side of a right angled triangle is 40, and the sum of the hypotenuse and the other side = 50 find the hypotenuse and the other side. Ans 41, 9.

4. The hypotenuse = 5 and the sum of the two sides = 7: find the sides. Ans 3 and 4

5. The difference of the hypotenuse and one side = 3; and the difference of the hypotenuse and the other side = 4; find the sides. Ans 12, 16 and 20

5a. The area of a right angled triangle = 24, and the hypotenuse = 10; determine the sides. Ans 8, 6.

6. Two sides of a triangle are 50 and 41, and the perpendicular from the vertex to the base = 40; find the base. Ans 39

7. The area of an equilateral triangle = $16\sqrt{3}$; find its perimeter. Ans 24

8. The difference between the two sides of a triangle = 2, and the segments into which the base is divided by a perpendicular from the vertex are 5 and 9; determine the sides. Ans 15, 13.

9. The sides of a triangle are $a$, $b$, and $c$, find the diameter of the circumscribed circle, if $a = 15$, $b = 15$, and $c = 18$ find the diameter, and the height of the segment on the chord 18. Ans $\frac{abc}{2 \text{ area of the triangle}} = 18\frac{3}{4}, 6\frac{3}{4}$

10. In a certain lake the tip of a bud of lotus was seen 1 foot above the surface of the water, forced by the wind it gradually advanced
and submerged at a distance of 5ft; calculate the depth of the water.

Ans 12 ft.

11. Find the length of a string which is tied in one corner on the floor of a room and stretched so as to reach the opposite corner under the ceiling; the length of the room = 32, breadth = 24 and the height = 9.

Ans 41

12. Find the radius of the circle inscribed in a triangle of which the sides are \( a, b \) and \( c \); if \( a = 8 \), \( b = 6 \) and \( c = 10 \), find the radius,

Ans \( \frac{\text{area}}{\text{Semiperimeter}} \), 2ft.

13. Find the side of a decagon inscribed in a circle of which the radius = 10; and hence shew the method of finding its area.

Ans 6.3 nearly

14. A ladder whose foot rests in a given position, just reaches a window on one side of a street, and when turned about its foot, just reaches a window on the other side. If the two positions of the ladder be at right angles to each other and the heights of the windows be 39 and 52 feet respectively, find the width of the street and the length of the ladder.

Ans 91 ft, 65ft.

15. What is the length of a diagonal of a square whose side = 10

Ans \( 10\sqrt{2} \)

16. If \( s = \text{semiperimeter} \) of a triangle prove that its area

\[ = \sqrt{s(s-a)(s-b)(s-c)} \]

where \( a, b \) and \( c \) are the sides.

17. If \( A B C \) be an equilateral triangle and the length of \( AD \), a perpendicular on \( BC \), be \( a \); find the length of \( AB \)

Ans \( \frac{2a}{\sqrt{3}} \)

18. In order to ascertain the height of a steeple I measured 150 feet from its base, and found the angle of elevation of its summit from that point to be 60°; what is the height of the tower?

Ans 259.8 ft nearly

19. From the top of a hill there are observed two consecutive milestones, on a horizontal road, running from the base. The angles of
depression are found to be $60^\circ$ and $30^\circ$. Find the height of the hill

\[ \text{Ans} \quad \frac{\sqrt{3}}{2} \text{ miles} \]

20. From the top of a lighthouse 200 ft above the sea the angle of depression of a ship's hull is found to be $30^\circ$. How far is the ship distant? \[ \text{Ans} \quad 346 \text{ ft nearly} \]

21. The radii of two circles which intersect one another are 30 and 25 ft, and the distance of their centres is 25 ft; find the length of their common chord.

\[ \text{Ans} \quad 48 \text{ ft.} \]

22. If from one of the angles of a rectangle, a perpendicular be drawn to its diagonal d, and from the point of their intersection, lines p, q be drawn perpendicular to the sides which contain the opposite angle; shew that \[ p^2 + q^2 = d^2. \]

23. The length of a kite string is 180 yds in length and the angle of elevation of the kite is $30^\circ$. Find the height of the kite.

\[ \text{Ans} \quad 90 \text{ yds} \]

24. A coconut tree measuring 50 ft, in height and standing upon level ground was broken in one place by a storm; the broken part instantly inclined towards the ground so that its extremity reached a distance of 10 ft. from the foot of the tree, at how many feet from the foot was the tree broken?

\[ \text{Ans} \quad 24 \text{ ft.} \]

25. A peacock perched on the top of a pillar 9 cubits in height. A snake's hole was at the foot of the pillar and at a distance equal to three times its height was seen a snake; seeing the snake glide towards the hole, the peacock pounced upon it at a place which was equidistant between the top of the pillar and the place where the snake was first seen; at how many cubits from the snake's hole did they meet.

\[ \text{Ans} \quad 12 \text{ cubits} \]

26. Two monkeys were sitting on the top of a tree 100 cubits high, and at the distance of 200 cubits from the foot of the tree there was a pool of water. One of the monkeys gradually descended
from the tree and went directly to the pool; the other vaulted to some height perpendicularly from the top of the tree and from thence leaped diagonally to the pool. Both monkeys went over the same space in these several ways. Required the height of the leap.

**Ans 50 cubits**

27. If $p$, $h$, $d$ be the sides of a regular pentagon, hexagon and decagon respectively, inscribed in the same circle; shew that $p^2 = h^2 + d^2$.

28. Show that the area of a dodecagon inscribed in a circle is equal to that of a square on the side of an equilateral triangle inscribed in the same circle.

29. ABC is a triangle $AB = 16$, $AC = 20$, $BC = 18$; D is the middle point of BC, find AD.

**Ans 15.7 nearly.**

30. Two straight rods AB and CD each 12 ft. and 6 ft. in length respectively are fixed perpendicularly in the ground at the points A and C; two strings are fastened one from A to D and the other from C to B; find the distance of the point where the two strings cross each other from the ground.

**Ans 4.**

31. Find the area of a quadrilateral whose diagonals are 40 and 50 ft. and which are inclined to each other at an angle of $45^\circ$.

**Ans $500\sqrt{2}$**

32. A bar 14 ft. long is bent into a right angle, so that the lengths of the portions which meet at the angle are 8 ft. and 6 ft. respectively; find the distance of the middle of the hypotenuse from the point of the bar which was the middle when the bar was straight.

**Ans $3\sqrt{2}$**

33. Find the area of a rhombus two of whose sides are inclined to each other at an angle of $30^\circ$; the side being 20 ft. **Ans 200$\sqrt{3}$ ft.**

34. A person stood on the top of a hill $\frac{1}{2}$ a mile high, from the sea level and observed the radius of his offing to be 63 miles; required the radius of the earth.

**Ans 3969 miles**

35. Find the side of a pentagon inscribed in a circle the radius
of which is 16 feet; and hence shew the method of finding its area

Ans 18. 7 feet, nearly

36. Two roads AB, AC diverge from the same town, making with each other an angle of 45°; the length of the road $AB = 40\sqrt{2}$ and of $AC = 70$ miles; find the distance of C from B. Ans 50 miles

37. A crow wishing to quench its thirst came to a vessel which contained 28 cubic inches of water. The crow being unable to reach the water picked up several small stones each $\frac{2}{3}$ of a cubic inch in size, and let them drop into the vessel until the water came to the top of the vessel. If the size of the vessel was such that it would exactly hold 73 cubic inches of water, find the number of stones dropped in by the crow.

Ans 60 stones.

38. The light of a light house on the Alguada reef is 120 ft. above the sea level and a spectator's eye 6 ft. above the same datum; find the greatest distance in miles at which the light is visible.

Ans 16 miles nearly

39. Find the height of a tree by the help of a mirror, by a bucket of water or by an artificial horizon.

40. The sides of a triangle are 13, 14 and 15; find the perpendicular on the side 14; in how many ways can you solve this problem?

Ans 12

41. ABC is a triangle $AB = 4$, $BC = 5$, $AC = 6$, AD is the line which bisects the vertical angle $\angle ABC$; find BD and DC.

Ans 2 and 3.

42. If r be the radius of the inscribed circle of a triangle whose sides are $a$, $b$ and $c$ and R the radius of the circumscribed circle, prove that $2Rr = \frac{abc}{a + b + c}$

43. ABCD is a quadrilateral inscribed in a circle, $AB = a$, $BC = b$, $CD = c$, $AD = d$; find the two diagonals.

Ans $AC = \sqrt{\left(\frac{(ac + bd)(ad + bc)}{ab + cd}\right)}$, $BD = \sqrt{\left(\frac{(ac + bd)(ab + cd)}{ad + bc}\right)}$
44. From the preceding ex shew that the rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle is equal to both the rectangles contained by its opposite sides. (Ptolemy's theorem Euclid Prop. D Book VI)

45. If \( s \) = semiperimeter of a quadrilateral inscribed in a circle and if \( a, b, c, d \) are the four sides, prove that its area =

\[
\sqrt{s(s-a)(s-b)(s-c)(s-d)}
\]

46. If a quadrilateral, whose sides are \( a, b, c, d \), is capable of having both a circle inscribed in it and one circumscribed about it, prove that its area = \( \sqrt{abcd} \)

47. Having given that two points, each 10 feet above the surface, cease to be visible from each other over still water at a distance of 8 miles, find the Earth's diameter.

Ans 8448 miles.

48. The sides of a triangle are \( a, b \), and \( c \), find the magnitude of the escribed circle which touches the side \( c \) and the other two produced.

\[ \text{Ans radius} = \frac{\text{area of the triangle}}{s-c}, \text{ where } s = \text{semiperimeter of the triangle.} \]

**Exercise 33.**

**ADVECTED QUADRATIC EQUATIONS.**

Solve the following equations,

1. \( x^2 - 3x = -2 \) \( x = 1 \) or \( 2 \)
2. \( x^2 + x = 12 \) \( x = -4 \) or \( 3 \)
3. \( x^4 + 4x = 12 \) \( x = 2 \) or \( -6 \)
4. \( x^2 - (a + b)x = -ab \) \( x = a \) or \( b \)
5. \( x^2 - 4ax = a^2 - 4ax^2 \) \( x = 2a \)
6. \( 12x^2 - 7x + 1 = 0 \) \( x = \frac{1}{3} \) or \( \frac{1}{4} \)
7. \( a^2 b^2 x^2 - (2b^2 + 3a^2)x = -6 \) \( x = \frac{2}{a^2} \) or \( \frac{3}{b^2} \)
8. \( 6x^2 - 5x = -1 \)  
   \( x = \frac{1}{2} \) or \( \frac{1}{3} \)

9. \( 5x^2 - 12x + 3 = 12 \)  
   \( x = 3 \) or \( -\frac{3}{5} \)

10. \( x^2 + 2x + 2\sqrt{x^2 + 2x + 1} = 47 \)  
    \( x = 5 \)

11. \( 3x^2 - 2x + \sqrt{3x^2 - 4x - 6} = 18 + 2x \)  
    \( x = 3 \) or \( -\frac{5}{3} \)

12. \( x^4 + 1 = 0 \)  
    \( x = \pm 1 \pm \sqrt{-1} \sqrt{2} \)

13. \( x^3 - 1 = 0 \)  
    \( x = 1 \) or \( \frac{-1 + \sqrt{-3}}{2} \)

14. \( x^2 + \frac{1}{x} = 2 \)  
    \( x = -\frac{1}{2} \pm \sqrt{5} \)

15. Form an equation whose roots are 4 and 5. Ans \( x^2 - 9x = -20 \)

16. Form an equation whose roots are \( a \) and \( b \). 
    Ans \( x^2 - (a + b)x = -ab \)

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**Exercise 34.**

**MAXIMA AND MINIMAS.**

1. Divide a given number 8 into two such parts that their product may be the greatest possible. Ans 4 and 4

2. To find such a value of \( x \) as may make \( \frac{x}{(x + 8)(x + 2)} \) a maximum. Ans \( x = 4 \).

* "The problems which relate to the maxima and minima, or the greatest or least values of variable quantities, are among the most interesting in the mathematics; they are connected with the highest attainments of wisdom and the greatest exertions of power; and seem like so many unmoveable columns erected in the infinity of space, to mark the eternal boundary which separates the regions of possibility and impossibility from one another."

2nd Diss Ency. Brit.
3. For what value of \( x \) the expression \( p - (x - 6)^2 \) a maximum.
   \[ \text{Ans} \quad x = 6 \]

4. To find such a value of \( x \) as will make \( \frac{x}{1 + x^2} \) a maximum
   \[ \text{Ans} \quad x = 1 \]

5. Divide the number 16 into two such factors that the sum of their squares shall be a minimum.
   \[ \text{Ans} \quad 4 \text{ and } 4 \]

6. What fraction exceeds its square by the greatest possible number.
   \[ \text{Ans} \quad \frac{1}{2} \]

7. For what value of \( x \) the expression \( m + \sqrt{m^2 - 2m^2x + mx^2} \) a minimum.
   \[ \text{Ans} \quad x = m \]

8. Determine the greatest rectangle that can be inscribed in a triangle.
   \[ \text{Ans} \quad \text{The greatest rectangle is that whose altitude } = \frac{1}{2} \text{ the altitude of the triangle.} \]

9. Of all right angled triangles of the same area, find that the sum of whose legs is the least possible.
   \[ \text{Ans} \quad \text{When the two legs are equal} \]

10. Of all triangles upon the same base 8ft and having the same perimeter of 18ft, find that which has the greatest area.
    \[ \text{Ans} \quad \text{An isosceles triangle whose equal sides are each 5ft} \]

11. Inscribe the greatest rectangle in a semicircle of which the radius = 2.
    \[ \text{Ans} \quad \text{The rectangle whose sides are } 2\sqrt{2} \text{ and } \sqrt{2} \]

12. Of all squares inscribed in a square whose side = 8ft, find that which is the least.
    \[ \text{Ans} \quad \text{The square found by joining the middle points of the sides of the given square.} \]

13. Find the least triangle which can be described about a given quadrant.
    \[ \text{Ans} \quad \text{When the triangle is isosceles.} \]

14. Find for what value of \( x \) the expression \( m^2 + n^2x - 2n^2x^2 \) becomes a maximum.
    \[ \text{Ans} \quad x = \frac{n^2}{2n^2} \]
15. In a given circle of which the radius = 4, inscribe the greatest rectangle possible. Ans. A square whose sides are $4\sqrt{2}$.

16. Find that number which being added to its reciprocal, the sum may be minimum. Ans. 1.

17. Find the value of $x$ when $\frac{x(m-x)}{m^2}$ is a maximum. Ans. $x = \frac{m}{2}$.

18. Find the value of $x$ when $\frac{(x+m)(x-n)}{x^2}$ is maximum. Ans. $x = \frac{2mn}{m-n}$.

19. Find the height at which a lamp should be placed so that the greatest quantity of light may be thrown on a book placed on the table, at a given horizontal distance of 4 feet. Ans. $2\sqrt{2}$ ft.

20. Find for what value of $x$ the expression $\frac{m^2x^2 + n^2}{(m^2 - n^2)x}$ is minimum. Ans. $\frac{n}{m}$.

21. Divide the number 16 into two such parts that if the square of one of these be subtracted from their product, the remainder is the greatest possible. Ans. 4 and 12.

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© This problem is for those who have gone through trigonometry. I have placed this here at the request of many F. A. students for its practical utility; students during their night studies and workmen generally will find this useful.
**Exercise 35.**

CALCUTTA UNIVERSITY ENTRANCE PAPERS.

1877—Rev. G. H. Rouse and Mr. A. M. Nash, M. A.

1. Simplify \( \frac{x + 2}{1 + x + x^2} \cdot \frac{x - 2}{1 - x + x^2} - \frac{2x^2 - 4}{1 - x^2 + x^4} \); multiply together \( a + b + c, b + c - a, c + a - b, a + b - c \); and divide \( x^4 + x^3 - 2x^2 - 35x + 57 \) by \( x^2 + 2x - 3 \). \( \text{Ans} \frac{4x^4 + 8}{x^2 + x^4 + 1} \); \( 2b^2c^2 + 2a^2c^2 + 2a^2b^2 - a^4 - b^4 - c^4 \); \( x^2 - x - 19 \).

2. Solve the equations:——

\[
(1) \quad \frac{2x - 3}{6} + \frac{3x - 8}{11} = \frac{4x + 15}{33} + \frac{1}{2}, \quad x = 4\frac{1}{2}
\]

\[
(2) \quad 2(x + 2) = 1 + \sqrt{4x^2 + 9x + 14}, \quad x = \frac{5}{3}
\]

\[
(3) \quad 3x + 4y - 11 = 0, 5y - 6z = -8, 7z - 8x - 13 = 0,
\]

\[ \text{Ans} x = 1, y = 2, z = 3 \]

3. Find the G. C. M. of \( x^4 + x^3 - 11x^2 - 9x + 18 \) \( x^4 - 10x^3 + 35x^2 - 50x + 24 \), \( \text{Ans} \ x^2 - 4x + 3 \)

4. Find the first four terms of the square root of \( a^2 + x^2 \) and from the result deduce the square root of 101 correct to six places of decimals.

\[ \text{Ans} \ \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \&c. \\text{10.0498756} \]

5. If \( a : b = c : d \) prove that \( a^2 + c^2 : b^2 + d^2 = ac : bd \).

6. A and B together can do a piece of work in 15 days; A can do it alone in 24 days; how long would B take to do it alone?

\[ \text{Ans 40} \]

7. Two passengers have together 5 cwt. of luggage, and are charged for the excess above the weight allowed 5s. 2d. and 9s. 10d.
respectively; but if the luggage had all belonged to one of them he would have been charged 19s. 2d. How much luggage is each passenger allowed to carry free of charge? and how much luggage had each passenger?

Ans \(\frac{3}{2}\) cwt, 2 cwt, 3 cwt.

1878—Mr. W. Booth, B.A. and Mr. Mowat, M.A.

1. Divide \(a(1 + y^2)(1 + z^2) + y(1 + z^2)(1 + a^2) + z(1 + x^2)(1 + y^2) + 4xyz \) by \(1 + xy + yz + wz\)

   Ans \(a + y + z + awz\)

2. Extract the square root of \((a^2 + b^2 + c^2)^2(a^2 + y^2 + z^2) - (bx - cy)^2 - (cx - az)^2 - (ay - bx)^2\)

   Ans \(aw + by + cz\)

3. If \(\frac{a}{b + c + a} = \frac{y}{c + a + b} = \frac{z}{a + b - c}\), find the value of \((b - c)x + (c - a)y + (a - b)z\)

   Ans 0

4. Solve the equations

   (a) \(\sqrt{4a^2 + 20a + 17} = \sqrt{16a^2 + 11a + 10} + 2(a + 2)\)

   Ans \(a = -3\)

   (b) \(\frac{4x + 3}{9} + \frac{13a}{108} = \frac{8x + 19}{18}\)

   Ans \(a = 6\)

5. See ex 43 page 64.

1879.

1. Rs 49 was divided amongst 150 children, each girl had 8 as and each boy 4 as; how many boys were there? Ans 104

2. Multiply \(a^{2n} - a^n a^n + a^{2n}\) by \(a^n + a^n\) and find the greatest common measure of \(a^n + a^n + a^{\frac{1}{2}}\) and \(a^{2n} + a^n + a^{\frac{1}{2}}\)

   Ans \(a^{2n} + a^{2n}\); \(a + \frac{1}{2}\)

3. Divide \(x^{2n} - y^{2n}\) by \(x^{2n-1} + y^{2n-1}\) and Simplify \(\frac{x^2 - y^2}{x - y} + \frac{x^2 - z^2}{x - z}\)

   Ans \(\frac{(y - z)^2}{(x - y)(x - z)}\); 2.
4. Solve the equations

\[ a - k + \sqrt{k^2 + x^2} = m \quad \quad \quad a = \frac{m(m + 2k)}{2(m + k)} \]

\[ a^2, a^3 + 1 = a^7 \]
\[ a^2y, a^{3x} + 5 = a^{20} \]
\[ y = 3 \quad \quad \quad y = 3 \]
\[ x = 4 \quad \quad \quad x = 4 \]
\[ \frac{2}{y} = 19 \quad \quad \quad \frac{2}{y} = 20 \]
\[ y = 10 \quad \quad \quad y = 10 \]

5. If \( \frac{a}{b} = \frac{c}{d} \) prove that \( \frac{a^2 + b^2}{a^2 - b^2} = \frac{ac + bd}{ac - bd} \)

6. Two armies number 11,000 and 7000 men respectively, before they fight each is reinforced by 1000 men: in favor of which army is the increase.

Ans. In favor of the latter.

7. From two towns 561 miles apart two men start, one from each at the same time, one goes 24 and the other 27 miles a day: in how many days will they meet.

Ans. 11

1880.

7. Simplify \( \left\{ \frac{x}{a} + \frac{2a^2}{a} \right\} \quad \left\{ \frac{a}{x} - \frac{2ax}{x} \right\} \)

Ans 1

8. Find the highest common factor and the l. c. m. of \( 3a^2 - 10ax + 7a^2 \) and \( x^3 - 5ax^2 + 7a^2x - 3a^3 \).

Ans II C. M. = \( x - a \); L. C. M = \( (x - a)(3x - 7a)(x - 3a)(x - a) \)

9. Solve the equations

(a) \( 15 + \sqrt{x + 7} = 19 \)

Ans 9

(b) \( 4x - \frac{x - 1}{2} = x + \frac{2x - 2}{5} + 21 \)

Ans 11
10. If $a : b : c : d$ shew that $ma + nb : mc + nd : : (a^2 + c^2)^{\frac{1}{2}}$.

$$(b^2 + d^2)^{\frac{1}{2}}$$

11. Extract the square root of $a^6 - 2a^{-3}x^{\frac{3}{2}} + 2a^{\frac{4}{5}}x^{\frac{4}{5}} + a^2$

$= a^{-\frac{3}{5}}x^{\frac{4}{5}} - 2a^{\frac{1}{5}}x^{\frac{1}{5}} + a^2$

Ans. $a^{-\frac{3}{5}}x^{\frac{4}{5}} - 2a^{\frac{1}{5}}x^{\frac{1}{5}} + a^2$

12. A boat goes up stream 30 miles and down stream 44 miles in 10 hours, it also goes up stream 40 miles and down stream 55 miles in 13 hours. Find the rate of the stream and of the boat.

Ans. Stream 3 miles and boat 8 miles.

1881.

1. What do you mean by a negative quantity?

Prove that $a - (b - c) = a - b + c$.

2. Simplify $\frac{1}{ab} + \frac{1}{a(a - b)(x - a)} + \frac{1}{b(b - a)(x - b)}$ and resolve into elementary factors the expressions:

$x^2 - 5ax - 66a^2$ and $(1 - c^2)(1 + a)^2 - (1 - a^2)(1 + c)^2$

Ans. $\frac{1}{x(x - a)(x - b)}$; $(x + 6a)(x - 11a)$

3. A man receives $\frac{x}{y}$ ths of 10 Rs., and afterwards $\frac{y}{x}$ ths of 10 Rs. He then gives away 20 Rs. Show that he cannot lose by the transaction.
4. What is an equation? Prove that a simple equation has only one root.

5. Solve the equations

\[ (1) \sqrt{x^2 + 11x + 20} - \sqrt{x^2 + 5x - 1} = 3 \quad x = 5 \]

\[ (2) \frac{4.05}{9.0} - \frac{\frac{1.8}{2}}{\frac{3.6}{6}} = \frac{3}{2} - 6x \quad x = 3 \]

\[ (3) \quad ax + by = c, \quad a^2 x + b^2 y = c^2 \quad x = \frac{c(c - b)}{a(a - b)}, \quad y = \frac{c(a - c)}{b(a - b)} \]

6. A challenged B to ride a bicycle race of 1040 yds. He first gave B 120 yds. start, but lost by 5 seconds: he then gave B 5 seconds start and won by 120 ft. How long does each take to ride the distance? Ans A, 1 min 55\(\frac{9}{11}\) sec; B, 2 min 5\(\frac{3}{11}\) sec.

**Exercise 36.**

**MISCELLANEOUS EXAMPLES.**

1. Multiply \(1 + x^{-1} + x^{-2}, \ 1 - x^{-1} + x^{-2} \) and \(1 + x^{-2} + x^{-4}\)

   Ans \(1 + x^{-4} + x^{-8}\).

2. If \(\frac{b^2 + c^2 - a^2}{2bc} + \frac{c^2 + a^2 - b^2}{2ac} + \frac{a^2 + b^2 - c^2}{2ab} = 1\) show that two of the 3 fractions on the left-hand side are each equal to 1 and the other to \(-1\).

3. If \(n\) is a positive whole number show that \(13^{2n+1} + 1\) is divisible by 14.

4. Solve \(\frac{2x + 3y - 4z}{x + 5} = \frac{3x + 4y - 2z}{5x} = \frac{4x + 2y - 3z}{4x - 1}\)

   \[\frac{x + y + z}{6}, \quad x = 5, \ y = 4, \ z = 2\]

5. If \(\frac{bx + cy - cz}{a^2 + b^2} = \frac{cy + bz - ca^2}{b^2 + c^2} = \frac{ax + cz - by}{c^2 + a^2}\) show that

\[\frac{ax + by + cz}{a + b + c} = \frac{bc + ca}{ab + bc + ac}\]
\textbf{ALGEBRAICAL EXERCISES.}

7. If \( a = \frac{1}{3}, \, x + y = \omega + y + z = 0 \), find the value of
\[(y^2 - z^2)(y^2 + z^2 - y(x - z))\] Ans \( \frac{1}{6} \)

8. Divide \( a^3 + b^3 + c^3 - 3ab^2c \) by \( a + b + c \)
\[\text{Ans } a^3 + b^3 + c^3 - a^2b - a^2c - b^2a - b^2c - c^2a - c^2b\]

9. Eliminate \( x \) and \( y \) from the equations
\[a = \frac{1}{x} - \omega, \quad b = \frac{1}{y} - \omega, \quad x^2 + y^2 = 1.\] Ans \( a^2b^2(a^2 + b^2) = 1 \)

10. If \( x^2 + y^2 + z^2 = \omega y + yz + \omega z \), show that \( x^3 + y^3 + z^3 = 3\omega y z \)

11. If \( a^2 + b^2 = c^2 + d^2 = 1 \), show that \( (ad + bc)(ad - bc) = a^2 - c^2 \)

12. If \( x = \sqrt[3]{a + \sqrt{a^2 + b^2}} + \sqrt[3]{a - \sqrt{a^2 + b^2}} \), show that \( x^3 + 3\omega x = 2a \)

13. If \( a = x^2 - 1, \, b = 2\omega \) and \( c = x^2 + 1 \), show that
\[(a + b + c)(b + c - a)(a + c - b)(a + b - c) = 4a^2b^2\]

14. If \( a + b + c = 0 \), show that \( 2(a^2b^2 + \omega^2b^2 + a^2c^2) = a^4 + b^4 + c^4 \)

15. If \( \omega = \frac{1}{3} \left\{ \sqrt[3]{a} - \sqrt[3]{b} \right\} \), show that \( \frac{2\sqrt{1 + \omega^3}}{x + \sqrt{1 + \omega^3}} = a + b \)

16. If \( x^2 - yz = -8, \, y^2 - xy = 1, \, z^2 - xz = 10, \) find \( x, \, y \) and \( z \)

17. Find a number of two digits such that its quotient by their sum exceeds the second digit by 1, and is 4 times the other. Ans 72

18. Simplify \( \left\{ \frac{1 + \omega}{1 - \omega} + \frac{4\omega}{1 + \omega^2} + \frac{8\omega}{1 + \omega^2} - \frac{1 - \omega}{1 + \omega^2}\right\} \)
\[+ \left\{ \frac{1 + \omega^2}{1 - \omega^2} + \frac{4\omega^2}{1 + \omega^4} - \frac{1 - \omega^2}{1 + \omega^4}\right\} \]
Ans \( 2\omega^{-1} \)

19. A flagstaff 5\( \frac{3}{4} \) ft. high stands at the top of a wall, whose height is 3 ft. Find the distance from the bottom of the wall, at which the flagstaff subtends the greatest angle. Ans 5 ft.

20. \[\frac{x + \sqrt{(x^2 - 25)}}{x - \sqrt{(x^2 - 25)}} = 125 \frac{\sqrt{(x + 5)} - \sqrt{(x - 5)}}{\sqrt{(x + 5)} + \sqrt{(x - 5)}}\] Ans \( x = 13 \)
21. Explain the following anomaly.

\[ 1 = \frac{w^2 - x^2}{w^2 - x^2} \frac{(x + w)(x - w)}{x(a - w)} = \frac{x + w}{w} = 2 \]

22. Solve \( 3^2 + 2^2 = 17 \) and \( 3^{x+1} + 2^{y+2} = 59 \) \hspace{1cm} \text{Ans} \hspace{0.5cm} w = 2, y = 3

23. Two boys start from the right angle of a triangular field, and run along the sides with velocities in the ratio of 13 : 11. They meet first in the middle of the opposite side and again 30 yds from the starting point. Find the length round the field. \hspace{1cm} \text{Ans} \hspace{0.5cm} 360 \text{ yds}

24. Solve the equation \( \left( \frac{x + a}{b} \right)^3 + \left( \frac{x + c}{d} \right)^3 = 2 \left( \frac{(x + a)(x + c)}{bd} \right)^{\frac{1}{3}} \)

\[ w = \frac{bc - ad}{d - b} \]

25. Two persons started at the same time from A. One rode on an elephant at the rate of 7\( \frac{1}{2} \) miles an hour and arrived at B 30 minutes later than the other who travelled the same distance by train at the rate of 30 miles an hour. Find the distance between A and B. \hspace{1cm} \text{Ans} \hspace{0.5cm} 5 \text{ miles.}

26. Find the l. c. m. of \( x^3 + x^2 y + xy^2 + y^3 \) and \( x^3 - x^2 y + xy^2 - y^3 \). \hspace{1cm} \text{Ans} \hspace{0.5cm} w^4 - y^4

27. Simplify \( \frac{w}{x + a} - \frac{w}{x - a} - \frac{x + a}{x - a} \). \hspace{1cm} \text{Ans} \hspace{0.5cm} \frac{4a^3w}{x^4 - a^4}

28. The third law of Kepler is that the squares of the periodic times of the planets are to each other as the cubes of their mean distances from the Sun, if the distance of the earth from the Sun be 95,000,000 miles and the periodic times of the Earth and Venus be respectively 365 and 224 days; find the distance of Venus from the Sun. \hspace{1cm} \text{Ans} \hspace{0.5cm} 68,610,000 \text{ miles}

29. Find for what value of \( n \) the expression \( 2a^8 - 6a^2 - 6m^2n \) is divisible by \( w - m \) without a remainder. \hspace{1cm} \text{Ans} \hspace{0.5cm} a = \frac{1}{2} m
30. If \( a + b + c = 0 \) prove that \( \frac{a^5 + b^5 + c^5}{a^2 + b^2 + c^2} = \frac{5(a^2 + b^5 + c^5)}{3(a^2 + b^2 + c^2)} \)

31. If \( \frac{x-y}{y-z} = \frac{3x-y}{y-3z} = \frac{4x-z}{z-4y} \) prove that \( \frac{19x - 3y - 9z}{-9y - 4z} = \frac{13x - 5y - z}{y - 8z} \)

32. Mr. Leivin an aeronaut who made a balloon ascent in Calcutta on the 29th December 1877, wishing to know the altitude of the balloon when it was at its maximum height, threw down a bottle and observed that the time occupied for its descent to the surface of the earth was half a minute. Find the height of the balloon. (see Ex. 62, page 89)

Ans. 14490 ft.

33. With a given perimeter of 32 feet, find what rectangle has the maximum area.

Ans. A square whose side is 8 feet.

34. Disintegrate \( 4a^4 - 37a^2b^2 + 49b^4 \) into its elementary factors.

Ans. \( (2a^2 - 7b^2 + 3ab)(2a^2 - 7b^2 - 3ab) \)

35. A man is allowed the option of choosing 36 sq feet of ground for a room, find the length and breadth of the room that the cost of the walls may be minimum.

Ans. A square whose side is 6 ft.

36. Simplify \( \frac{x^2 - yz}{(x + y)(x + z)} + \frac{y^2 - xz}{(y + z)(y + z)} + \frac{z^2 - xy}{(x + z)(y + z)} \)

Ans. 0

37. Find the G. C. M. of \( 2x^5 + x^2 + x - 4 \) and \( 4x^5 + 6x^3 + 10x^2 + 3x + 4 \).

Ans. \( 2x^2 + 3x + 4 \)

38. If \( x^2 + \frac{1}{x} = 2 \) shew that \( x(x + 1) = 1 \)

39. Solve \( 8a^2 - 24a = 65 \)

Ans. \( a = 8 \frac{1}{3}, \frac{1}{3} \)

40. Solve \( \frac{\sqrt{2} + \sqrt{a}}{\sqrt{a}} + \frac{\sqrt{2} - \sqrt{a}}{\sqrt{a}} = 2 \sqrt{a} \)

Ans. \( a = 4 \)

41. Two men travel in the same direction in the circumference of a circular island, they both start at the same instant and it is known
that one can travel the circumference in 24 hours and the other in 27 days 7 hours; after what time they will be together again.

\[ \text{Ans } 24 \text{ hours 55 min nearly} \]

42. From the foregoing example deduce that if a man has the Moon in his meridian at a certain time of the day, it will be in his meridian again after 24 hours 55 minutes.

43. Investigate the rule for finding two square numbers whose sum is a square number.

44. Find the value of \( \frac{a^3}{3(a^2 - x)} + \frac{b^3}{3(b^2 - a)} \) when \( x = \frac{a^2 + b^3}{2} \)

45. Simplify \( \frac{x^4}{y^2} \) when \( x = \frac{m^4}{y^4} \)

\[ \text{Ans } \left( \frac{x}{y} \right)^{\frac{1}{2}} \]

46. The trinomial \( \alpha x^2 + \alpha x + \alpha \) becomes 9, 18 and 31 if \( x \) is equal to 1, 2 and 3 respectively what will its value be if \( x = 4 \)

\[ \text{Ans } 48 \]

47. \( \alpha + y + z = 0 \)
\( \alpha(a + b) + y(a + c) + z(c + a) = 0 \)
\( x^2 + y^2 + z^2 = 3(a - b)(b - c)(c - a) \)
\[ \text{Ans } x = b - c, y = c - a, z = a - b \]

48. \( x(b - c) + y(c - a) + z(a - b) = 0 \) prove that
\[ \frac{bz - cy}{b - c} = \frac{cx - az}{c - a} = \frac{ay - bx}{a - b} \]

49. If \( \frac{\alpha_0}{\alpha_1} = \frac{\alpha_1}{\alpha_2} = \frac{\alpha_2}{\alpha_3} = \ldots = \frac{\alpha_{r-2}}{\alpha_r} \)

\[ \text{shew that each of the expressions } \]
\[ \frac{x_0}{x_1} + \frac{x_1}{x_2} + \frac{x_2}{x_3} + \ldots + \frac{x_{r-2}}{x_{r-1}} + \frac{x_{r-1}}{x_r} \]

\[ \text{is } \frac{x_0}{x_1} + \frac{x_1}{x_2} + \frac{x_2}{x_3} + \ldots + \frac{x_{r-2}}{x_{r-1}} + \frac{x_{r-1}}{x_r} \]

50. If the sides of a triangle be \( a, b, c \), prove that
\[ (a + b + c - a)^{-1} + (c + a - b)^{-1} + (a + b - c)^{-1} > a^{-1} + b^{-1} + c^{-1} \]

51. The area of an equilateral triangle is \( 16\sqrt{3} \); find the side.
\[ \text{Ans } 8 \]
52. Eliminate $m$ and $n$ from the equations $y = mx + \sqrt{(a^2 m^2 + b^2)} \leq n \leq \sqrt{(a^2 n^2 + b^2)}$ and $mn + 1 = 0$. Ans $a^2 + b^2 = x^2 + y^2$

53. If $x = a^2 - bc$, $y = b^2 - ac$, $z = c^2 - ab$ show that

$$\frac{x^2 - yz}{a} = \frac{y^2 - wz}{b} = \frac{z^2 - wx}{c} = (a + b + c)(x + y + z)$$

54. Divide $(a + b)^3 + 3(a + b)^2 c + 3(a + b)c^2 + c^3$ by $a^2 + 2a(b + c) + (b + c)^2$. Ans $a + b + c$

55. Eliminate $x$ and $y$ from the equations $ay = x^2 + 3y^2$, $bx = y^2 + 3x^2$, $xy = c$. Ans $(a + b)^3 - (a - b)^3 = 4c^3$

56. Solve \( \frac{1}{x + a + b} = \frac{1}{x} + \frac{1}{a} + \frac{1}{b} \). Ans $a = -b$ or $-a$

57. Multiply $w^{3} + x^{3} - y^{3} - z^{3}$ by $\frac{w^2 - x^2 - y^2 - z^2}{w^2 + x^2 + y^2 + z^2}$. Ans $w^2 + x^2 - y^2 - z^2$

58. Solve $w^2 + x^2 + y^2 = 25$. Ans $w = 2$, $y = 3$, $z = 4$

59. Solve $x^2 + 2^2 - x = 1$. Ans $x = 1$

60. Eliminate $x = a^2 + 3y^2$, $b = y^2 + 3x^2$, $xy = c$. Ans $\frac{a + y + z}{2}$

61. $\sqrt{3} + 1 = \frac{2}{3}$. Ans $\frac{2}{3}$

62. Simplify $\frac{x - a}{x - b} + \frac{x - b}{x - a} - \frac{(a - b)^2}{(x - a)(x - b)}$. Ans $2$

63. Simplify $\frac{a^3 + b^3 + c^3 - 3abc}{(a^2 + b^2 + c^2)}$. Ans $\frac{a + b + c}{2}$

64. The sum of two numbers is 4225 and their G. C. M. is 845; find the numbers. Shew that there are two pairs of numbers satisfying the condition. Ans 845, 3380; 1690, 2535

65. A market woman bought eggs at 2 a penny and as many more at 3 a penny; and thinking to make her-money again, she sold them
at 5 for two pence; she lost however 4d by the business; how much did she lay out?

\[ \text{Ans} \quad 8s. 4d \]

66. Find the value of \( \frac{a + 2a}{a - 2a} + \frac{x + 2b}{x - 2b} \), when \( a = \frac{4ab}{a + b} \)

\[ \text{Ans} \quad 2 \]

67. Solve \( \sqrt{a^3 - 3ax + 2a^2} + \sqrt{a^3 - 7ax + 5a^2} = \sqrt{a^3 - 6ax - 6a^2} + \sqrt{a^3 - 10ax - 3a^2} \)

\[ \omega = -\frac{3a}{2} \]

68. A waterman rowed 3½ miles down a river and up again in 100'; supposing the stream to have a current of 2 miles an hour, find at what rate he would row in still water.

\[ \text{Ans} \quad 5 \text{ miles} \]

69. A gamester loses \( \frac{1}{3} \) of his money, and then wins 10s; he loses \( \frac{1}{3} \) of this and then wins £1, when he leaves of as he began, what had he at first.

\[ \text{Ans} \quad £2, 8s. \]

70. Divide \( (x^3 - 1)a^2 - (x^3 + x^2 - 2)x^2 + (4x^2 + 3x + 2)x - 3(x + 1) \) by \( (x - 1)a^2 - (x - 1)x + 3 \)

\[ \text{Ans} \quad (x^2 + x + 1)a - (x + 1) \]

71. Reduce to its lowest term,

\[ \frac{\omega^4 + \omega^3 + 1}{\omega^5 - \omega^3 - \omega^2 + \omega + \omega^3 - 1} \]

\[ \text{Ans} \quad \frac{x^2 + x + 1}{x^3 - 1} \]

72. Solve \( \left( \frac{2\omega + 3}{2\omega - 3} \right)^{\frac{3}{2}} + \left( \frac{2\omega - 3}{2\omega + 3} \right)^{\frac{3}{2}} = \frac{8}{16} \cdot \frac{4x^2 + 9}{4x^2 - 9} \), \( x = -\frac{3}{4} \)

73. Solve \( \frac{x^{(m-a)^3} + x^{-(m-a)^3}}{x^{(m-a)^3} - x^{-(m-a)^3}} = \frac{a}{b^b-1} \)

\[ \omega = \left( \frac{p + 1}{p^b - 1} \right)^{\frac{1}{m+n}} \]

74. Divide \( x^6 + \frac{1}{3}x^4 + \frac{1}{3} \) by \( x^6 + y^6 \)

\[ \text{Ans} \quad x^2 + y^2 \]

75. If \( x^2 + ax + b \) and \( x^2 + a'x + b' \) have a common measure prove that \( (a - a')(a'b' - a'b) + (b - b')^2 = 0 \)

76. Find the square root of \( \sqrt{8} \cdot \frac{12a}{b} + \frac{3a}{2}\sqrt{b^2} + \frac{49b^2}{4a^2} - \frac{28b}{2a} \)

\[ \text{Ans} \quad \frac{3a}{5b} - \frac{7b}{2a} \]
77. Solve \( \frac{243 + 324\sqrt{3x}}{16x - 3} = (4\sqrt{x} - \sqrt{3})^2 \)  
\( \text{Ans } x = 3 \)

78. Find the G. C. M. of \( x^3 + (y + z)x^2 - (y - z)^2x - (y + z)(y - z) \) and \( x^3 + y^3 + z^3 - 3xyz \)  
\( \text{Ans } x + y + z \)

79. Solve \( \frac{\sqrt{(x + 1)} - \sqrt{(x - 1)}}{\sqrt{(x + 1)} + \sqrt{(x - 1)}} = \frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} \)  
\( \text{Ans } x = -\frac{1}{2} \)

80. Solve \( \frac{x - a}{b} + \frac{x - b}{a} + \frac{x - a - b}{c} = \frac{2x}{a + b} \)  
\( \text{Ans } x = a + b \)

81. Solve \( 5 - x = 2(\sqrt{xy}) - \sqrt{6} + y \)  
\( \text{Ans } x = 3, y = 2 \)

82. Find the value of \( x \) which will make \( x^3 + 3cx^2 + 2c^2x + 5c^3 \) equal to the cube of \( x + c \)  
\( \text{Ans } x = 4c \)

83. An officer can form the men in his battalion into a solid square and also into a hollow square 12 deep: if the front in the latter formation exceed the front in the former by 3, find the number of men in the battalion.  
\( \text{Ans } 1296 \)

84. Simplify \( \frac{x^2 + xy + y^2}{x\sqrt{x} + y\sqrt{y}} \)  
\( \text{Ans } \frac{x + \sqrt{xy} + y}{\sqrt{x} + \sqrt{y}} \)

85. If \( x + m \) be the G. C. M of \( ax + b \) and \( cx + d \) shew that \( m = \frac{b \cdot d}{a - c} \)

86. Two trains start at the same time from two towns and each proceeds at a uniform rate towards the other town. When they meet it is found that one train has run 108 miles more than the other and that if they continue to run at the same rates, they will finish the journey in 9 and 16 hours respectively. Find the distance between the towns and the rates of the trains.  
\( \text{Ans } 756 \text{ miles, } 86 \text{ and } 27 \text{ miles} \)

87. The height of a light house on the Cocos reef is 132 ft. How far is it visible from a ship at sea, the radius of the earth=4000 miles  
\( \text{Ans } 10\sqrt{2} \text{ miles nearly} \)
88. Simplify \( \frac{x^2 + y^2 + z^2 + 1}{(x-y)(w-z) + (w-y)(z-x)} \) 
\[ \text{Ans } 1 \]

89. Find the g.c.m. of \( w^4 + (m+n)w^2 - (m+n)w + n \) and \( w^4 - (p+q)w^2 + (p+q)w - q \) 
\[ \text{Ans } w^2 - 1 \]

90. Eliminate \( x \) and \( y \) from the equations \((m+n)x^2 + n^2 = m^2 + n^2\), 
\[ x^2 + y^2 = 1, \quad \frac{x^2}{m^2} + \frac{y^2}{n^2} = \frac{1}{m^2 + n^2} \] 
\[ \text{Ans } \frac{m^2}{m^2 + n^2} + \frac{n^2}{m^2 + n^2} = 1 \]

91. Find the condition that \( x^2 + my + n^2 \) may be a multiple of \( x + p \) 
\[ \text{Ans } n^2 + p^2 = mp \]

92. Find the g. c. m of \( \frac{x^2}{3} + \frac{11x}{6}(x+1) - x - 1 \) and \( w^2 - \frac{x}{3} - \frac{1}{4} \) 
\[ \text{Ans } w - \frac{1}{2}(x+1) \]

93. If \( \frac{2b}{c-a} = \frac{a}{b-c} + \frac{c}{a-b} \) prove that \( 2(a^3 + c^3 - 3b^3) = (a+b+c)(a^2 + c^2 - 2b^2) \)

94. Find the length of a zig-zag road ascending by a gradient of 1 in 8, to the top of Parishmoth which is 4500 ft. high. \[ \text{Ans } 36000 \text{ ft.} \]

95. Divide \( 3 + xy + wz - 1 + wy - 1 + yz - 1 + x - 1 \) by \( x^{-1} + y^{-1} + z^{-1} \) 
\[ \text{Ans } w + y + z \]

96. If \( w^2 + x^2 + y^2 = 2 \) prove that \( \frac{w^2 - 6w}{1 + w^2} = \frac{y^2 - 6y}{1 + y^2} = \frac{x^2 - 6x}{1 + x^2} \)

97. If \( yz + wz + xz = 1 \) prove that \( (1 + yz)(1 + wz)(1 + wy) = (1 + x^2)(1 + y^2)(1 + z^2) \)

98. If \( w + y + z + w = \) prove that \( 6(w^5 + y^5 + z^5 + w^5) = 5(w^5 + y^5 + z^5 + w^5)(w^5 + y^5 + z^5 + w^5) \)

99. Shew that \( (ab)^n - (bc)^n + (ad)^n - (ad)^n \) is divisible by \( ab - bc + cd - ad \), if \( n \) be any positive integer.
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ALGEBRAICAL EXERCISES.

100. If \( 3z = a + b + c \) prove that
\[
(\omega - a)^2 + (\omega - b)^2 + (\omega - c)^2 = 2z(z - b)(\omega - c)^2 + (\omega - a)^2 + (\omega - a)(\omega - b)^z
\]

101. Simplify \( \frac{2\omega^9 - 2\omega^4}{2\omega^5 + 1} \times 4 \) \( \text{Ans} \frac{7}{8} \)

102. Solve \( \frac{a + b}{a + b} + \frac{\alpha + c}{\alpha + b + c} = \frac{2(\alpha + b + c)}{\alpha + b + c} \)
\( \alpha = \frac{b^2 + c^2}{b + c} \)

103. A and B run a mile. First A gives B a start of 44 yds and beats him by 51 seconds; at the second heat A gives B a start of 1 min. 15 sec. and is beaten by 88 yds; find the time in which A and B can run a mile respectively. \( \text{Ans} \) A in 5, and B in 6 min.

104. Add together \( \omega^2 - (\omega - y + z)(\omega + y - z), y^2 - (y - y + z)(y + \omega - z) \)
\( (y + \omega - z)z^2 - (z - x + y)(z + \omega - y), \text{Ans} 2(\omega^2 + y^2 + z^2 - xy - xz - yz) \)

105. Simplify \( \frac{\omega^8}{(\omega - y)(\omega - z)} + \frac{y^8}{(y - \omega)(y - z)} + \frac{z^8}{(z - \omega)(z - y)} \)
\( \text{Ans} \omega + y + z \)

106. AB is a railway 220 miles long, and three trains P, Q, R, travel upon it at the rate of 25, 20 and 30 miles per hour respectively, P and Q leave A at 7 a.m. and 8 15 a.m. respectively, and R leaves B at 10 30 a.m. When and where will P be equidistant from Q & R
\( \text{Ans} \) 12 O'clock; 125 miles from A.

107. Simplify \( \frac{1 + \omega + \omega^2}{1 - \omega^8} + \frac{1 - \omega + \omega^2}{1 + \omega^8} - \left( \frac{\omega}{1 + \omega} + \frac{1 - \omega}{1 + \omega} + \frac{1 + \omega}{1 - \omega} \right) \times \frac{1}{1 - \omega} \)
\( \text{Ans} \frac{1}{1 - \omega^8} \)

108. Simplify \( \frac{\omega^2 - b^2}{\omega - b} \times \frac{1}{\frac{1}{b - a}} \)
\( \text{Ans} \omega - b \)

109. If \( \frac{\omega}{a^2 + b^2} = \frac{2y}{a^2 + b^2} = \frac{4z}{a^2 + b^2} \) shew that \( \omega(y^2 - z^2) + 2y(a^2 - x^2) + 4z(x^2 - y^2) = 0 \)
110. If \( x + y + z = 0 \) show that
\[
\left( \frac{y - z}{x} + \frac{z - x}{y} + \frac{x - y}{z} \right) = 0
\]
proceed that
\[
\left( \frac{x}{y - z} + \frac{y}{z - x} + \frac{z}{x - y} \right) = 0
\]

111. If \( x + y + z = x^2 + y^2 + z^2 = 2 \) prove that
\[
(x - 1)^2 = (1 - y)^2 = (1 - z)^2
\]

112. Simplify \( \{x\sqrt{(1 - y^2)} + y\sqrt{(1 - w^2)}\}^2 + \{w\sqrt{(1 - x^2)}(1 - y^2)\}^2 \)

\[\text{Ans } 1\]

113. If \( \frac{y}{x} = m, \frac{z}{x} = n, \frac{y}{w} = p \)

prove that \( m + n + p^2 = 2m^2n^2 + 2m^2p^2 + 2m^2n^2 + m^2n^2 + p^2 \)

114. Simplify \( (4x^2 - 3x)^2 + (3\sqrt{1 - x^2} - 4(1 - x^2)^2) \)

\[\text{Ans } 1\]

115. If \( w^2 + xy + y^2 = c^2, w^2 + wz + z^2 = b^2, y^2 + yz + z^2 = a^2 \) show that
\( x^2 + y^2 + z^2 = \sqrt{\left( \frac{1}{4} \left( 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4 \right) \right)} \)

116. Divide \( x^2 - 3ax^2 + 3a^2x - a^2 + b^2 \) by \( x - a + b \)

\[\text{Ans } (x - a)^2 - b(x - a) + b^2\]

117. Eliminate \( 4(x^2 + y^2) = ax + by, 2(a^2 - y^2) = ax - by, x^2 = a^2 \)

118. Prove that \( x^n - y^{n-1} + x + (n-1)ax \) is divisible by \( (x - a)^2 \) if \( n \) be a positive whole number.

119. Solve \( \frac{x + 4a + m}{x + a + m} + \frac{4x + a + 2m}{x + a - m} = 5 \)

\[x = -m\]

120. The slope of Baharinoth hill in Ranigunge is 1 in 8. A man whose usual pace on level ground is 4 miles an hour, ascends and descends in 5 hours. His pace uphill is pace on level : : 3 : 4 and pace down hill to pace on level : : 5 : 4. What is the height of the hill?

\[\text{Ans' } 2062\frac{1}{2} \text{ yds.}\]

121. If \( \frac{\omega_1}{\omega_n} = \frac{\omega_2}{\omega_n} = \omega c, \frac{\omega_n}{v_n + 1} \)

\[\text{prove that } \left( \frac{\omega_1 + \omega_2 + \omega_3 + \cdots + \omega_n}{\omega_2 + \omega_3 + \omega_4 + \cdots + \omega_n + 1} \right)^n = \frac{\omega_1}{\omega_n} + \cdots + \omega_n \]
122. Prove that $(1 + \sqrt{(-1)}) + (1 - \sqrt{(-1)}) = 0$

123. Find the G. C. M. of $a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$ and
$a^2 - b^2 - c^2 - 2bc$

Ans $a - b - c$

124. If $T = \text{the time between two successive inferior or superior conjunctions of a planet}$, $E = \text{periodic time of the earth}$, $P = \text{periodic time of the planet}$ prove that $P = \frac{T \times E}{1 + E}$

125. If the interval between the inferior conjunctions of Venus be 584 days and the periodic time of the earth be 365 days find by ex 25 the periodic time of Venus.

Ans 224 days nearly

126. Solve $\frac{2x + 11}{x + 3} - \frac{9x - 9}{3x - 4} = \frac{3x + 13}{x + 3} - \frac{15x - 47}{3x - 10}$

Ans $-\frac{5}{2}$

127. If $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$ and $\frac{x^2}{m^2} + \frac{y^2}{n^2} + \frac{z^2}{p^2} = 1$ prove that

$\frac{a^2}{m^2} + \frac{b^2}{n^2} + \frac{c^2}{p^2} = \frac{a^2 + b^2 + c^2}{x^2 + y^2 + z^2}$

128. Divide $x^{n+1} + y^{n-1}$ by $x^n + \frac{1}{3} y^{n-1}$

Ans $x^2(n+1) - x^n + \frac{1}{3} y^{n-1} + y^{2(n-1)}$

129. Solve $x^2 + 2x = 3x^2 + 10x$

$x = 2, -3$ and $4$

130. Divide $a^4(b - c) + b^4(c - a) + c^4(a - b)$ by $(a - b)(b - c)(c - a)$

Ans $a^4 + b^4 + c^4 + ab + ac + bc$

131. Supposing that $(2x - a)(2x - b)(2x - c)$ is an exact cube

Show that $(ab + ac + bc) \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} \right)$ is an exact square

132. Simplify $\frac{a + b + c - abc}{1 - ab - ac - bc} - \frac{b + c}{1 - bc}$

Ans $a$

133. Solve $\frac{ax}{a + x} + \frac{by}{b + y} = \frac{c(a + b)}{a + b + c}$, $x + y = c$, $x = \frac{ac}{a + b}$, $y = \frac{bc}{a + b}$
RECREATIONS, AMUSING PROBLEMS.

134. A man has a wolf, a goat and a cabbage, to carry over a river, but being obliged to transport them one by one on account of the smallness of the boat, in what manner is this to be done, that the wolf may not be left with the goat, nor the goat with the cabbage?

135. "Tell me, illustrious Pythagoras, how many pupils frequent thy school? One half, replied the philosopher, study mathematics, one fourth natural philosophy, one seventh observe silence, and there are three females besides." Ans 28

136. A mule and an ass travelling together, the ass began to complain that her burden was too heavy, "Lazy animal" said the mule you have little reason to complain, for if I take one of your bags, I shall have twice as many as you, and if I give you one of mine we shall then have only an equal number," With how many bags was each loaded.

Ans Mule 7 bags, ass 5 bags

137. Three gentlemen, with their servants, having to cross a river at a ferry, find a boat without a boatman; but the boat is so small that it can contain no more than two of them at once. How can these six persons cross the river, two and two, so that none of the servants shall be left in company with any of the gentlemen, unless when his master is present.
SOLUTIONS.

NOTATION.

Ex. 1. 73. \((-\frac{1}{2})^4 - \left(\frac{3}{2}\right)(\frac{1}{2})^3 + \frac{3}{2} \cdot 0 = \frac{1}{16} - \frac{9}{8} + 0 = \frac{1}{16} - \frac{9}{8}
\]

74. \(a \cdot b = \alpha \therefore a = b + x = b + 3\) also from the 2nd \(a + (b + \omega) = \alpha + \omega \therefore 2a = 2 \therefore a = 1 \therefore b = -2 \therefore\) the expression =

\[(1 + 2)\{9 - 2 \cdot 1 \cdot 9 + 1 \cdot 3 - (1 - 2)4\} = 3 \{9 - 18 + 3 + 4\} = -6
\]

ADDITION.

Ex. 2. 1. \(4a - 6b\) 2. \(5x^2 + 4y^2\) 13. \(2x^2 + y^4 + 2z^6\)

\[
\begin{array}{ccc}
2a + 3b & 4x^2 - 8y^2 & -4x^2 - 5z^6 \\
-a + 2b & -2a^2 + y^2 & 4x^2 - 3y^4 + 7z^8 \\
a + b & 4y^2 - 9z^6 & 2x^2 - 2y^4 + 4z^8 \\
8a - 4b & 11x^2 - 4y^2 & \\
\end{array}
\]

17. \(2\sqrt{(a + x)} - 4\sqrt{(a - \omega)} + 6\sqrt{(a^2 - \omega^2)}
\]

\[-3\sqrt{(a + x)} - 6\sqrt{(a - \omega)} + 2\sqrt{(a^2 - \omega^2)}
\]

\[-4\sqrt{(a + x)} + 6\sqrt{(a - \omega)} + 5\sqrt{(a^2 - \omega^2)}
\]

19. Add the coefficients of \(x^4, x^3, x^2, w\) and \(y\) and arrange.

20. Proceed as in the foregoing example.

29. \((a + b)(x + y)^2 + (c + d)(x + y) + c\omega
\]

\[(a - b)(x + y)^2 + (c - d)(\omega + y) + c\omega
\]

\[-2c(x + y)^2
\]

\[(a + b + b - 2a)\omega + (c + d + c - d)(\omega + y) + c(\omega + y)
\]

or \(2c(x + y) + c(\omega + y) - 3c(\omega + y)\)
SOLUTIONS.

SUBTRACTION.

Ex. 3.  1. \( 8a + 16b \)  
2. \( 3a + 10b \)  
\[ \frac{5a + 6b}{\text{diff} = 2c} \]  
12. Sum of the 1st two = \( 2a + 2c \),  
and the sum of the last two is \( 2a \),  
\[ \{2a + b - c - (a + b - c)\} x + \{(a + b + c) - (a + c)\} y + \{a + b + c - (a + b)\} = ax^2 + by + c \]

BRACKETS.

Ex. 4.  1. \( 3a - 2b + 2a - b = 5a - 3b \)  
2. \( a + b - c - a + b + c = 2b \)  
3. \( 1 - a + a^2 - 1 - a^2 - a + 1 = 2 + 2a \)  
11. \( 10a - \{4a - 5 + 6a - 2a\} = 10a - 4a + 5 - 6a + 2a = 2a + 5 \)  
12. \( 2 - \{1 - 1 + a\} - 4 = 2 - 1 + 1 - a - 4 = - a - 2 \)  
13. \( x - \{-1 - a - 1 + a\} = x + 1 + a + 1 - a = x + 2 \)  
14. \( 5a - \{2a - \{1 - 2a + 1\}\} = 5a - [2a - 1 + 2a - 1] \)  
\[ = 5a - 2a + 1 - 2a + 1 = a + 2 \]
20. \(-[-1 - \{1 - (a - 1 + a) - a\}] = -[-1 - \{1 - a + 1 - 2a\}] \)  
\[ = -[-1 - 1 + a - 1 + 2a] = 1 + 1 - a + 1 - 2a = 3 - 3a \]
21. \(-[-(-a + 1)]\} = -[-\{a - 1\}] = -[-a + 1] = a - 1 \)  
23. \(a - [-a - 1 + \{-1 - a + 1 - 2\} - 1] \)  
\[ = a - [-a - 2 - a + 1 - 2 - 1] = a + a + 2 + a - 1 + 2 + 1 = 3a + 4 \]
24. \(-[-1 - \{-1 + 2 - 1\}]\} = -[-1 + 1 - 2 + 1] = 1 - 1 + 2 - 1 = 1 \)

MULTIPLICATION.

Ex. 5.  21. \( \frac{a^2 - ab + b^2}{a + b} \)  
54. \( \frac{x^3 + y^3}{x + y^3} \)  
\[ \frac{x^3 - ab + b^2}{a^2 - b^2} \]  
\[ = \frac{a^3 - ab + b^2}{a^2 - ab^2 + b^2} \]  
\[ \frac{w}{u^2 + c^2} \]  
\[ \frac{x^3 + y^3}{x + y} \]  
\[ \frac{x^3 - y^3}{x + y} \]  
\[ = -\frac{x^3 y^3}{x - y} \]