6. Show how to find the greatest term in the expansion of \((x+a)^n\) when \(n\) is a positive integer. Expand \((a^3 - 2ax)^\frac{3}{2}\) in ascending powers of \(x\) as far as \(x^3\), and write down the general term.

1890.

1. Show that \(x^2 - a^n\) is divisible by \(x - a\) for all positive integral values of \(n\), and hence, or otherwise, show that if any rational and integral expression which contains \(x\) vanish when \(a\) is put for \(x\), the expression is divisible by \(x - a\).

If \(x - a\) is a factor of \(a_1 x^2 + 2b_1 x + c_1\) and \(x + a\) of \(a_2 x^2 + 2b_2 x + c_2\), prove that \((c_2 a_1 - c_1 a_2)^3 + 4(a_1 b_2 + a_2 b_1)(b_1 c_2 + b_2 c_1) = 0\).

(See page 170.)

2. Solve the equations :-
   
   (1) \(x(x - \sqrt{2} - \sqrt{3} - 3) + \sqrt{2} + \sqrt{3} + 2 = 0\);
   
   (2) \(x + \frac{3}{y} = 2, y + \frac{3}{x} = -2\).

3. If \(a, b, c, x\) be positive quantities and \(a < b\), prove that \(\frac{a + x}{b + x} = \frac{a}{b}\).

If \(a, b, c\) be in continued proportion, prove that

\[\sqrt{a b} = \sqrt{b c} + \sqrt{c a} = \sqrt{(a - b + c)(b - c + d)}\].

(See page 101.)

4. Show how to insert \(n\) Geometric means between two quantities \(a\) and \(b\).

\(a, \beta, \gamma\) are the Geometric means between \(ca, ab; ab, bc; bc, ca\) respectively. Prove that if \(a, b, c\) are in \(A. P., a^2, b^2, \gamma^2\) are also in \(A. P., \) and \(\beta + \gamma, \gamma + a, a + \beta\) are in \(H. P.\).

(See page 258.)

5. Show how to find the number of permutations of \(n\) things taken \(r\) at a time. State how many combinations of \(r\) things each can be formed from \(n\) different things.

A cricket team consisting of eleven players is to be selected from two sets consisting of six and eight players respectively. In how many ways can the selection be made, on the supposition that the set of six shall contribute not fewer than four players?

(See page 276.)

6. Assuming the truth of the Binomial Theorem for a positive integral index, prove the truth of the theorem when the index is any positive quantity.

Find the co-efficient of \(x^r\) in the expansion of \(\left(1 + \frac{x}{1 - x}\right)^\frac{1}{2}\) in ascending powers of \(x\).

1891.

1. Find the G. C. M. of \(4x^6 - 209x^5 + 15\) and \(15x^6 - 209x^5 + 4\).

2. Between two given numbers \(a\) and \(b\) insert :-(1) two arithmetic means; (2) two harmonic means; (3) two geometric means.

If \(x_1, x_2\) be the arithmetic means, \(y_1, y_2\) the harmonic means and \(z_1, z_2\) the geometric means, show that \(x_1 y_2 = x_2 y_1 = z_1 z_2\).
3. Solve the equations—

(1) \( \sqrt{x^2 + a} + \sqrt{x^2 - a} = \sqrt{2a + b} + \sqrt{b} \).

(2) \( \frac{x}{y+1} + \frac{y}{x+1} = \frac{5}{3} ; x^2 + y^2 = 2. \)

4. Prove that the number of permutations of \( n \) things taken all together, of which \( p \) are alike and all the rest unlike is \( \frac{n!}{p!} \).

In how many ways can the letters forming the word plantain be arranged so that the two \( a \)'s do not come together?

5. Write down the five terms in the expansion of \( (a^2 + x)^\frac{1}{2} \).

Prove that if \( M \) differ from \( N \) by a small quantity the square root of \( M \) is approximately equal to \( \frac{3}{2} N \left( 3N^2 - M \right)^{\frac{1}{8}} \) (See page 352.)

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BOMBAY UNIVERSITY PREVIOUS EXAMINATION PAPERS.

1882.

1. Define an expression. When is an expression said to be homogeneous?

Show that the value of the expression \( y^2 z - z^2 y + z^2 x - x^2 z + x^2 y - xy^2 \) is not altered if any the same quantity be added to or subtracted from each of the quantities \( x, y \) and \( z \).

2. Prove that the square root of a binomial, one of whose terms is a quadratic surd and the other rational, may sometimes be expressed by a binomial, one or each of whose terms is a quadratic surd. In what case is it useless to employ this method?

Find the square root of \( a + b + \sqrt{ab} + b^2 \).

3. Find the sum and product of the root of the quadratic—

\( px^2 + qx + r = 0. \)

If \( a \) and \( \beta \) be the roots of this equation, shew that the roots of the equation \( qRx^2 + (pr + q^2)x + pq = 0 \), are \( \frac{1}{a + \beta} \) and \( \frac{1}{a} + \frac{1}{\beta} \).

4. Prove that a ratio of greater inequality is diminished and of less inequality increased by adding any quantity to both its terms.

If four numbers be proportionals, shew that there is no number which being added to each will leave the resulting four numbers proportionals.
5. Find the sum of a G. P. to \( n \) terms and when possible to infinity.
   If \( s_1, s_2, s_3 \) be the sums to \( n, 2n, 3n \) terms respectively, prove that
   \[ s_1(s_3 - s_2) = (s_2 - s_1)^3. \]

6. If \( (n)_r \) represent the number of combinations of \( n \) things taken \( r \) together, prove that independently of any formula that \( (n)_r = n(n-1)_{r-1}. \)

Four persons are chosen by lot out of ten; in how many ways can this be done and how often would any one person be chosen? (See page 276.)

7. If \( m \) be any quantity whatever and \( f(m) \) represent the series
   \[ 1 + mx + \frac{m(m-1)}{2}x^2 + \cdots \frac{m(m-1)(m-2)}{2}\cdots x^3 + \&c, \]
   prove that \( \frac{f(m)}{f(n)} = f(m+n) \). By what consideration does Fuller prove this relation and on what principle does he base his demonstration? By the aid of this formula prove the Binomial Theorem for positive fractional exponent.

1883.

1. Prove that \( (n+1)(n+2)(n+3) \ldots \) to \( n \) factors
   \[ = 2n \times 1.3.5 \ldots \) to \( n \) factors.

2. Show that a factor may be found which will rationalise any binomial.

   Reduce \[ \sqrt{1+x} - \frac{\sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \] where \( x = \frac{\sqrt{3}}{2}. \) (See page 85.)

3. Distinguish between a quadratic equation and quadratic expression and show that a quadratic has two and only two roots.

   Solve the equation \( \sqrt{x^2 - 3x + 8} = \frac{1}{2}(x^2 - 7x + 8). \)

4. Define ratio, variation, proportional, commensurable.

   If \( A \) vary as \( B \) when \( C \) is constant, and \( A \) vary as \( C \) when \( B \) is constant, prove that \( A \) will vary as the product \( BC \) when both \( B \) and \( C \) are variable. Give fully a Geometrical illustration.

5. Sum the series:
   (1) \( 17\frac{1}{2}, 14\frac{1}{2}, 10\frac{1}{2}, \ldots \) to 24 terms.
   (2) \( 1, 3, 6, 10, 15, \ldots \) to \( n \) terms.
   (3) \( \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \) to infinity.

6. Investigate a formula for the number of permutations of \( n \) things taken all together which are not all different.

   How many different numbers can be made out of all the figures of 111223 ?

7. Enunciate the Binomial Theorem and prove it for a positive integral exponent.

   Write down the \((r+1)\)th term in the expansion of \( \sqrt{x^2 - x^2} \) and the co-efficient of \( x^0 \) in \( \frac{1+x}{(1-x)^3}. \)
1884.

1. If \( s = a + b + c \), prove that \( (s - a)^2 + (s - b)^2 + (s - c)^2 \)
   \[= 2((s - a)(s - b) + (s - b)(s - c) + (s - c)(s - a)).\]

2. Shew that if two quadratic surds cannot be reduced to others which
   have the same irrational part, their product is irrational, and also that one
   quadratic surd cannot be made up of two others which have not the same
   irrational part.

3. If \( a : b = c : d \), shew that
   
   \[(a)\] \( \frac{ma + nb}{n} = \frac{ma + nb}{n} : \frac{pc + qd}{q} \).
   
   \[(\beta)\] \( \frac{1}{n} + \frac{1}{q} = \frac{1}{l} \left\{ \frac{a}{q} + \frac{b}{q} + \frac{c}{q} + \frac{d}{q} \right\} \) \( . \) (See Page 102.)

4. What do you understand by the limit of an infinite series in Geometrical Progression? Are Arithmetical series susceptible of limits? Find the sum of an infinite \( G. \ P. \)

Determine the value of \( \sqrt[3]{a^2} \sqrt[b]{a^3} / \sqrt[3]{b} \sqrt[3]{a^3} / \sqrt{b} \). . . continued to infinity.

5. Insert a given number of Harmonical means between two given terms.

If \( a, b, c \) be in \( G. \ P. \), and \( a^x = b^y = c^z \), prove that \( x, y, z \), are in \( H. \ P. \).

6. Find for what value of \( r \) the number of combinations of \( n \) things
   taken \( r \) at a time is greatest.

A Brahman, hospitably disposed, wishes to make up as many different
paries as he can out of 40 friends, each party consisting of the same number
; how many should he invite at a time?

7. Investigate the sum of the co-efficients of the terms in the
   expansion of \( (1 + x)^n \) and shew that the sum of the co-efficients of the odd terms
   is equal to the sum of the co-efficients of the even terms, \( n \) being a positive integer.

Prove that if \( n \) be a positive integer, \( (1 + x)^n \ (1 + x^n) > 2^n + 1 \cdot x^n \).

1885.

1. If \( a + b + c + d = 2s \), prove that \( 4(ab + cd) - (a^2 + b^2 - c^2 - d^2)^2 \)
   \[= 16(s - a)(s - b)(s - c)(s - d).\]

2. Shew that in approximating to a cube root the number of figures
   may be nearly doubled by ordinary division.

If \( a \) be the greatest integer contained in \( N^\frac{1}{3} \) and the difference be so
small that its cube may be neglected, prove that a nearer approximate value
of \( N^\frac{1}{3} \) will be

\[\frac{1}{4} \left\{ a + \left( \frac{4N - a^3}{3a} \right) \right\}. \] (See page 48.)

3. If \( a, \beta \) be the roots of the quadratic equation \( x^2 - px + q = 0 \), shew
   that \( x^2 - px + q = (x - a)(x - \beta) \).

Shew also that \( \frac{a^2 + \beta^2}{a^4 + \beta^4} = \frac{p^2 - 4q^2}{q^2} + 2. \)
If \( a \propto b \) when \( c \) is invariable and \( a \propto c \) when \( b \) is invariable, prove that \( a \propto bc \) when both \( b \) and \( c \) are variable.

4. The value of a silver coin varies directly as the square of its diameter while its thickness remains the same, and directly as its thickness while its diameter remains the same. Two silver coins have their diameters in the ratio of \( 4 : 3 \); find the ratio of their thickness if the value of the first be four times the value of the second. (See page 115.)

5. Investigate the sum of the squares of the first \( n \) natural numbers.

Sum the following series to \( n \) terms and write down the general term of each:

\[
\begin{align*}
(a) & \quad 2 + 7 + 14 + 23 + \ldots . \\
(b) & \quad 2 + 5 + 10 + 17 + \ldots .
\end{align*}
\]

6. Derive the number of permutations of \( n \) things \( r \) at a time from the number of their permutations \( r - 1 \) at a time.

Find the number of different arrangements that can be made of bars of the seven prismatic colours so that the blue and the green shall never come together.

7. Investigate the greatest term in the expansion of \((x+a)^n\), where \( n \) is a positive integer.

Find the greatest term in \((1+x)^n\) when \( x = \frac{1}{2} \).

1886.

1. If \( x+y = p \) and \( xy = q \), express \( x^2 + y^2, x^3 + y^3, x^4 + y^4 \) in terms of \( p \) and \( q \).

2. Simplify \( \frac{a^2}{(a-b)(a-c)} - \frac{b^2}{(a-b)(b-c)} + \frac{c^2}{(a-c)(b-c)} \).

Show that \( \left( \frac{1}{a-b} + \frac{1}{b-c} + \frac{1}{c-a} \right)^2 = \frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} \).

3. Find \( x \) and \( y \) from the equations—
\[
\begin{align*}
\sqrt{x} - \sqrt{y} &= \sqrt{\frac{4}{3}} \\
\sqrt{xy} &= \frac{2}{3}.
\end{align*}
\]

4. What is the condition that \( ax^2 + bx + c \) shall be a perfect square with respect to \( x \)?

For what value of \( n \) will the expression \( x^2 - (n-1)x + n + \frac{1}{4} \) be a perfect square.

5. If \( a, b, c, d \) are proportionals—
\[
a + b : a - b : : c + d : c - d ;
\]
\[
\frac{a^2 + c^2}{b^2 + d^2} = \frac{2}{b} \cdot \frac{a}{d} ;
\]
\[
(a + d) - (b + c) = \frac{(a - b)(a - c)}{a}.
\]
6. Sum the series—

\[3 - 1 + \frac{1}{3} - \frac{1}{9} + \&c., \text{ad infinitum.}\]

\[3 + 6 + 11 + 20 + \&c., \text{to } n \text{ terms.}\]

If \(a, b, c\) be in Arithmetical Progression, \(b, c, d\) in Geometrical Progression and \(c, d, e\) in Harmonical Progression, prove that \(a, c, e\) are in Geometrical Progression.  

(See page 258.)

7. Show by mathematical induction that the sum of the squares of the first \(n\) natural numbers is \(\frac{n(n+1)(2n+1)}{6}\).  

(See page 313.)

Hence obtain an expression for the sum of the squares of the odd numbers in this series.

8. Prove that the number of combinations of \(n\) things taken \(r\) at a time is equal to the number of combinations taken \(n - r\) at a time.

Find the number of different signals that can be made with six flags, two of which are white, two black, and two red; six flags to be used in each signal.

9. Write down the co-efficient of \(x^r\) in the expansion of \((1+x)^n\) and \((1+x)^{-n}\) and the middle term of \(\left(\frac{1}{x} + x\right)^n\).

If \(P\) denote the sum of the odd terms and \(Q\) the sum of the even terms in the expansion of \((a+b)^n\), \(P^2 - Q^2 = (a^2 - b^2)^n\),

and \(4PQ = (a+b)^n - (a-b)^n\).

10. Find the 5th root of 35 correct to five places of decimals.

1887,

1. Find a value of \(x\) which will make the expression \(x^5 - 8x^3 + 11x^2 + 7x - 1789\) exactly divisible by \(x^2 + 7x - 1\).

2. When \(n + 1\) figures of a square root have been obtained by the ordinary method, shew that \(n\) more may be obtained by division only, supposing the whole number of figures in the square root to be \(2n + 1\).

By this method extract the square root of 5294745225.

3. Shew that a factor may be found which will rationalise any binomial.

\[\text{Simplify } \frac{2\sqrt[3]{8} + \sqrt[3]{3}}{4 + \sqrt{10} - \sqrt{2}}.\]

(See page 84.)

4. Shew that a quadratic equation cannot have more or less than two roots.

If \(a, \beta\) be the roots of \(x^2 + px + q = 0\), form the equation whose roots are \(a^2 + a\beta\) and \(\beta^2 + a\beta\).  

(See page 159.)

5. If \(\frac{a}{b} = \frac{c}{d} = \frac{e}{f}\), prove that \(\left(\frac{a + 2c + 3e}{b + 2d + 3f}\right)^2 = \frac{ac + ce}{bd + df}\).
Solve the simultaneous equations:
\[
\begin{align*}
\frac{1}{x} + \frac{1}{y} &= \frac{12}{11} ; \\
x + y &= \frac{7}{12} ; \\
x + y - 5 &= \frac{7}{12}.
\end{align*}
\]

6. Investigate an expression for the \(n\)th term of an Harmonical Progression of which the first and last terms and the number of terms are given.

If \(a, b, c\) are the \(p\)th, \(q\)th, \(r\)th terms of an \(A.\ P.,\) show that \((q - r)a + (r - p)b + (p - q)c = 0\), and if \(a, b, c\) are the \(p\)th, \(q\)th, \(r\)th terms of an \(H.\ P.,\) then \((q - r)bc + (r - p)ca + (p - q)ab = 0\).

7. Find the sum of a Geometrical Progression when the number of terms is indefinitely great and the common ratio is a proper fraction.

Prove that any term is \(\ldots = \frac{\ldots}{\ldots}\), or \(\ldots\); the sum of all the succeeding terms according as the ratio is \(\ldots = \frac{\ldots}{\ldots}\), or \(\ldots\).

8. Investigate the number of permutations of \(n\) things taken \(r\) at a time.

Six examination papers are to be set in a certain order not to be divulged; it being discovered that this order has leaked out, in how many ways can the order be changed?

9. Write down the expansion of \((1 + x)^n\), and find the sum of the coefficients of all the terms.

If \(p_1, p_2, \ldots, p_n\) denote the coefficients of \(x, x^2, \ldots, x^n\) in this expansion, show that \(p_1 + 2p_2 + 3p_3 + \ldots + np_n = 2^{n-1} n\).

10. Obtain the general term in the expansion of \((1 - x)^{-n}\).

Find the coefficient of \(x^n\) in the expansion of \(\frac{(1 + x)^2}{(1 - x)^4}\).

1888.

1. Simplify: \(-4\sqrt{147} - \frac{10}{\sqrt{3}} - 3\sqrt{75} - 2\sqrt{3}\),

\[
\frac{2 + \sqrt{3}}{\sqrt{2} + \sqrt{2} + \sqrt{3}} + \frac{2 - \sqrt{3}}{\sqrt{2} - \sqrt{2} - \sqrt{3}}.
\]

2. Prove that the product of any four consecutive even integers increased by 16 is a perfect square.

3. Show how to obtain the sum and product of the two roots of a quadratic equation in the terms of the co-efficients and the last term.

If \(a\) and \(b\) are the roots of \(ax^2 + bx + c = 0\), find the values of the sum and product of \(aa + b\) and \(ab + b\).

4. Define ratio and ratio of greater inequality.

If \(6x^3 + 6y^3 = 13xy\), what is the ratio of \(x\) to \(y\)?

If \(a : b\) is a ratio of greater inequality, shew that \(a : b\) is greater than \(a^2 + b^2 : 2ab\).
5. Define direct, inverse and joint variation; and give an illustration of each.

Given that \( x+y \) varies as \( z + \frac{1}{z} \), and \( x-y \) varies as \( z - \frac{1}{z} \), find the relation between \( x \) and \( z \), if \( z = 2 \), when \( x=3 \) and \( y=1 \). (See page 116.)

6. Find the \( n \)th term in an Arithmetical, a Geometrical, and an Harmonical Progression.

In a Geometrical progression, if the \((p+q)\)th term is \( m \), and the \((p-q)\)th term is \( n \), find the \( p \)th and \( q \)th terms.

7. Find the sum of an infinite Geometrical Progression, the common ratio being less than unity.

If \( S_1, S_2, S_3, \ldots S_p \) are the sums of infinite Geometrical series, whose first terms are \( 1, 2, 3, \ldots p \), and whose common ratios are \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \)

respectively, prove that \( S_1 + S_2 + S_3 + \ldots + S_p = \frac{p}{p+1}(p+3) \).

8. Find the number of ways in which it is possible to make an arrangement of \( r \) things out of \( n \), when in each permutation any of the things may be repeated once, twice, \ldots \( r \) times.

There are 3 candidates for a Professorship, and one is to be elected by the votes of 5 men; in how many ways can the votes be given? (See page 289.)

9. Prove the Binomial Theorem, when the exponent is any negative quantity.

Find the \((r+1)\)th term of \((1+x)^{-n}\) and \((1-nx)^{-\frac{1}{m}}\).

10. Find the greatest term in the expansion of \((1+x)^n\) when \( n \) is fractional and positive.

Find the greatest term in \( \left(1 + \frac{2}{3}\right)^n \).

1889.

1. If \( \frac{1}{1+l+m} + \frac{1}{1+m+mn} + \frac{1}{1+n+nl} = 1 \), prove that either \( lmn = 1 \), or \((1+l)(1+m)(1+n) = -1 \).

2. Prove that a factor can be found which will rationalise any binomial surd.

If \( (x + \sqrt{z^2} - bc)(y + \sqrt{y^2} - ca)(z + \sqrt{z^2} - ab) \)

\[ = (x - \sqrt{z^2} - bc)(y - \sqrt{y^2} - ca)(z - \sqrt{z^2} - ab), \]

show that each of these expressions = \( \pm abc \). (See page 82.)

3. When \( n+2 \) figures of a cube root have been obtained by the ordinary method, \( n \) more can be obtained by division only, supposing \( 2n+2 \) to be the whole number.
If \( x^4 + 3dx^2 + ex + fx + gx + h + k = 0 \) be a perfect cube find its cube root and determine the co-efficients \( e, f, g, h \) in terms of \( d \) and \( k \).

(See page 47.)

4. If \( \alpha, \beta \) are the roots of the equation \( Ax^2 + Bx + C = 0 \), then \( \alpha + \beta = \frac{B}{A} \) and \( \alpha \beta = \frac{C}{A} \).

If \( \alpha \) be a root of the equation \( 4x^2 + 2x - 1 = 0 \), prove that \( 4\alpha^2 - 3\alpha \) is the other root.

(See page 158.)

5. If \( a : b = c : d \), then \( ma + nb : pa + qb = mc + nd : pr + qd \).

If \( ab = cd = ef \), show that

\[
\frac{ae + ce + ea}{bd(fb + ad + f)} = \frac{a^2 + c^2 + e^2}{d^2f^2 + f^2b^2 + b^2d^2}.
\]

(See page 100.)

6. Find the sum of \( n \) terms of an Arithmetical Progression.

If \( s_1, s_2, s_3 \), denote the sum of \( n \) terms of three arithmetical series whose first terms are unity and their common differences in harmonic progression, prove that \( n = \frac{s_2s_1 - s_1s_3 - s_3s_2}{s_1 - 2s_2 + s_3} \).

(See page 258.)

7. Find the sum of \( n \) terms of the following series:

\[ a, (a + b)r, (a + 2b)r^2, (a + 3b)r^3, \ldots \]

If \( P, Q, R \) be the \( p \)th, \( q \)th, and \( r \)th terms of a Geometrical Progression, show that \( Pr^{-r}, Qr^{-q}, Rr^{-q} = 1 \).

8. Find for what value of \( r \) the number of combinations of \( n \) things taken \( r \) at a time is greatest.

There are \( n \) points in a plane of which no three are in a straight line except \( n \), which are all on a straight line. Find the number of triangles formed by joining the points.

(See page 277.)

9. Write down the expansion of \( (1 - x)^{-n} \) and hence show that

\[
\sqrt{3} = 1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + \cdots.
\]

(See page 350.)

10. Find the sum of the co-efficients of the odd terms in the expansion of \( (1 + x)^n \) where \( n \) is a positive integer.

(See page 324.)

If \( c_0, c_1, c_2, c_3, \ldots, c_n \) be the co-efficients in the expansion of \( (1 + x)^n \) where \( n \) is a positive integer, prove that

\[
c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \frac{c_3}{4} + \ldots + \frac{c_n}{n+1} = \frac{2^{n+1} - 1}{n+1}.
\]

1890.

1. If \( x = \frac{1}{4}(b + c - a), \ y = \frac{1}{4}(c + a - b), \ z = \frac{1}{4}(a + b - c) \), show that \( x^3 + y^3 + z^3 - 3xyz = \frac{1}{4}(a^3 + b^3 + c^3 - 3abc) \).

2. Find a factor which will rationalise a given binomial.

Bring \( \frac{7}{2^6 + 2^3 + 1} \) to a form with a rational denominator. (See page 70.)
3. Find the sum and product of the roots of the quadratic \( ax^2 + bx + c = 0 \). When are the roots real and when imaginary?

If \( x_1, x_2 \) be the roots of the equation \( x^2 + px + q = 0 \), prove that \( x_1^4 + x_1^2x_2^2 + x_2^4 = n^2(2m^2 + 3n^2) \). *(See page 160.)*

4. Define proportion, and if four quantities are in proportion, shew that the product of the means is equal to the product of the extremes.

If \( (pa + qb + rc + sd)(pa - qb - rc - sd) = (pa - qb + rc - sd)(pa + qb - rc - sd) \), shew that \( bc, ad, ps, qr \) are in proportion. *(See page 95.)*

5. Insert a given number of arithmetical means between two given quantities.

An Arithmetical Progression and an Harmonical Progression have the same first term, the same last term, and the same number of terms; prove that the product of the \( r \)th term from the beginning in one series and the \( r \)th term from the end in the other is independent of \( r \). *(See page 257.)*

6. Define Geometrical Progression, and obtain the relation which must hold among three quantities which are in Geometrical Progression.

From three numbers which are in \( G. P. \) three other numbers in \( G. P. \) are subtracted, and the remainders are found to be also in \( G. P. \); prove that the three series have the same common ratio.

7. Investigate the number of permutations of \( n \) things taken \( r \) at a time.

Find how many significant numbers can be formed by using the digits \( 0, 1, 2, 3, 4 \), but using each not more than once in any number.

8. Describe the method of proof called Mathematical Induction, and apply this method to prove that every even power of every odd number when divided by \( 8 \) leaves \( 1 \) for a remainder. *(See page 311.)*

9. The Binomial Theorem being proved for any positive integral value of the index, prove it for any negative integral value.

In the expansion of \( \frac{3x - 8}{4 - 4x + x^2} \) in ascending powers of \( x \), prove that the co-efficient of \( x^4 \) is \(-\frac{1}{4} \), and find the co-efficient of \( x^r \). *(See page 347.)*

10. Write down the general term in the expansion of \((1 + x)^n\).

If \( a, b, c \) be three consecutive co-efficients in the expansion of a power of \( 1 + x \), prove that the index of the power is \( \frac{2ac + b(a + c)}{b^2 - ac} \) and that the number of the term of which \( a \) is the co-efficient is \( \frac{a(b + c)}{b^2 - ac} \). *(See page 330.)*

1891.

1. Prove that if \( x + y^{-1} = a, y + z^{-1} = b, z + x^{-1} = c \), then

\[
(1 - bc)x + (1 - ab)x^{-1} + 2b = (1 - ca)y + (1 - bc)y^{-1} + 2c = (1 - ab)z + (1 - ca)z^{-1} + 2a.
\]
2. Prove the rule for finding the L.C.M. of two Algebraical expressions.
   If \( x^2 + ax + b, \ x^2 + a'x + b' \) have an L. C. M. of the form \( x^3 + px + q \)
   prove that \( ab = a'b' = -aa'(a + a') \).

3. Show how to find the square root of a binomial of the form \( a + \sqrt{b} \)
   where \( \sqrt{b} \) is a surd.

   Simplify \( \frac{2\sqrt{5} - \sqrt{8}}{\sqrt{10} - 2 + \sqrt{(7 - 2\sqrt{10})}} \).

4. Find two independent relations between the roots and the co-efficients in a quadratic equation.

   If \( y \) and \( z \) be the roots of the equation \( A(x^2 + m^2) + Ax + Cx^2 m^2 = 0 \),
   show that \( A(y^2 + z^2) + Ay + Cy^2 z^2 = 0 \).

5. If \( A \propto B \) when \( C \) is constant, and \( A \propto C \) when \( B \) is constant,
   prove that \( A \propto BC \) when both \( B \) and \( C \) vary.

   If \( m \) sovereigns in a row stretch as far \( n \) pennies, and \( p \) sovereigns
   in a pile are as high as \( q \) pennies, compare the values of equal bulks of gold
   and copper, assuming that the area of a circle varies as the square of its
   radius.

6. Shew how to find the sum of \( n \) terms of an Arithmetical Progression.

   If \( A_p, G_p \) be respectively the \( p^{th} \) terms of the series—
   \[ a, a + b, a + 2b, \ldots \ldots \]
   \[ a, ar, ar^2, \ldots \ldots \]
   find the sum of the first \( n \) terms of the series whose \( p^{th} \) term is \( A_p, G_p \).

7. When are three quantities said to be in Harmonical Progression ?
   Prove that the reciprocals of such quantities are in Arithmetical Progression.

   If \( p, q, r \) be in Arithmetical Progression, prove that
   \( \frac{q^r - r^q}{rq + pr^r}, \frac{r^p - p^r}{rp + qr^r}, \frac{p^q - q^p}{qp + qr^q} \)
   are in Harmonical Progression.

8. Explain the method of proof known as Mathematical Induction.

   Shew by this method that the sum of \( n \) terms of the series,
   \[ 1 + 3 + 6 + 10 + 15 + \ldots \ldots \]
   \( \frac{1}{2} n(n + 1)(n + 2) \). \( \text{ (See page 310).} \)

9. Find an expression for the number of combinations of \( n \) dissimilar
   things taken \( r \) at a time.

   There are \( m \) men and \( n \) monkeys, \( n \) being greater than \( m \); find the
   number of ways in which each man may become the owner of one monkey.
   If a man may have any number of monkeys, in how many ways may every
   monkey have a master ? \( \text{ (See pages 267 and 290).} \)

10. Assuming the Binomial Theorem for positive exponents, shew it to
    be true for any exponent.

    Shew that the co-efficient of \( x^r \) in the expansion of \( \frac{(1 + 3x)^s}{(1 + 2x)^3} \)
    is \( (-2)^{r-3}.(r-8) \).
ALGEBRA MADE EASY.

1892.

1. If \( a + b + c = 0 \), prove that \( a^5 + b^5 + c^5 = -5abc(ab + ca + ab) \).

2. In any equation which involves rational quantities and quadratic surds, the rational parts on each side are equal, and also the irrational parts.

Shew that \( \frac{1}{(6\sqrt{3}+10)} - \frac{1}{(6\sqrt{3}-10)} = 2 \).

3. Solve the equation \( (x + 2)(3x + 1)(x - 1)(3x + 2) = 224 \).

4. Shew how to resolve \( ax^2 + bx + c \) into factors of the first degree in \( x \). What will the factors be when \( b^2 = 4ac \) ?

Find the values of \( a \) which make the expression \( x^2 - ax + 1 - 2a^2 \) always positive for real values of \( x \).

5. If \( \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \), prove that each of these ratios is equal to

\[
\frac{pa + qc + re}{pb + qd + rf}.
\]

If \( \frac{x}{y} = \frac{a}{b} \), and \( x \) and \( y \) be unequal, then each of these ratios is equal to \( x + y + z \) or \( x^{-1} + y^{-1} + z^{-1} \).

6. Find the sum of the squares of the first \( n \) natural numbers.

Shew that \( 1 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots \) to \( n \) terms

\[
= \frac{n(n+1)(2n^2 + n + 3)}{6} \quad \text{or} \quad \frac{n^2(n^2 + 9n + 4)}{6},
\]

according as \( n \) is odd or even.

7. Shew how to insert a number of harmonic means between two given terms.

If \( x, y, z \) be in A. P., \( ax, by, cz \) in G. P., and \( a, b, c \) in H. P., then the harmonic mean of \( a \) and \( c \) is to the Geometric mean of \( a \) and \( c \) as the harmonic mean of \( x \) and \( z \) is to the Geometric mean of \( x \) and \( z \).

8. Find the number of permutations of \( n \) things taken all together, of which \( p \) are of one sort, \( q \) of another, and the rest all different.

How many combinations and how many permutations can be made with the letters of the word \( \text{parabola} \) taken three at a time?

9. Describe carefully the method of \( \text{Mathematical Induction} \), and apply it to prove the Binomial Theorem for a positive integral exponent

Shew that \( 2^{4n} - 2^n (7n + 1) \) is some multiple of the square of 14, where \( n \) is a positive integer greater than unity.

10. Find in its simplest form the general term in the expansion of \( (1 + x)^{-n} \).

If \( s \) = the sum of two quantities, \( p \) = their product, \( q \) = their quotient, shew that

\[
p^s = s\left(q^2 + \frac{4.5}{1.2}q^5 + \frac{4.5}{1.2.3}q^8 + \ldots \right).
\]
1893.

1. If \( a^2 + a + 1 = 0 \), shew that \( x^3 - 1 = (x - 1)(x - a)(x - a^2) \).

2. Prove that if two algebraical expressions have a common factor, the factor will divide the difference of the expressions.

   The expressions \( x^2 + 6x + a \) and \( x^3 + 12x + 3a \) have a common factor; what numerical values can \( a \) have?

3. From an equation whose roots are \( \alpha, \beta \). How can you tell, without solving, whether the roots, supposed real, of a quadratic equation are positive or negative?

   Prove that the positive root of \( x^2 - 8x - 8 = 0 \) is greater than 8.

4. If \( a, b, x \) be positive quantities of which \( a \) is greater than \( b \), prove that the ratio \( a + x : b + x \) is less than the ratio \( a : b \).

   A is 24 years old, B is 15 years old; what is the least number of years after which the ratio of their ages will be less than \( 7 : 5 \)?

5. If \( A \) vary as \( B \) when \( C \) is constant and vary as \( C \) when \( B \) is constant, prove that \( A \) varies as \( BC \) when both \( B \) and \( C \) vary.

   Supposing that the velocity of a steamer varies inversely as the area of its greatest section when the tonnage is constant, and inversely as the tonnage when the area is constant, and that a steamer whose section is 200 sq. ft. and tonnage 1000, goes 15 miles per hour, find the velocity of a steamer whose section is 250 sq. ft. and tonnage 1200.

6. Find an expression for the sum of \( n \) terms of a geometrical progression and prove that in the continued product

   \((1 + x + x^2 + \ldots + x^n)(1 - x + x^2 - \ldots + x^n)\) the coefficients of odd powers of \( x \) are zero and of even powers unity.

7. Insert \( n \) harmonic means between \( a \) and \( b \).

   If \( \frac{a - x}{px} = \frac{a - y}{qy} = \frac{a - z}{rz} \), and \( p, q, r \) be in A. P., shew that \( x, y, z \) are in H. P.

8. Explain the method of Mathematical Induction. By this method prove that either \( 2^n + 1 \) or \( 2^n - 1 \) is divisible by 3, where \( n \) is a positive integer.

9. Find the number of combinations of \( n \) things taken \( r \) at a time.

   A gentleman invites a party of \( m + n \) friends to dinner and places \( m \) at one round table and \( n \) at another; find the number of ways in which he can arrange them among themselves. (See page 299.)

10. Assuming the Binomial theorem to be true for a positive exponent, prove it for a negative exponent.

    Prove that the coefficient of \( x^n \) in the expansion of

    \((1 - 9x + 20x^2)^{-1}\) is \( 5^{n+1} - 4^{n+1} \).
1. Solve the equations:
   
   (i) \[ 4x^2 + 7y^2 = 148 \]
   \[ 12(x^2 + y^2) = 25xy \] \hspace{1cm} (See page 179.)

   (ii) \[ x + y + 3\sqrt{x+y} = x^2 + y^2 = 10. \] \hspace{1cm} (See page 202.)

2. The termini of a railway 126 miles long are at A and C, and the station B, at which a certain train stops 15 minutes, is 70 miles from A. The whole journey from A to C takes 15 minutes less than twice as long as the journey from A to B. Determine the average rate of the train, including all stoppages except that at B.

3. Prove that if the same quantity be added to the antecedent and consequent terms of a ratio of lesser inequality the ratio is increased.

   A man engages a servant for 18 days in the month of January at Rs. 16 per mensem; in paying him his wages, the man gives him \[ \frac{19}{32} \] (instead of \[ \frac{18}{31} \]) of the month's pay. How much does he pay in excess of the stipulated amount?

4. Prove the Binomial Theorem for positive integral indices. Write down the co-efficient of \[ \frac{1}{y^4} \] in the expansion of \[ \left( y + \frac{c^3}{y^2} \right)^{10}. \]

1888.

1. Solve the equations:
   
   (i) \[ \frac{2x - b}{b} = \frac{2y + a}{b} = \frac{3x + y}{a + 2b} \]

   (ii) \[ \left( \frac{x - a}{x + a} \right)^3 - 5\left( \frac{x - a}{x + a} \right) + 6 = 0. \]

2. Find the sum of \( n \) terms in a Geometrical Progression.

   Find the sum of the following series:
   
   \[ 1 - \frac{2}{3} + \frac{4}{9} - \&c. \ldots \] to 6 terms.
   
   \[ 4 + 8 + 16 + \&c. \ldots \] to infinity.

3. Enunciate and prove the exponential Theorem. Prove that the limit when \( n \) is infinite of \( \left( 1 + \frac{x}{n} \right)^n \) is \( e^x \).
1889.

1. Solve the equations:
   
   (i) \[ \frac{2}{x} + \frac{7}{y} = 28 \] \[ \frac{5}{x} - \frac{6}{y} = 2 \]
   
   (ii) \[ bx + ay = a^2 + b^2 \] \[ x^2 + y^2 = b^2 \] \[ a^2 + b^2 = a^2 + b^2 \]

2. What must be added to \( a^2x^2 + abxy + c^2y^2 \) in order to make it a perfect square?

3. Find the sum of any number of terms of an arithmetical Progression. How many terms of the series 5 + 7 + 9 + &c. must be taken in order that the sum may be 480?

4. Prove the Binomial Theorem for positive integral indices. Show that the co-efficient of \( x^n \) in the expansion of \( (1+x)^{2n} \) is double the co-efficient of \( x^n \) in the expansion of \( (1+x)^{n-1} \).

5. What is logarithm and a mantissa? How is the characteristic of a logarithm determined by inspection?

1890.

1. Find the sum of the roots of the equation \( ax^2 + 2bx + c = 0 \).

   If \( \alpha \) and \( \beta \) be the roots of the equation \( ax^2 + 2bx + c = 0 \), form the equation whose roots are \( \frac{\alpha + \beta}{a\beta} \) and \( \frac{1}{a\beta} \).

2. Solve the equations:
   
   (i) \[ \sqrt{x} + 2 + \sqrt{x} - 3 = 5 \]
   
   (ii) \[ \frac{3 + 2x}{x - \frac{5+2x}{1 + 2x}} = 1 - \frac{4x^2 - 2}{7 + 16x + 4x^2} \]

3. Find the sum of \( n \) terms of a Geometrical series, and show that, when the number of terms is even, the product of two terms equidistant from the beginning and end is equal to the product of the two middle terms.

   Sum the series: \[ 2 + \sqrt{2} + 1 + &c. \] to \( n \) terms.

4. Find the number of permutations of \( n \) things taken \( r \) at a time.

   A company of 80 men are to be selected from a regiment of 900, find the number of ways in which it can be done so that the same ten men may be always included.

   Prove that
   
   \[ a^x = 1 + (\log a)x + \frac{(\log a)^2}{2} \cdot x^2 + &c. \] In what base is \( \log a \) taken?
1891.

117. Trace the changes in the value of $ax^3 + bx + c$ as $x$ varies from a very large positive to a very large negative quantity.

Construct a quadratic equation with rational co-efficients of whose roots one will be $p + \sqrt{q}$.

2. (Solve the equations:—
   (i) $\frac{x - 7}{x - 9} - \frac{x - 9}{x - 11} = \frac{x - 13}{x - 15}$;
   (ii) $3 \sqrt{x - 1} = \frac{5}{3\sqrt{x + 7}} + 6$;
   (iii) $\frac{x - a}{x - b} + \frac{x - b}{x - c} = \frac{a}{b} + \frac{b}{c}$.

3. Show how to insert two harmonical means between $a$ and $b$.

Sum the series:—
   (i) $\frac{1}{\sqrt{3}}$, $1$, $\frac{3}{\sqrt{3}}$, &c., to 18 terms;
   (ii) $(a + x)^3$, $a^3 + x^3$, $(a - x)$, &c., to $n$ terms.

4. Prove the Binomial Theorem for any exponent.

Expand $(4 + 3x)^3$ to four terms, and write down the general term.

1892.

1. Prove that the algebraical expression $x^3 + 2bx + c$ is greater than, equal to, or less than $(x + b)^3$ according as the roots of the equation $x^2 + 2bx + c = 0$ are imaginary, equal or real.

Form the equation whose roots are $\frac{1}{3 + \sqrt{5}}$, $\frac{1}{3 - \sqrt{5}}$.

2. Solve the equations:—
   (i) $2\sqrt{4x + 5} - \sqrt{8x - 4} = \sqrt{2x + 11}$;
   (ii) $\frac{a}{x + a} + \frac{b}{x + b} = \frac{a - c}{x + a - c} + \frac{b + c}{x + b + c}$.

3. Show how to insert $n$ Geometrical means between $a$ and $b$.

If $a$, $b$, $c$ be in Geometric Progression, and $x$, $y$ be the Arithmetic means between $a$, $b$ and $b$, $c$ respectively, prove that

$$\frac{2}{b} = \frac{1}{x} + \frac{1}{y}, \text{ and } 2 = \frac{a}{x} + \frac{c}{y}.$$  

4. Find the number of permutations of $n$ things taken all together, which are not all different.

A person has 15 acquaintances of whom 5 are relatives. In how many ways may he invite 13 guests from among them so that 3 of these may be relatives?

5. Write down the two middle terms of $(a + x)^3a + 1$. 

6. Prove that
\[ \log_2 \frac{n+1}{n} = 2 \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^3} + \text{&c.} \right\} . \]

1893.

1. (a) Prove that a quadratic equation cannot have more than two roots.

(b) What do you understand by the roots being real, imaginary or impossible, and state the conditions of each.

2. Solve the following equations:
   
   (a) \( x^2 + \sqrt{x^2 - 5} = 11 \);
   
   (b) \( \frac{2x(a - x)}{3a - 2x} = \frac{a}{4} \);
   
   (c) \( x^3 - 3x = 2 \).

3. When are quantities said to be—
   
   (a) in Arithmetical Progression;
   
   (b) in Geometrical Progression;
   
   (c) in Harmonical Progression.

4. If there are 6 terms, prove algebraically—
   
   (a) That in the case of Arithmetical Progression the sum of 1st and last is equal to the sum of 3rd and 4th.
   
   (b) That in the case of Geometrical Progression the product of 1st and last is equal to the product of 3rd and 4th.

5. At an Election where every voter may vote for any number of candidates not greater than the number to be voted, there are 6 candidates and 3 members to be chosen; in how many ways may a man vote?

6. Find the relation between the logarithms of the same number to different bases.

ALLAHABAD UNIVERSITY INTERMEDIATE EXAMINATION PAPERS.

1889.

1. Solve the equations:
   
   (a) \( x(z+1)+3\sqrt{2x^2+6x+5} = 2(12-x)+1 \);
   
   (b) \( x^3 + y^3 = 3z, \quad x^3 + y^3 = z \);
   
   (c) \( x(y+z) = 100, \quad y(z+x) = 144, \quad z(x+y) = 154 \).

2. Transform the equation \( x^4 - 4x^3 + 13x^2 - 18x - 52 = 0 \) into one whose roots shall be the same except that each is increased by unity, and by this means solve the original equation.

2—32
3. The Geometrical mean between \( a \) and \( b \) is to their Arithmetical mean as \( m \) is to \( n \); show that
\[
\frac{a}{b} : n + \sqrt{n^2 - m^2} : n - \sqrt{n^2 - m^2}.
\]

4. Out of 7 white and 8 black sailors, 5 are to be selected for a boat's crew, which must always consist of three white and 2 black men; in how many ways may the crew be formed?

5. Expand by the Binomial Theorem to 5 places—
\[
\frac{1}{(1 + \sqrt{x})^6}.
\]

6. State and prove the Exponential Theorem.

7. Find the square root of \( 12 - 6 \sqrt{3} \).

8. Water is admitted into a cistern by three cocks, two of which are exactly equal. When they are all open, five-twelfths of the cistern is filled in four hours; and if one of the equal cocks is stopped, seven-ninths of the cistern is filled in ten hours and forty minutes. In how many hours would each cock fill the cistern?

1890.

1. Show how to determine whether the roots of the equation \( ax^2 + bx + c = 0 \) are real, equal, or impossible.

If \( \alpha, \beta \) are the roots of the equation \( x^2 + px + q = 0 \), then \( \alpha + \frac{1}{\beta} \) and \( \beta + \frac{1}{\alpha} \) are the roots of the equation \( \eta x^2 + p(1+\eta)x + (1+\eta)^2 = 0 \).

2. Solve the equations :

(i) \[
\frac{x + 2}{x - 2} + \frac{2x - 3}{2(x - 1)} = \frac{23}{6};
\]

(ii) \[
\begin{align*}
xy + \frac{x}{y} &= 10, \\
xy^2 - x &= 6y.
\end{align*}
\]

3. If \( a : b : : c : d \), prove that
\[
a^2 + c^2 : b^2 + d^2 : : ac : bd.
\]

4. Shew how to find a Harmonic mean between \( a \) and \( b \). If \( 2(y - a) \) is a Harmonic mean between \( y - x \) and \( y - z \), then \( x - a, y - a, z - a \) form a Geometrical Progression. (See page 258.)

5. Find the total number of permutations of \( n \) things taken all together when there are \( p \) of one kind, \( q \) of another kind, and the rest are unlike.

6. Show that when a Binomial is expanded, terms equally distant from the beginning and the end have the same co-efficients.

Write down the two middle terms of the expansion of \((a - b)^9\).

1891.

1. Prove that \( x^4 + ax^3 + bx^2 + cx + d \) will be a perfect square for all values of \( x \), if \((a^2 - 4b)^2 = 64d\), and \( c^2 = a^2d \).
2. Solve the equations:
   (i) \( 3(1 + x + x^2) = \sqrt{21(1 + x^3 + x^4)} \);
   (ii) \( xy + x + y = 27 \) \[
   \begin{align*}
   1 + \frac{1}{x^2} + \frac{1}{y} &= \frac{1}{3} \\
   \end{align*}
\]

3. A man travels 84 miles, and finds that he could have made the journey in 5 hours less if he had travelled 5 miles an hour faster; at what rate did he travel?

4. Find the sum of a Geometrical Progression in terms of the first term, the last term, and the common ratio.

The sum of 3 quantities in Geometrical Progression is 24\(\frac{2}{3}\), and their product is 61; find them.

(See page 250.)

5. How many different words can be made out of the letters which form the word Allahabad? In how many will the vowels occupy the even places?

(See page 287.)

6. In the expansion of \((1 + x)^n\) where \(n\) is a positive integer, both the sum of the co-efficients of the odd terms and the sum of the co-efficients of the even terms are equal to \(2^{n-1}\).

Find the greatest term in the expansion of \((1 + 5x)^9\), when \(x = \frac{1}{5}\).  

1892.

1. If \(a, \beta\) be the roots of \(ax^2 + bx + c = 0\), form the quadratic whose roots are \(\frac{1}{a + \beta}\) and \(\frac{1}{a} + \frac{1}{\beta}\).

Prove that \((y - 1)(y - 3)(y - 4)(y - 6) + 10\) is positive for all real values of \(y\).

2. Solve, giving all the roots:
   (i) \( \frac{a - x}{a + \sqrt{a} - x} + \frac{a + x}{\sqrt{a} + a + x} = \sqrt{a} \);
   (ii) \( (1 + x)(1 + y) = 10 \)
        \[
        \begin{align*}
        x^3y + xy^3 &= 18 \\
        \end{align*}
\]
   (iii) \( xyz = \frac{x + y}{3} = \frac{y + z}{4} = \frac{z + x}{5} \).

3. Insert four Harmonical means between 1 and 30.

In a Harmonical progression if the \(p^{th}\) term = \(qr\), the \(q^{th}\) term = \(pr\), prove that the \(r^{th}\) term = \(pq\).

4. Write down the \(n^{th}\) terms and sum the following series:
   (i) \( 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \ldots + \&c. \) to \(n\) terms;
   (ii) \( 1^3 + 2^3 + 3^3 + 4^3 + \&c. \) to \(n\) terms; (See page 231.)
   (iii) \( 2.1^2 + 3.2^2 + 4.3^2 + 5.4^2 + \&c. \) to \(n\) terms.
5. Find the number of permutations of \( n \) letters all together, when \( p \) of them are of one sort, \( q \) of another sort, and the rest all different.

6. If the \( r^{th} \) term in the expansion of \((x+1)^{p+q}\) has its co-efficient equal to that of the \((r+4)^{th}\) term, find \( r \).

7. State (without proof) the Exponential Theorem and assuming its truth, prove that

\[
\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \&c. \text{ to infinity.}
\]

1893.

1. Eliminate \( a, b, c \) from the following equations:

\[
bz + cy = a, \quad ex + az = b, \quad ay + bx = c.
\]

2. Solve the following equations:

(i) \[
\frac{ax + b}{cx + a} + \frac{bx + a}{cx + b} = \frac{(a + b)(x + 2)}{ex + a + b};
\]

(ii) \[
\frac{x}{b + y} = \frac{y}{a + x}, \quad ax + by = (x + y)^2.
\]

3. Find the sum of any number of terms of a series in Arithmetical progression of which two particular terms are known.

4. Investigate the order of magnitude of the Arithmetical, Geometrical and Harmonical means between two given numbers.

5. Prove that the number of combinations of \( n \) things taken \( r \) together is equal to the number taken \( n - r \) together.

Prove that if the number of combinations of \( n \) things taken \( r \) together be equal to the number taken \( r \) together, either \( r = s \), or \( r + s = n \).

6. Take any number, the one next to it, and a third equal to the product of the first two. Add together the squares of the three numbers and prove that the result will always be a perfect square, whatever the number you choose to start with.

7. Prove the Exponential Theorem.

1894.

1. (a) Write down the roots of the equation \( ax^3 + bx + c = 0 \) and show when the roots are (1) real and unequal, (2) real and equal, (3) imaginary, and (4) rational and unequal.

(b) Solve the equations:

(i) \[
\sqrt{(x^2 - 8x + 15)} + \sqrt{(x^2 + 2x - 15)} = \sqrt{(4x^2 - 18x + 18)};
\]

(ii) \[
(x^2 + y^2)(x + y) = 272, \quad (x^2 - y^2)(x - y) = 32.
\]

2. (a) Deduce the formula for the sum of \( n \) terms of an Arithmetical progression of which the first term is \( a \) and common difference \( d \).

(b) On the ground are placed a basket and 12 stones in a straight line. The first stone is one yard from the basket, the second stone is
3 yards from the first, the third stone 5 yards from the second, and so on, the distances between the stones increasing in Arithmetical progression. How many yards will a man run, who starting from the basket, picks up the stones one by one, returning each time he picks up a stone to deposit it in the basket?

3. (a) Find for what value of \( r \) the number of combinations of \( n \) things taken \( r \) at a time is the greatest.

(b) A police post, consisting of 5 mounted men and 9 foot policemen, has to furnish a daily guard consisting of 2 from each class. How many days will elapse before the same guard recurs after all possible selections have been made?

4. (a) In the expansion of \((1 + x)^n\) prove that the sum of the co-efficients of the odd terms = the sum of the co-efficients of the even terms = \(2^{n-1}\).

(b) Find the co-efficient of \(x^6\) in the expansion of \((1 + 2x)^6\).

(c) Having given

\[
\log 2 = 0.30103, \\
\log 3 = 0.47712, \\
\text{and } \log 7 = 0.84510,
\]

solve the equations \(2^x7^y = 80,000\); \(3^y = 500\); the values of \(x\) and \(y\) to be given correct to 5 decimal places.
ANSWERS TO UNIVERSITY PAPERS.

Cal. 1888.

1. (i) 31 or 17; (ii) \( x = \frac{14}{29}, \ y = \frac{22}{29} \); 
   (iii) \( x = 2 \) or \(-3 \pm \sqrt{-16}\).

Cal. 1890.

1. (i) \(-24, 37\); (ii) \( x = 3 \) or \(-3 \), \( y = 2 \) or \(-2 \); 
   \[ x = \frac{\sqrt{195}}{7}, \quad x = -\frac{\sqrt{105}}{7} \]
   \[ y = 5\sqrt{\frac{15}{7}}, \quad y = -5\sqrt{\frac{15}{7}} \]; 
   (iii) \((A_1A_2 - C_1C_2)^2 = (A_1B_2 - A_2B_1)(B_1C_1 - B_2C_1)\).

4. \(\sqrt[n]{3}\).

Cal. 1891.

1. (a) 6 \(\frac{1}{2}\) or \(-1\); (b) \( x = \pm \frac{2ab}{\sqrt{a^2 + 4b^2} + a} \), 
   \[ y = \frac{2ab}{\pm \sqrt{a^2 + 4b^2} - a} \].

Cal. 1892.

1. (i) \(\frac{12}{5}\) or \(-\frac{17}{7}\); 
   (ii) \( x = \pm 1 \), \( x = \pm \frac{13}{\sqrt{21}} \), \( y = \pm \frac{2}{\sqrt{21}} \).

4. 10, 8, 6, \&c. 5. \( \frac{\left(\frac{\Delta}{n}\right)^2}{n} \).

Cal. 1893.

1. \( \frac{2b}{a^2}(3c - 4b) \); \( -\frac{2b}{a^2}(4b^2 - ac)^2 - ac(4b^2 - 2ac) - 4a^2b^2 \).

5. \(\frac{2n}{(n)^2} \).
Cal. 1894.

1. (i) \(5\frac{1}{2}\) or \(-3\frac{3}{4}\); (ii) \(\frac{25}{147}\) or 27. 3. 1728; 25.

Cal. 1895.

2. (i) \(\frac{-(a+b+c)\pm \sqrt{a^2+b^2+c^2-2ab-2ac-2bc}}{2}\) or 0;
   (ii) \(\begin{array}{l} x = 0 \\ y = 2 \\ x = 2 \\ y = 0 \\ x = 3 \\ y = 3 \end{array}\).

Cal. 1896.

1. (i) \(-2 \pm 2\sqrt{2}, 1, -5\); (ii) \(-1, 1 \pm \sqrt{30}\).
2. \(x^2 + x - 30 = 0\). 4. (ii) \(2(1 + 6x + x^2)\).

Cal. 1897.

2. (i) \(-5\) or 3; (ii) \(\begin{array}{l} x = 2 \\ y = 1 \end{array}\), \(\begin{array}{l} x = \frac{1}{2} \\ y = \frac{1}{2} \end{array}\).
4. \(a + (n+1)b\).
6. \(\frac{6\sqrt{21.35}\ldots}{4\pi^n}\) to \((r-2)\) terms.

Cal. 1898.

1. (i) \(\frac{a+b+2(\sqrt{ab+2})}{a+b-2(\sqrt{ab+2})}\) or 1; (ii) \(x = \pm 4, y = \pm 9\).
3. \(\frac{(h-2a)\pm \sqrt{(h-2a)^2+8ab}}{2b}\).
4. (i) \(\frac{n}{6}(2n^2+3n+7)\). 7. 1.04139.

Cal. 1899.

1. (i) \(2u \text{ or } b\); (ii) \(\begin{array}{l} x = \pm 1 \text{ or } \pm 3 \\ y = \pm \frac{1}{2} \text{ or } \pm 1 \end{array}\). 2. \(x^2 - x + 2 = 0\).
3. (2) -75; -5; -18. 4. 36.
5. (i) \(\frac{(n+1)(2n+1)(3n+1)\ldots}{n!}\frac{\{(r-1)n+1\}}{x^r}\); (ii) 1.006578.

Cal. 1900.

1. \(x^2 + 2(2q-p^2)x + p^2(4q-p^2) = 0\).
2. (i) \(\pm 5\); (ii) \(\begin{array}{l} x = \pm 3 \text{ or } \pm 2 \\ y = \pm 2 \text{ or } \pm 1 \end{array}\).

Cal. 1901.

1. \(\pm(a + \sqrt{-1})\); -1 or \(\frac{1 \pm \sqrt{-3}}{2}\).
2. (1) \(-\frac{a}{3}, a, \frac{a}{2}\); (2) \(\begin{array}{l} x = 9 \text{ or } 1 \\ y = 1 \text{ or } 9 \end{array}\).
5. \(1 - 4x + 13x^2\).
Cal. 1902.

1. \( a^2c^2x^3 + (2ac-b^2)(a^2+c^2)x + (2ac-b^2)^2 = 0. \)

2. (1) \( 3a \) or \(-4a \); (2) \( x = 0, 3, \) or \( 9 \).
   \[ y = 0, 9, \) or \( 3 \) \( .\)

Cal. 1903.

1. (1) \( b = 0 \); (2) \( c = a \); (a) \( bx^2 - 2ax + a = 0. \)

2. (1) \( 1 \); (2) \( x = 3 \), \( x = 1 \), \( x = -3 \), \( x = -1 \).
   \[ y = 1 \), \( y = 3 \), \( y = -1 \), \( y = -3 \) \( .\)

3. \( \frac{n(n+1)(2n+7)}{6} \).

Cal. 1904.

1. \( b = 0 \); (a) the expression under the radical sign \( = -(ab'-a'b)^2 \),
   which shews that \( x \) is real only when \( ab'-a'b = 0. \)

2. (1) \( \frac{-1 \pm \sqrt{3}}{2}, \frac{a \mp \sqrt{a^2 - 4}}{2} \);
   (2) \( x = 0 \), \( y = 0 \), \( x = \frac{1}{a} \sqrt{a^2 + 1} \), \( y = \pm \sqrt{a^2 + 1} \).

4. \( \frac{m+n+p+q}{m-n+p-q} \).

5. 2 or 7.

Cal. 1905.

1. \( 2(a-x)(x+\sqrt{x^2+b^2}) = a^2 + b^2 - (a-x)^2 - (x^2+b^2) + 2(a-x)\sqrt{x^2+b^2} = a^2 + b^2 - \{(a-x) - \sqrt{x^2+b^2}\}^2 \), which shews &c.

Otherwise, putting \( m \) for \( 2(a-x)(x+\sqrt{x^2+b^2}) \), we get \( 4x^2(m-b^2) - 4ax(m-2b^2) + m^2 - 4a^2b^2 = 0 \); whence, the expression under the radical sign in the value of \( x = m^2(a^2+b^2-m^2) \); hence, &c.

2. (1) \( u, -9a, \) \( (4 \pm \sqrt{15})u \);
   (2) \( x = 1 \), \( y = \frac{1}{2} \), \( x = -1 \), \( y = -2 \).

4. \( \frac{2n}{n^2} \) or \( \frac{2n}{4n^2} \), according as every two arrangements in which all the persons sitting at a round table have the same neighbours, be counted as two or one.

Cal. 1906.

2. (1) (i) \( a, \frac{1}{a} \); (ii) \( 1, -3, \) \( -\frac{7}{5} \);
   (ii) \( x = 3 \), \( y = 6 \), \( x = 6 \), \( y = 3 \).
   (2) -50.
Cal. 1907.

2. (1) \( \frac{2ab}{b-a} \); (2) \( x = \pm \frac{a^2}{\sqrt{a^2 + b^2 + c^2}} \), &c.

3. 5. \( 1 + \frac{1}{4}x + \frac{1.3}{4.8}x^2 + \frac{1.3.5}{4.8.12}x^3 + \ldots \)
\[ + \frac{1.3.5.7\ldots(2r-1)}{4.8.12.16\ldots4r}x^r + \text{&c.} \]

Cal. 1908.

2. \( ax^2 + (2a+b)x + (a+b+c) = 0 \).

3. (1) \( 1, \frac{-1 \pm \sqrt{-3}}{2} \); (2) \( 1, -2, \frac{-1 \pm \sqrt{-19}}{2} \);
(3) \( x = \frac{4}{y} = 2, x = \frac{2}{y} = 4 \).
4. \( p+q\text{-}\mu \).

6. \( 1 - \frac{1}{3}x - \frac{1.2}{3^2}x^2 - \frac{1.2.5}{3^3}x^3 - \ldots - \frac{1.2.5.8.11\ldots(3r-4)}{3^r}x^r + \text{&c.} \).

Cal. 1909.

2. (i) \( x = a \), \( y = b \); (ii) \( x = 1 \), \( y = 2 \).

3. \( \frac{120}{4} \).

4. (i) \( \frac{3.7.11.15.19.23.27}{4.8.12.16.20.24.28} \).
(ii) \( \frac{3}{4} - \frac{5}{4.3} \).

5. (i) \(-238\); (ii) 1.

Mad. 1881.

1. \(-\frac{1}{3}, -3, 1\).

6. \( 2^r \).

Mad. 1882.

1. \( 0, 1, \left(\frac{x^2 + 3q^2}{p^2 - q^2}\right)^{12} \).

2. \( a^2 + b^2 + c^2 - 2ab - 2bc - 2ca \) not less than zero.

\[ a \frac{192}{48^4} \cdot \frac{16}{4^4} \]

Mad. 1883.

1. \( x = 0, -\frac{32a}{17}, -a \).
\( y = 0, -\frac{8a}{17}, a \).
4. \( \frac{ac(n-1)}{a(m-1) - c(m-2)} \).

5. (b) \( \left( \frac{3n}{2n} \right)^{n^2} \).

Mad. 1885.

1. (b) \( \frac{x(x - b)(b - c)(c - a) - 3abc}{(a - b)(b - c)(c - a)(x - a)(x - b)(x - c)} \).

2. (a) \( 0, -5, \frac{-15 \pm \sqrt{265}}{6} \); (b) \( x = 7, -7 \) \( y = 3, -3 \).

4. 8 days.

6. \(-220x^2y^2\).

Mad. 1886.

2. (b) (i) \( 0, 1, -2 \); (ii) \( x = \pm 3 \), \( y = \pm 2 \), \( x = \pm \frac{2\sqrt{6}}{\sqrt{11}}, y = \pm \frac{3\sqrt{6}}{\sqrt{11}} \).

4. (2) 1.

6. 1.00039857.

Mad. 1887.

1. \( 2(1 + x)(1 + y)(1 - xy)(x - y) \).

2. (1) \( \pm 2 \) or \( \pm \sqrt{2} \).

5. 560.

6. \( 2^{r-3} \left\{ 9r^2 + 15r + 8 \right\} \).

Mad. 1888.

1. (a) \( (a + b)(b + c)(c + a)(a + b + c) \).

Mad. 1889.

2. (a) \( x = \frac{\pm 1}{y = \frac{2}{3}} \), \( x = \pm \frac{3}{2}, y = \frac{3}{2} \); (b) \( x = 220, y = 165 \).

3. \( xy^{\frac{1}{2}} = \frac{1}{4}x^2y^2 + y^3 \).

6. \( \frac{3}{2} \left\{ 1, 3, 5, \ldots, (2r - 5) \right\} a^{\frac{3}{2} - r} x^r \).

Mad. 1890.

2. (i) \( 1, \sqrt{2} + \sqrt{3} + 2 \); (ii) \( x = 3 \), \( y = -3 \), \( x = -1 \), \( y = 1 \).

6. \( \frac{1, 3, 5, \ldots \text{ to } \frac{r}{2} \text{ terms}}{2^{\frac{r}{2}} \frac{r}{2}} \), when \( r \) is even.

1. \( 3, 5, \ldots \text{ to } \frac{r-1}{2} \text{ terms} \), when \( r \) is odd.
Mad. 1891.

1. \( x^3 - 4x + 1. \)

3. \( (1) \pm \sqrt{a+b}; \)

(2) \[
\begin{align*}
x &= -1 \\
y &= -1
\end{align*}
\]

(3) \[
\begin{align*}
x &= \frac{7}{5} \\
y &= -\frac{1}{5}
\end{align*}
\]

(4) \[
\begin{align*}
x &= -\frac{1}{5} \\
y &= \frac{7}{5}
\end{align*}
\]

4. 7560.

Bom. 1882.

2. \( \sqrt{\frac{2a+b}{2}} + \sqrt{\frac{b}{2}}. \)

Bom. 1883.

3. \( 7, \frac{8}{3}, \frac{13 + \sqrt{73}}{6}. \)

5. (i) 710; (iii) 1\%.

6. 60.

7. \[
\frac{1}{2^r \cdot r^{2r-1} \cdot 3^{2r}} \cdot \frac{2^{2r}}{x^2}
\]

Bom. 1884.

4. \( a^2 b^4. \)

6. 20.

Bom. 1885.

6. 3600.

7. 3rd term.

Bom. 1886.

1. \( p^2 - 2q; p^3 - 3pq; (p^2 - 2q)^2 - 2q^2. \)

2. 1.

3. \( x = 6, y = \frac{3}{2}. \)

4. \( b^2 - 4ac = 0; 0 \text{ or } 6. \)

6. (i) 41.

8. 90.

9. \[
\frac{n}{n-r}; \quad (-1)^r \frac{n(n+1)(n+2) \ldots (n+r-1)}{r}; \quad \frac{2^n}{n^n}
\]

10. 202345.

Bom. 1887.

1. \( x = 1. \)

2. 72765.

5. \[
\begin{align*}
x &= 11 \\
y &= 11
\end{align*}
\]

(3) \[
\begin{align*}
x &= -\frac{7}{2} \pm 2 \sqrt{11} \\
y &= -\frac{7}{2} \pm 2 \sqrt{11}
\end{align*}
\]

8. 719.

9. \( 2^n. \)

10. 231.
1. (i) $9 \sqrt{3}$.  
2. $b; ac$.  
3. $\frac{m}{n}$.  
4. $\frac{5}{3}$ or $\frac{3}{5}$.  
5. $a\left(\frac{n}{m}\right)^{\frac{r-1}{2a}}$.  
6. $a\left(\frac{n}{m}\right)^{\frac{r-1}{2a}}$.  
7. $06$.  
8. $2(\sqrt{2} - 1)$.  
9. \[ (-1)^{r} \left(\frac{r+1}{r}\right)^{x} \]
10. $240p^{2}m^{2} = qu^{2}$.

Bom. 1892.

3. $2, -3, \frac{-3 \pm \sqrt{-143}}{6}$.
4. $(x + \frac{b}{2a})^{2}$. $a$ must lie between $\frac{2}{3}$ and $\frac{2}{3}$.
5. $26; 136$.
6. $(-1)^{r} \frac{n(n+1)(n+2) \ldots (n+r-1)x^{r}}{r}$.

Bom. 1893.

2. 0 or 9.
4. 8.
5. 10 miles per hour.

Pun. 1887.

2. $\frac{61}{10}$ miles per hour. 3. $\frac{13}{62}$ Rs. 4. $210c^{12}$.

Pun. 1888.

1. (i) $x = \frac{3b^{2} + ab}{4a}$, $y = \frac{3b^{2} - ab - 2a^{2}}{2a}$; 
(ii) $-3a, -2a$. 2. $\frac{133}{243}$; 5.

Pun. 1889.

1. (i) $x = \frac{47}{182}$, $y = \frac{47}{136}$; (ii) $x = \frac{a^{3}}{b}$; $y = \frac{b^{3}}{a}$; 2. $-abxy \pm 2axxy$.

or, $x = b$, $y = a$. 3. 24.
Pun. 1890.

1. \( e^2x^2 + (2b-a)cx - 2ab = 0. \)

2. (i) \( 7; \) (ii) \( \frac{7}{8}. \)

3. \( 2^3 + 2^\frac{3}{2} - 2^\frac{3}{2} - 2^\frac{3}{2}. \)

4. \( \frac{1890}{70.820}. \)

Pun. 1891.

1. \( x^2 - 2px + p^2 - q^2 = 0. \)

2. (i) 13; (ii) 6; (iii) \( a+b \) or 0.

3. (i) \( 9841(\sqrt{3}+1); \) (ii) \( n(a^2 + x^2 + 3ax - nx). \)

4. \( (-1)^r \cdot 5.3.1.3.5... (2r-7)_{25} - 3r(3x)^r. \)

Pun. 1892.

1. \( 4x^2 - 6x + 1 = 0. \)

2. (i) \( 2^\frac{1}{7} \) or \( 1^\frac{9}{14}; \) (ii) 0 or \( \frac{a+b}{2}. \)

4. 10.

5. \( \frac{2n+1}{n} \cdot n^{n+1}x^n; \) \( \frac{2n+1}{n} \cdot n^{n+1}x^n. \)

Pun. 1893.

2. (a) \( \frac{8}{11}; \) (b) \( \frac{3a}{2} \) or \( \frac{a}{2}; \) (c) \( -1 \) or 2.

5. 41.

All. 1889.

1. (a) \( \frac{-3 \pm \sqrt{241}}{2}, -5, 2; \)

(b) \( \begin{cases} x = 0 \\ y = 0 \end{cases}; \) \( \begin{cases} x = 4 \\ y = 8 \end{cases}; \) (c) \( x = \pm 5, y = \pm 9, z = \pm 11. \)

2. \( x^4 - 8x^3 + 13x^2 - 60x - 16 = 0; \) \( 1 \pm \sqrt{3} \) or \( 1 \pm 2\sqrt{-2}. \)

4. 980.

5. \( 1 - 6x^3 + 21x^3 + 56x + 126x^3. \)

7. \( \pm \sqrt{3}(\sqrt{3} - 1). \)

8. 32 and 24.

All. 1890.

2. (i) \( \frac{27 \pm \sqrt{29}}{11}; \) (ii) \( \begin{cases} x = 0 \\ y = 0 \end{cases}; \) \( \begin{cases} x = 4 \\ y = 4 \end{cases}; \) \( \begin{cases} x = -4 \\ y = -4 \end{cases}. \)

6. \( -126a^4b^4. \)
Al. 1891.

2. (i) \( \frac{-1 \pm \sqrt{-3}}{2} \), \( \frac{7 \pm 3 \sqrt{-7}}{14} \);
   (ii) \( x = 6 \), \( x = 3 \), \( y = 3 \), \( y = 6 \).

3. 7 miles per hour

6. \( \frac{267}{512} \).

Al. 1892.

1. \( bex^2 + (b^2 + ac)x + ab = 0 \).

2. (ii) \( x = 3 + \sqrt{6} \), \( y = 8 - \sqrt{6} \); \( x = 3 - \sqrt{6} \), \( y = 3 + \sqrt{6} \);
   (iii) \( x = 0 \), \( x = 2 \sqrt{6} \), \( x = -2 \sqrt{6} \); \( y = 0 \), \( y = \sqrt{6} \), \( y = -\sqrt{6} \); \( z = 0 \), \( z = 3 \sqrt{6} \), \( z = -3 \sqrt{6} \).

3. \( \frac{129}{121}, \frac{129}{46}, \frac{28}{21}, \frac{47}{17} \).

4. (i) \( \frac{3,2n - 2n - 3}{2^n - 1} \); (iii) \( \frac{n(n + 1)(n + 2)(3n + 1)}{12} \).

Al. 1893.

1. \( x^2 + y^2 + z^2 + 2xyz = 1 \).

2. (i) \( 0, \frac{c(a^2 + b^2) - (a^2 + b^2)}{c(a^2 + b^2) - c^2(a + b)} \);
   (ii) \( x = 0 \), \( x = \frac{b \sqrt{a}}{\sqrt{a} + \sqrt{b}} \); \( y = 0 \), \( y = \frac{a \sqrt{b}}{\sqrt{a} + \sqrt{b}} \).

Al. 1894.

1. (i) 0, 3 or \( 5\frac{1}{2} \); (ii) \( x = 5, y = 3 \).

2. 1800 yds.

3. 360.

4. (b) \( -\frac{5}{16} \); (c) \( x = 40706, y = 5.65879 \).
APPENDIX.

CHAPTER I.

GRAPHS.

1. **Instruments required.** The student should first of all provide himself with the following instruments and acquire skill in manipulating them with accuracy and neatness.

   (1) **A Hard Pencil.**

   **Note.** It must be well sharpened so that the lines drawn may be very fine.

   (2) **A Pair of Compasses** (also called Dividers).

   (3) **Two Set-squares.**
(4) A graduated Flat Ruler (of moderate length) shewing tenths of an inch.

(5) A Diagonal Scale, giving hundredths of an inch.

Example 1. Through the point A draw a straight line parallel to BC.
Place the Set-square DEF in such a way that the edge DE may fall along BC. Then slip the other Set-square GHK into the position shewn in the diagram, so that HG may pass by A. Now trace a line along HG, which will evidently be parallel to BC.

**Example 2.** Through the point A in the straight line BC draw a straight line perpendicular to BC.

First trace a line DE parallel to BC. Then place the Set square GHK in such a way that HK may fall along DE and GH may pass by A. Now trace a line along HG, which will evidently be perpendicular to BC.

**Example 3.** Find the lengths of the straight lines AB and CD:

(1) By means of the Pair of Compasses and the Diagonal Scale we find that the length of AB is equal to the distance between the two points marked on the line 4—4 in the diagram. Hence the required length = 2·24 inches.

(2) The length of CD is found to be equal to the distance between the two points marked on the line 9—9 in the diagram. Hence the required length = 1·69 inches.

2—33
Exercise (1).

1. Produce the straight line AB to double its length:

   A    ————    B

2. On a given straight line AB a point D is taken supposing it to be the middle point. By means of a Pair of Compasses however it is found that AD is a trifle shorter than BD. How is the mistake to be corrected?

3. ABC is a triangle and D a point on AC, as in the following diagram. Through D draw, towards AB, a straight line parallel to CB.

   A ———— D    ———— B
       \  |
        |
        C

4. In the same diagram, through D draw, away from AB, a straight line parallel to BC.

5. In the diagram of example 3, through B draw a straight line parallel to AC.

6. From the vertices of a given triangle draw perpendiculars to its opposite sides.

7. In example 3, measure the lengths of the sides of the triangle, and also measure the lengths of AD and DC.

2. Squared paper. A specimen of a sheet of squared paper is given on the next page.
We have two sets of parallel straight lines on the paper. One set being parallel to the length, and the other, parallel to the breadth, of the paper, it is clear that every line of the first set is perpendicular to every line of the second. The distance between every two consecutive parallels is one-tenth of an inch, whilst every two consecutive thick parallels are half an inch apart. The whole paper is thus divided into a large number of small squares which are equal to one another, each side of each square being one-tenth of an inch in length. The paper is also divided into a number of thick-bordered squares, each side of each such square being half an inch in length. It is clear also that twentyfive of the small squares are contained in each of the thick-bordered squares.

**Note 1.** Lines parallel to $AB$ may be regarded as east-and-west lines, and those parallel to $AD$, as north-and-south lines. They may also be considered as horizontal and vertical lines respectively.

**Note 2.** For the sake of convenience the length of a side of a small square may be denoted by the symbol $a$.

**Note 3.** The paper may also be ruled so that the length of a side of a small square is only one-tenth of a centimetre (i.e., a millimetre) instead of one-tenth of an inch. In that case the distance between every two consecutive thick parallels is evidently half a centimetre or 5 millimetres. (One centimetre is approximately equal to .39 of an inch).
Example 1. P, Q, R, S are four stations such that Q is 7 miles east of P, R is 11 miles south of P and S is 13 miles north of Q. Find the distance between R and S.

Taking the length of a side of a small square (i.e., \( \alpha \)) to represent one mile, we have P, Q, R, S as in the above figure, where \( PQ = 7\alpha \), \( PR = 11\alpha \) and \( QS = 13\alpha \).

With R as centre and RS as radius describe an arc of a circle cutting the east-and-west line through R at T.

Now as \( RT = 25\alpha \), we have RS also = \( 25\alpha \). Hence the required distance = \( 25 \) miles.

Example 2. An upright post is 8 feet high. A string of length \( 8\frac{2}{3} \) feet has one end attached to the top of the post and is held tight with the other end in contact with the ground. How far is this end from the foot of the post?

Let \( 3\alpha \) (i.e., 3 times the length of a side of a small square) represent one foot. Then 8 feet will be represented by \( 24\alpha \) and \( 8\frac{2}{3} \) feet by \( 26\alpha \).
Let $AB$ represent the post, so that $AB = 24a$. Take a point $C$ on the horizontal line through $B$ such that $BC = 26a$.

With $B$ as centre and $BC$ as radius describe an arc of a circle cutting the horizontal line through $A$ at $D$. Join $BD$; then $BD$ represents the string.

Now, $AD$ is equal to $10a$, which is $9a + a$. Hence the required distance $= 3\frac{1}{3}$ feet.

**Exercise (2).**

1. $A$ is $5\frac{1}{3}$ units of length east of $O$, and $P$ is 4 units of length north of $A$. How far is $P$ from $O$?

2. B is 3 feet west of $O$, and $Q$ is $7\frac{1}{2}$ feet south of $B$. How far is $Q$ from $O$?

3. $C$ is 2 yards north of $O$, and $R$ is $6\frac{2}{7}$ yards west of $C$. How far is $R$ from $O$?

4. $D$ is 2.1 inches south of $O$, and $S$ is 2.8 inches east of $D$. How far is $S$ from $O$?
5. A is 2.7 feet east of O. P is north of A and 4.5 feet from O. How far is P from A?

6. Q is 2.4 feet south of B. O is east of B and 2.5 feet from Q. How far is B from O?

7. B is 4.5 yards east of A. C is 3/5 yard north of A, and D is 2 yards north of B. How far is D from C?

8. B is 25 feet north of A. P is 40 feet west of A, and Q is 20 feet east of B. How far is Q from P?

9. Two vertical posts, 14 feet and 3.5 feet high, are 13.5 feet apart. Find the distance between the tops of the posts.

10. A ladder 30 feet long has its foot at a distance of 10 feet from a vertical wall. How far up the wall does it reach? (The Diagonal Scale may be used if necessary).

3. If in a plane, a point and two straight lines passing through it at right angles to each other be given, the position of any point in the plane can be easily defined.
In the plane of the paper as shown in the last diagram, let XOX' and YOY' be the two given straight lines at right angles to each other. If P be any point in the plane, how to know its position?

We may regard XOX' as the east-and-west line, and YOY' as the north-and-south line. Draw PM parallel to YOY' meeting XOX' at M. Evidently then M is due east of O, and P, due north of M. Hence if OM and MP be known, we know the position of P at once.

Taking the length of a side of a small square as the unit of length, we have OM = 9 units of length and MP = 12 units of length. Hence the position of P may be briefly defined as follows:

9 units east, 12 units north.

Note 1. If Q be a point whose position is defined to be 5 units east, 8 units north, to fi 1 Q all that we have to do is to take a point 5 units due east of O and thence proceed 8 units northwards.

Note 2. If R be a point whose position is defined to be 7 units west, 4 units south, to find R all that we have to do is to take a point 7 units due west of O and thence proceed 4 units southwards.

**Exercise (3).**

[Squared paper is to be used in every case.]

1. Find the points whose positions are defined as follows:

   (1) 5 units east, 7 units north.
   (2) 8 units west, 5 units north.
   (3) 10 units west, 12 units south.
   (4) 15 units east, 6 units south.
   (5) 8 units west, 13 units north.
   (6) 14 units east, 15 units south.

2. It is clear that "6 units west" is the same as "−6 units east", and "8 units south" is the same as "−8 units north". Hence find the points whose positions are defined as follows:

   (1) 7 units east, −8 units north.
   (2) −10 units east, 6 units north.
   (3) −9 units east, −13 units north.
3. In defining the position of a point the words "east" and "north" may be omitted if it is accepted as a rule that the distance measured towards the east should invariably be mentioned first. On this convention, find the points whose positions are defined as follows:

(1) 8 units, 9 units. (2) 6 units, −11 units.
(2) −12 units, 15 units. (4) −10 units, −14 units.

4. We may define the position of a point still more briefly if the word "units" be omitted. Find, then, the points whose positions are defined as follows:

(1) 6, 4. (2) 13, 8. (3) −7, 6.
(4) 8, −6. (5) −10, −13. (9) −9, −15.

4. Definitions. The student is referred to the diagram of the last article. The given lines XOX' and YOY' with reference to which the positions of all points in the plane are defined, are called the axes of co-ordinates; and the point O, where these lines intersect, is called the origin.

The straight line XOX' is called the axis of x and the straight line YOY', the axis of y.

The lengths OM and MP which define the position of the point P are called its co-ordinates, OM being called the abscissa (or x co-ordinate) and MP, the ordinate (or y co-ordinate).

"The point (x, y)" or simply "(x, y)" means "the point whose abscissa = x units of length, and ordinate = y units of length."

Note 1. When we speak of the "x and y" of a point, we mean its "abscissa and ordinate".

Note 2. The abscissa is positive or negative according as M is on the right or on the left of O. The ordinate is positive or negative according as P is above or below XOX'.

Note 3. "To plot a point" is to find the position of a point when its co-ordinates are given.

Example 1. In the diagram given on the next page, write down the co-ordinates of the points P₁, P₂, P₃, P₄.

The figure explains itself. Take the length of a side of a small square as the unit of length.
(1) \(OM_1 = 8\) units and \(M_1\) is on the right of \(O\); \(M_1P_1 = 10\) units and \(P_1\) is above the line \(XOX'\). Hence the co-ordinates of \(P_1\) are 8 and 10.

(2) \(OM_2 = -5\) units and \(M_2\) is on the left of \(O\); \(M_2P_2 = 13\) units and \(P_2\) is above the line \(XOX'\). Hence the co-ordinates of \(P_2\) are -5 and 13.

(3) \(OM_3 = 10\) units and \(M_3\) is on the left of \(O\); \(M_3P_3 = 11\) units and \(P_3\) is below the line \(XOX'\). Hence the co-ordinates of \(P_3\) are -10 and -11.

(4) \(OM_4 = 15\) units and \(M_4\) is on the right of \(O\); \(M_4P_4 = 10\) units and \(P_4\) is below the line \(XOX'\). Hence the co-ordinates of \(P_4\) are 15 and -10.

**Example 2.** Plot the points \((-1, 0), (0, 1), (1, 2)\) and \((2, 3)\); and shew that they all lie in a straight line.
Let 5 times the side of a small square represent the unit of length, and let $P_1$, $P_2$, $P_3$, $P_4$, respectively denote the four given points. Then the positions of the points will be as shown in the figure.

Now we find that a flat ruler may be so placed that its edge will pass through all the four points. Hence they all lie in the same straight line.

**Exercise (4).**

1. In the diagram given on the next page, what are the co-ordinates of the points $P_1$, $P_2$, $P_3$, $P_4$, (i) when the unit of length is represented by a side of a small square, (ii) when the unit of length is represented by 5 times the side of a small square?
2. What will be the co-ordinates of the same points if the unit of length be represented by three times the side of a small square?

3. Plot the points \((-4, -4), (7, 7), (13, 13)\) and satisfy yourself that they lie in a straight line passing through the origin.

4. Plot the points \((-8, 4)\) and \((10, -5)\), and satisfy yourself that the straight line joining them passes through the origin.

5. Plot the points \((8, 5)\) and \((-4, -11)\), and find the distance between them.

6. Plot the points \((-7, 9)\) and \((-12, 21)\), and find the distance between them.
7. Plot the points \((-11, 13)\) and \((3, -35)\), and find the distance between them.

8. Join the points \((0, 0)\) and \((5, 5)\), and produce the straight line both ways. Find the ordinate of the point on this straight line whose abscissa is 11, and the abscissa of the point whose ordinate is \(-13\).

9. Join the points \((0, 7)\) and \((12, 0)\), and produce the straight line both ways. Find the ordinate of the point on the straight line whose abscissa is \(-18\), and the abscissa of the point whose ordinate is \(-14\).

10. Join the points \((-4, 0)\) and \((0, -8)\), and produce the straight line both ways. Find the ordinate of the point on the straight line whose abscissa is \(-10\), and the abscissa of the point whose ordinate is \(-24\).

5. Graphs of Simple Equations. The following examples will make the subject clear.

Example 1. If a point moves in such a manner that its abscissa is always equal to 5 units of length, find the path along which the point will move.
Let twice the side of a small square represent the unit of length. (The figure is on page 14).

On $OX$ take the point $M$ such that $OM = 5$ units of length; through $M$ draw the straight line $PMP'$ parallel to $YOY'$.

Now, if any point be taken on the straight line $PMP'$ its $x$ will evidently be equal to 5 units of length; but this will not be so if the point be taken on either side of the line $PMP'$.

Hence the moving point will always be on the line $PMP'$.

We see therefore that if a point moves in such a manner that its $x$ is always equal to 5 units of length, the path along which the point will move is the straight line $PMP'$. This fact is briefly expressed by saying that the straight line $PMP'$ is the Graph of the equation $x = 5$.

Note 1. From the above it is clear that the graph of the equation $y = 5$ is a straight line parallel to $XOX'$.

Note 2. Generally speaking, the graph of the equation $x = a$ is a straight line parallel to the axis of $y$ and passing through a point on the axis of $x$ which is at a distance of $a$ units of length from the origin, and the graph of the equation $y = b$ is a straight line parallel to the axis of $x$ and passing through a point on the axis of $y$ which is at a distance of $b$ units of length from the origin.

Note 3. Evidently therefore the graph of the equation $x = 0$ is the axis of $y$ itself, and the graph of the equation $y = 0$ is the axis of $x$ itself.

Example 2. If a point moves in such a manner that its $x$ and $y$ are always connected by the relation $y = 3x$, find the path along which the point will move.

Since $y = 3x$, when $x = 0$ and when $x = 3$ we have $y = 0$ and we have $y = 9$.

Evidently therefore $(0, 0)$ and $(3, 9)$ are two positions of the moving point.

Take the length of a side of a small square as the unit of length. (The figure is on the next page).

Join the points $(0, 0)$ and $(3, 9)$, and produce the straight line both ways. Then this straight line will be the required path.

Take any point $P$ on this straight line. The co-ordinates of $P$ are found to be 5 and 15, which evidently satisfy the given relation. Similarly the co-ordinates of any other point on this straight line may be shewn to satisfy the given relation. But the co-ordinates of a point which is outside the line $OP$ will not satisfy the given relation, as can be easily verified.