Hence the moving point will always be on the line OP and never stray out of it.

Thus it is found that if a point moves in such a way that its \(x\) and \(y\) are invariably connected by the relation \(y = 3x\), the path along which the point will move is the straight line OP. In other words the line OP is the Graph of the equation \(y = 3x\).

**Note** Generally speaking, the graph of the equation \(y = mx\), where \(m\) is any given number, is a straight line passing through the origin.

**Example 3** If a point moves in such a way that its \(x\) and \(y\) are invariably connected by the relation \(y = -4x + 5\), find the path along which the point will move.

From the given relation,

- when \(x = 0\) we have \(y = 5\),
- and when \(x = 3\) we have \(y = -7\).

Evidently therefore \((0, 5)\) and \((3, -7)\) are two positions of the moving point.
Let twice the side of a small square represent the unit of length. Join the points \((0, 5)\) and \((3, -7)\), and produce the straight line both ways. Then this straight line will be the required path.

Take a point \(P\) on this straight line. The co-ordinates of \(P\), which are found to be \(-1\) and \(9\), satisfy the given relation.

Take another point \(Q\) on the straight line; its co-ordinates also, which are found to be \(2\) and \(-3\), satisfy the given relation. Similarly the co-ordinates of any other point on this straight line may be shewn to satisfy the given relation. But if a point
be taken outside the line PQ, its co-ordinates will *not* satisfy the given relation, as can be easily seen. Hence the moving point will always be on the line PQ and never stray out of it.

Thus it is found that if a point moves in such a manner that its co-ordinates always satisfy the equation \( y = -4x + 5 \), the path along which the point will move is the line PQ. In other words, the line PQ is the Graph of the equation \( y = -4x + 5 \).

**Note 1.** Generally speaking, the graph of the equation \( y = mx + c \) where \( m \) and \( c \) are any given numbers, is a straight line passing through the point \((a, c)\).

**Note 2.** As every equation of the first degree in \( x \) and \( y \) can be reduced to the form \( y = mx + c \), it is clear that graphs of all simple equations are straight lines.

**Note 3.** The graph of the equation \( y = mx + c \) is also said to be the graph of the expression \( mx + c \).

**Note 4.** The graph of any given equation may be defined to be the path described by a point which moves in such a manner that in every position of the point its co-ordinates satisfy the given equation.

**Example 4.** Draw the graph of the equation \( 7x + 3y = 11 \).

When \( x = 0 \)
\[
\begin{align*}
    y &= \frac{11}{3} \\
\end{align*}
\]
when \( x = 1 \)
\[
\begin{align*}
    y &= \frac{11}{3} \\
\end{align*}
\]

Evidently therefore \((0, \frac{11}{3})\) and \((1, \frac{11}{3})\) are two points on the graph.

Let 3 times the side of a small square represent the unit of length. Join the points \((0, \frac{11}{3})\) and \((1, \frac{11}{3})\), and produce the straight line both ways. Then this straight line will be the required graph. (See diagram on page 529).

Take any point \( P \) on the line; its co-ordinates, which are found to be 3 and \(-3\frac{1}{3}\), satisfy the given relation. Take any other point \( Q \) on the line; its co-ordinates also, which are found to be \(-1\) and 6, satisfy the given relation. Similarly it may be shown that the co-ordinates of any point that may be taken on the line PQ will satisfy the given relation, but the co-ordinates of any point which is outside PQ will *not*. Hence the line PQ is the required graph.
Note 1. The graph of the equation $7x + 3y = 11$ is also said to be the graph of the expression $\frac{11 - 7x}{3}$.

Note 2. The straight line PQ being the graph of the equation $7x + 3y = 11$, this equation is said to be the equation of the straight line PQ.

Note 3. Hence the equation of a given straight line means the equation which is satisfied by the co-ordinates of every point on that straight line.

Example 5. Find the equation of the straight line which passes through the points $(1, 1)$ and $(3, -\frac{1}{2})$.

Let $y = mx + c$ be the required equation.

$2-34$
This equation being satisfied by \((1, 1)\) and also by \((3, \, -\frac{1}{2})\), we must have
\[
1 = m + c \quad \text{Hence, } 2m = -\frac{3}{2}, \text{ and } \therefore m = -\frac{3}{4};
\]
and \(-\frac{1}{2} = 3m + c\) \quad \text{whence } c = 1 + \frac{3}{4} = \frac{7}{4}.
Thus the required equation is \(y = -\frac{3}{4}x + \frac{7}{4}\),
or \(3x + 4y = 7\).

**Exercise (5).**

1. Draw the graphs of the following equations:
   (1) \(x = 8\). (2) \(x = 13\). (3) \(x + 11 = 0\).
   (4) \(y = -7\). (5) \(y - 9 = 0\). (6) \(y + 10 = 0\).

2. Draw the graphs of the following equations:
   (1) \(y = x\). (2) \(y = -x\). (3) \(y = 2x\).
   (4) \(y + 2x = 0\). (5) \(y = -3x\). (6) \(3y = 5x\).
   (7) \(7y + 8x = 0\). (8) \(6y + 13x = 0\).

3. Draw the graphs of the following equations:
   (1) \(y = 3x + 4\). (2) \(y = 7x - 8\). (3) \(y = -5x + 9\).
   (4) \(y = -8x - 11\). (5) \(3y = 7x + 4\). (6) \(-6y = 7x - 10\).

4. Draw the graphs of the following equations:
   (1) \(2x + 7y = 10\). (2) \(4x - 5y - 7 = 0\).
   (3) \(5x + 6y + 8 = 0\). (4) \(-3x + 7y + 8 = 0\).
   (5) \(10y - 9x = 13\). (6) \(8x - 11y + 13 = 0\).

5. Draw the graphs of the following equations:
   (1) \(\frac{x}{3} + \frac{y}{4} = 1\). (2) \(\frac{x}{7} + \frac{y}{-9} = 1\). (3) \(-\frac{x}{8} + \frac{y}{13} = 1\).
   (4) \(y = \frac{5 - 7x}{6}\). (5) \(y = \frac{9x - 13}{4}\). (6) \(\frac{3x}{4} - \frac{4y}{3} = 1\).

6. Draw the graphs of the following equations:
   (1) \(x - 3\). (2) \(3x + 4\). (3) \(-7x + 8\). (4) \(\frac{7 - 4x}{3}\).
   (5) \(\frac{5x - 9}{4}\). (6) \(\frac{8x + 11}{5}\).
7. Find the equation of the straight line which passes through each of the following pairs of points:

(1) (0, 0), (5, 6).    (2) (0, 5), (7, 0).
(3) (6, −8), (−7, 5).    (4) (−4, 8), (−9, −13).
(5) (−11, 0), (7, −10).


Example. Solve graphically:

\[
\begin{align*}
2x - 7y + 12 & = 0 \quad \text{1st. equation;} \\
3x + 2y & = 32 \quad \text{2nd. equation.}
\end{align*}
\]

Let us draw the graphs of the two equations.

We find that

\[
\begin{align*}
x = -6 & \quad \text{are points on the graph of the} \\
y = 0, & y = 2
\end{align*}
\]

whilst

\[
\begin{align*}
x = 0 & \quad \text{are points on the graph of the} \\
y = 16, & y = 7
\end{align*}
\]

Hence, taking the length of a side of a small square as the unit of length, the two graphs are as shewn on the next page.

Let \( P \) be the point where the two graphs intersect. \( P \) being common to the graphs, its co-ordinates will satisfy both the given equations.

Now the co-ordinates of \( P \) are found to be 8 and 4.

Hence, \( x = 8 \) \( y = 4 \) is the required solution.

Note 1. By actual verification we find that both the equations are satisfied when \( x = 8 \) and \( y = 1 \).

Note 2. If it is required to "solve graphically the equation

\[
\frac{x - 3}{5} = \frac{3x - 22}{2},
\]

all that we have to do is to draw the graphs of the expressions \( \frac{x - 3}{5} \) and \( \frac{3x - 22}{2} \) and take the abscissa of the point common to the two graphs.
Exercise (6).

Solve the following equations graphically:

1. \( x + y = 9 \), \( 3x - 2y = 7 \).
2. \( 4x + 3y = 13 \), \( 3x + 2y = 11 \).
3. \( \frac{x}{4} + \frac{y}{3} = 4 \), \( 4x - 5y = 2 \).
4. \( y - x = 2 \), \( 3x - 2y = 5 \).
5. \( 5x - 3y = 11 \), \( 2y - 3x + 4 = 0 \).
6. \( \frac{x - 2}{-5} = \frac{-5x + 4}{4} \).
7. \( \frac{2x + 7}{3} = \frac{3x - 7}{2} \).
8. \( \frac{4x - 3}{5} = \frac{6x}{7} - 1 \).

Example 1. Given that the price of a seer of rice is three annas, show that a straight line can be drawn such that if any point be taken on it, the abscissa of the point will represent the quantity of rice of which the price is represented by the ordinate.

In the above figure let the length of a side of a small square measured along OX represent one seer, and let an equal length measured along OY represent one anna. Then the meaning of the figures along OX and OY is clear.
Since the price of a seer is 3 annas, the price of 8 seers must be 24 annas. Clearly therefore \( P \) is a point such that its abscissa OM represents a quantity of rice of which the price is represented by the ordinate \( PM \).

Join \( OP \) and produce it. Then this is the straight line every point on which will satisfy a condition similar to that satisfied by \( P \).

\( Q \) is the point \((10, 30)\); consequently its abscissa represents a quantity of rice of which the price is represented by its ordinate. \( R \) is the point \((3, 9)\); its abscissa therefore represents a quantity of rice of which the price is represented by its ordinate. Similarly this is true of every point on the line \( OP \).

Hence \( OP \) is the required straight line.

**Note 1.** The line \( OP \) is called the graph of the price of rice, or more simply the **price-graph** of rice.

**Note 2.** The graph enables us to determine readily the price of any given number of seers of rice. For instance, if the abscissa be taken to be 12, the ordinate is immediately found to be 36; thus we know that the price of 12 seers of rice is 36 annas. Similarly for any other abscissa the corresponding ordinate can be immediately found.

**Note 3.** The graph also enables us to determine quickly the number of seers of rice that can be bought for any given price. For instance, if the ordinate is taken to be 27, the corresponding abscissa is immediately found to be 9, which shows that we can have 9 seers of rice for 27 annas.

**Example 2.** A person, named B, starting from a given place, travels at the rate of 5 miles an hour. Shew that a straight line can be drawn such that if any point be taken on it, the abscissa of the point will represent the number of miles that B travels in the time represented by the ordinate.

In the figure on the next page let the length of a side of a small square measured along \( OX \) represent one mile, and let an equal length measured along \( OY \) represent 12 minutes. Then the meaning of the figures along \( OX \) and \( OY \) is clear.

Since B travels 5 miles in one hour, he travels 10 miles in 2 hours. Clearly therefore \( P \) is a point such that its abscissa represents the number of miles that the person travels in the time represented by its ordinate.

Join \( OP \) and produce it. Then this is the straight line every point on which will satisfy a condition similar to that satisfied by \( P \).

Let \( Q \) be any point on the line. Its abscissa represents 6 miles and ordinate represents 1 hour 12 minutes; but we know
that the person travels 6 miles in 1 hour 12 minutes. Hence Q satisfies the condition above mentioned.

Let R be some other point on the line. Its abscissa represents 20 miles and ordinate represents 4 hours; but we know that the person travels 20 miles in 4 hours. Hence R also satisfies the proposed condition.

Similarly for any other point on the line. Hence OP is the required straight line.

**Note 1**. The line OP is called the graph of B's motion, or the motion-graph of B.

**Note 2**. The graph enables us to determine readily the time in which B travels any given number of miles. For instance, if the abscissa be taken which represents 13 miles, the corresponding ordinate is immediately found to be that which represents 2 hours 36 minutes; thus it is known that the time taken by the person to travel 13 miles is 2 hours 36 minutes.

**Note 3**. The graph also enables us to determine readily the number of miles that the person travels in any given time. For instance, if the ordinate be taken which represents 3 hours 24 minutes, the corresponding abscissa is immediately found to be that which represents 17 miles; thus it is known that in 3 hours and 24 minutes the person travels 17 miles.

**Example 3**. If one inch be equal in length to 2.5 centimetres, show that a straight line can be drawn such that the abscissa of any point on the line will represent the number of inches that are equivalent to the number of centimetres represented by the ordinate.

In the figure on the next page let the length of a side of a small square measured along $OX$ represent one inch, and let an equal length measured along $OY$ represent one centimetre. Then the meaning of the figures along $OX$ and $OY$ is clear.
Since 1 inch = 2.5 centimetres, we have 8 inches = 20 centimetres. Clearly therefore P is a point such that its abscissa represents the number of inches that are equivalent to the number of centimetres represented by its ordinate.

Join OP and produce it. Then this is the straight line every point on which will satisfy a condition similar to that satisfied by P.

Let Q be any point on the line. Its abscissa represents 12 inches, whilst its ordinate represents 30 centimetres; but we know that these two are equivalent. Hence Q satisfies the condition above mentioned.

Let R be some other point on the line. Its abscissa represents 2 inches, whilst its ordinate represents 5 centimetres; but we know that these two are equivalent. Hence R also satisfies the proposed condition.

Similarly for any other point on the line. Hence OP is the required straight line.

**Note 1.** The line OP is called the graph for converting inches into centimetres and vice versa, or more briefly the conversion graph for inches and centimetres.

**Note 2.** The graph enables us to determine readily the number of centimetres that are equivalent to any given number of inches. For instance, if the abscissa be taken which represents 10 inches, the corresponding ordinate is immediately found to be that which represents 25 centimetres; thus it is known that 10 inches are equivalent to 25 centimetres.

**Note 3.** The graph also enables us to determine readily the number of inches that are equivalent to any given number of centimetres. For instance, if the ordinate be taken which represents 15 centimetres, the corresponding abscissa is immediately found to be that which represents 6 inches; thus it is known that 15 centimetres are equivalent to 6 inches.
Example 4. A and B are two stations 30 miles apart. P starts from A and travels towards B at the rate of 5 miles an hour; at the end of 2 hours he takes rest for one hour, and then resumes his journey at the rate of 3 miles an hour. Q leaves B 2 hours 40 minutes after P leaves A, and travels towards A, without stoppage, at the rate of 4 miles an hour. When and where will the two travellers meet?

Let the length of a side of a small square measured horizontally represent one mile, and let an equal length measured vertically represent 10 minutes. Then the meaning of the figures along the lines in the above diagram is clear.
(i) \(P\) starts from \(A\), and travelling at the rate of 5 miles an hour, completes 10 miles in 2 hours. Hence if the point \(C\) be taken such that its co-ordinates respectively represent 10 miles and 2 hours, \(AC\) is the graph of \(P\)'s motion for the first two hours.

The graph for the 3rd hour must be such that the abscissa of any point on it may represent 10 miles, because \(P\) is supposed to be at rest throughout this hour. Hence \(CD\) drawn vertically to represent one hour, as in the diagram, will be the graph of \(P\)'s rest.

After the 3rd hour \(P\) travels at the rate of 3 miles an hour. Hence if \(DM\) be taken to represent 6 miles and \(ME\) to represent 2 hours, the straight line \(DE\) is the graph of \(P\)'s motion after the 3rd hour.

Thus the broken line \(ACDE\) is the complete graph of \(P\)'s motion.

(ii) \(Q\) starts from \(B\) 2 hours 40 minutes after \(P\) leaves \(A\). Hence if \(BF\) be measured vertically to represent 2 hours 40 minutes, \(BF\) may be regarded as the graph of \(Q\)'s rest at \(B\).

When \(Q\) leaves \(B\) he moves towards \(A\) at the rate of 4 miles an hour. Hence if \(FN\) be taken to represent 8 miles and \(GT\) to represent 2 hours, the straight line \(FG\) will be the graph of \(Q\)'s motion.

(iii) Let the two graphs intersect at \(H\), and draw \(HK\) perpendicular to \(AB\). Produce \(FN\) to meet \(HK\) at \(V\).

Now it is clear that at the end of time \(HK\), \(P\) will have gone a distance \(AK\) towards \(B\), and \(Q\) will have gone a distance \(BK\) (i.e. \(FV\)) towards \(A\). Hence they will meet at this instant. Thus the required time of meeting = that represented by \(HK = 5\) hours 40 minutes after the commencement of \(P\)'s motion.

Also, the distance of the place of meeting from \(A = \) that represented by \(AK = 18\) miles.

**Note 1.** As \(HV\) represents 3 hours, it is clear that \(P\) and \(Q\) meet at the end of 3 hours after \(Q\) starts from \(B\).

**Note 2.** The horizontal line through \(L\) meets the graphs at the points \(S\) and \(T\). As \(AL\) represents 4 hours 10 minutes and \(ST\) represents 10\(\frac{1}{2}\) miles, it is clear that at the end of 4 hours 10 minutes from the commencement of \(P\)'s motion, \(P\) and \(Q\) are at a distance of 10\(\frac{1}{2}\) miles from each other.
Exercise (7).

1. If milk sells for 4 annas per seer, construct the price-graph of milk, giving the price of any quantity of milk up to 5 seers. From the graph read off the price of 3 seers and 5 chattacks of milk, and also the quantity of milk that can be had for 10 annas and 9 pies.

2. If Fazli mangoes be worth one rupee two annas a dozen, construct a price-graph for mangoes, giving the price of any number up to 30. Read off from the graph the price of 17 mangoes and also the number of mangoes that can be had for 1 Rs. 12 as. 6 p.

3. If a man walks at the rate of 4 miles an hour, construct a graph of his motion. Read off from the graph the time in which he travels 13 miles and also the number of miles he travels in 4 $\frac{3}{4}$ hours.

4. If one cubit be equal to 1 $\frac{1}{4}$ feet, construct a conversion-graph for cubits and feet. Read off from the graph the number of feet that are equivalent to 5 $\frac{3}{4}$ cubits and also the number of cubits that are equivalent to 6 $\frac{3}{4}$ feet.

5. A starts from a place and walks in a given direction at the rate of 3 miles an hour; B starts from the same place one hour later and moves in the same direction at the rate of 5 miles an hour. Draw the motion-graphs of A and B, and find when and where B overtakes A.

6. A and B are two stations 20 miles apart. P starts from A and travels towards B at the rate of 3 miles an hour; whilst Q starting from B travels towards A at the rate of 2 miles an hour. Construct the motion-graphs of P and Q, and find when and where they meet.

7. Fifty articles of the same kind cost 3 Rs. 2 as. Construct a graph from which you can read off the cost of any number of articles up to 50. Hence find the cost of 19 articles, and the number of articles that you would get for 2 Rs. 7 as.

8. Given that 1 kilogramme = 2 $\frac{1}{4}$ lbs., construct a graph which will enable you to read off the number of kilogrammes that are equivalent to any given number of lbs. up to 15 lbs. Read off the number of kilogrammes in 11 lbs.

9. A man travels for 3 hours at the rate of 2 miles an hour, at the end of which he takes rest for an hour and a half,
and then starts to walk at the rate of two and a half miles an hour. Construct the graph of his motion.

10. A man starts from a place B to walk towards C at the rate of 4 miles an hour. After 3 hours he changes his mind and walks back towards B at the rate of 3 miles an hour. At the end of 2 hours again he suddenly changes his mind and begins to run towards C at the rate of 7 miles an hour. Draw a graph of his motion.

11. A, B and C are three stations in order on the same road, the distance between A and B being 6 miles. Q starts from B at noon to walk towards C at the rate of 3 miles an hour, and at 1:30 p.m. P starts from A to run towards B at the rate of $6\frac{1}{2}$ miles an hour. Draw graphs of their motion, and find when and where P will overtake Q.

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CHAPTER II.

GRAPHS (Continued).

1. Draw the graph of the equation $x^2 + y^2 = 36$.

Let twice the length of a side of a small square represent the unit of length.

With centre $O$ and a radius equal to 6 units of length describe a circle, as in the diagram on the next page. Then this circle will be the required graph.

Take any point $P$ on the circle, and let its co-ordinates, be denoted by $x$ and $y$; evidently then $x^2 + y^2 = OP^2 = 36$. But if a point, such as $Q$, be taken anywhere not on the circle, it is easy to see that its co-ordinates will not satisfy the given equation.
Thus it is shown that the co-ordinates of every point on the circle, and of no other point, satisfy the given equation. Hence the circle drawn is the required graph.

2. Draw the graph of the equation \((x - 3)^2 + (y - 2)^2 = 25\).

Let twice the length of a side of a small square represent the unit of length.

Let \(A\) be the point \((3, 2)\). With centre \(A\) and a radius equal to 5 units of length describe a circle as in the diagram on the next page. Then this circle will be the required graph.

Take any point \(P\) on the circle, and let its co-ordinates be denoted by \(x\) and \(y\). Now from the diagram it is clear that \(AP\) is the hypotenuse of a right-angled triangle of which the sides are \((x - 3)\) and \((y - 2)\) units of length respectively.
Hence $(x - 3)^2 + (y - 2)^2 = AP^2 = 25$, which shows that the co-ordinates of $P$ satisfy the given equation. But if a point such as $Q$, be taken anywhere not on the circle, it is easy to see that its co-ordinates will not satisfy the given equation.

Thus it is clear that the co-ordinates of every point on the circle and of no other point, satisfy the given equation. Hence the circle described is the required graphs.

**Note 1.** It may be similarly shown that the graph of the equation $(x + 2)^2 + (y + 5)^2 = 49$ is a circle of which the centre is the point $(-2, -5)$ and the radius is equal to 7 units of length.

**Note 2.** The equation $x^2 + y^2 - 8x + 10y + 25 = 0$ can be easily reduced to the form $(x - 4)^2 + (y + 5)^2 = 16$. Hence its graph is a circle of which the centre is the point $(4, -5)$ and the radius is equal to 4 units of length.

3. Draw the graph of the equation $y^2 = 4x^2$.

From the given equation we have

$$y^2 - 4x^2 = 0$$

or, $(y + 2x)(y - 2x) = 0$.

Hence, it is clear that the given equation is satisfied by (1) all those points which satisfy the equation $y + 2x = 0$, and also (2) by all those points which satisfy the equation $y - 2x = 0$.

Hence, the required graph consists of two straight lines, one being the graph of the equation $y + 2x = 0$, and the other being the graph of the equation $y - 2x = 0$. 
Hence the required graph is as shewn below:

4. Draw the graph of the equation $4x^2 + 9y^2 = 36$.

(1) When $x = 0$, we have $y^2 = 4$, and therefore $y = \pm 2$. Hence the points $(0, 2)$ and $(0, -2)$ are on the required graph.

(2) When $y = 0$, we have $x^2 = 9$, and therefore $x = \pm 3$. Hence the points $(3, 0)$ and $(-3, 0)$ are on the required graph.
(3) When \( x = \pm 1 \), we have \( 9y^2 = 32 \), and therefore
\[
y = \pm \frac{4}{3} \sqrt{2} = \pm \frac{4 \times 1.414...}{3} = \pm \frac{5.656...}{3} = \pm 1.885... = \pm 1.9
\]
approximately. Hence the four points \((1, 1.9), (1, -1.9), (-1, 1.9)\) and \((-1, -1.9)\) are on the required graph.

(4) When \( x = \pm 2 \), we have \( 9y^2 = 20 \), and therefore
\[
y = \pm \frac{2}{3} \sqrt{5} = \pm \frac{2 \times 2.236...}{3} = \pm \frac{4.472...}{3} = \pm 1.490... = \pm 1.5
\]
approximately. Hence the four points \((2, 1.5), (2, -1.5), (-2, 1.5)\) and \((-2, -1.5)\) are on the required graph.
Let us now plot the twelve points as found above (taking 5 times the side of a small square as the unit of length) and draw a free-hand curve through them, as in the diagram on the last page.

The curve so drawn is the required graph.

Note 1. Evidently the curve is symmetrical about the axis of \( x \), i.e. every chord at right angles to the axis of \( x \) is bisected by it. Similarly, the curve is also symmetrical about the axis of \( y \).

Note 2. The curve lies entirely within the space enclosed by the four straight lines \( x = 3, x = -3, y = 2, y = -2 \). A curve of this class is called an Ellipse.

5. Draw the graph of the equation \( x^2 - y^2 = 1 \).
(1) When \( x = 0 \), we have \( y^2 = -1 \), and therefore \( y \) is imaginary. This shews that the graph does not cut the axis of \( y \).

(2) When \( y = 0 \), we have \( x^2 = 1 \), and therefore \( x = \pm 1 \). Hence the points \((1, 0)\) and \((-1, 0)\) are on the required graph.

(3) When \( x = \pm 2 \), we have \( y^2 = 3 \), and therefore \( y = \pm \sqrt{3} = \pm 1.732... = \pm 1.7 \) approximately. Hence the four points \((2, 1.7)\), \((2, -1.7)\), \((-2, 1.7)\) and \((-2, -1.7)\) are on the required graph.

(4) When \( x = \pm 3 \), we have \( y^2 = 8 \), and therefore \( y = \pm 2\sqrt{2} = \pm 2 \times 1.414... = \pm 2.828... = \pm 2.8 \) approximately. Hence the four points \((3, 2.8)\), \((3, -2.8)\), \((-3, 2.8)\) and \((-3, -2.8)\) are on the required graph.

Let us now plot the ten points as found above (taking 5 times the side of a small square as the unit of length) and draw a free-hand curve through them, as in the diagram on the last page.

The curve so drawn is the required graph.

**Note 1.** The curve so drawn is evidently symmetrical about the axis of \( x \) and also about the axis of \( y \).

**Note 2.** The curve consists of two branches, one lying entirely on the right of the line \( x = 1 \) and the other lying entirely on the left of the line \( x = -1 \). A curve of this class is called a **Hyperbola**.

**Note 3.** From articles 1, 3, 4, and 5 it may be easily seen that if the equation \( ax^2 + by^2 = c \) be taken, (1) the graph is two straight lines passing through the origin when \( c \) is zero, and \( a \) and \( b \) are of different signs; (2) the graph is a circle when \( \frac{a}{c} \) and \( \frac{b}{c} \) are positive and equal; (3) the graph is an **Ellipse** when \( \frac{a}{c} \) and \( \frac{b}{c} \) are positive and unequal; and (4) the graph is a Hyperbola when \( \frac{a}{c} \) and \( \frac{b}{c} \) are of different signs (their absolute values being either equal or unequal).

6. Draw the graph of the equation \( y = x^2 \), taking the unit for measuring \( y \) equal to half that for measuring \( x \).

Evidently the following points are on the required graph:

\[
\begin{align*}
&x = 0 \quad y = 0 \quad \begin{cases} x = 1 \quad y = 1 \\ x = -1 \quad y = 1 \end{cases} \\
&y = 0 \quad \begin{cases} x = 1 \quad y = 1 \\ x = -1 \quad y = 1 \end{cases}
\end{align*}
\]
\[ x = \frac{3}{2}, \quad y = \frac{9}{4} \]
\[ x = -\frac{3}{2}, \quad y = \frac{9}{4} \]
\[ y = \frac{9}{4} \]
\[ y = \frac{16}{4} \]

Let four times the side of a small square (i.e., \( \frac{4}{3} \) of an inch) be the unit for measuring \( x \) and twice the side of a small square (i.e., \( \frac{2}{3} \) of an inch), the unit for measuring \( y \).

Let us now plot the points found above and draw a curve through them free-hand, as in the following diagram:

The curve so drawn is the required graph.
Note 1. If the unit for measuring y were the same as that for measuring x (i.e. 4 of an inch), the curve drawn would be the graph of the equation \( y = \frac{1}{4}x^2 \), or that of \( 2y = x^2 \).

Note 2. Every chord drawn perpendicular to OY is bisected by it, as can be easily verified. Hence the curve is symmetrical about the axis of y. This is also evident from the fact that if the paper be folded about OY the left-hand portion of the curve entirely coincides with the right-hand portion.

Note 3. The curve lies entirely above the axis of x, and extends upwards to infinity. It is easy to see that the graph of the equation \( y = x^2 \) would be a curve lying entirely below the axis of x and extending downwards to infinity.

Note 4. The abscissa of any point on the curve is evidently the square root of the ordinate. Hence when the graph of the equation \( y = x^2 \) is drawn, by measuring the abscissa of any point on the graph we can determine the square root of the number which represents the ordinate.

Note 5. A curve of this class is called a Parabola.

7. Draw the graph of the expression:

\[ 3 - 4x - 2x^2. \]

The required graph is the same as that of the equation \( y = 3 - 4x - 2x^2 \).

It is easy to see that the following points are on the required graph:

\[
\begin{align*}
& x = 0 \quad x = 1 \quad x = 1.5 \quad x = -1 \quad x = -3 \quad x = -3.5 \quad x = 1.5 \quad x = -3.5 \\
& y = 3 \quad y = -3 \quad y = -7.5 \quad y = 5 \quad y = 3 \quad y = -3 \quad y = -7.5 \quad y = 5.
\end{align*}
\]

Take one inch as the unit for measuring x, and one-tenth of an inch as the unit for measuring y.

Let us now plot the above points and draw a curve through them free-hand, as in the following diagram:
The curve so drawn is the required graph.
Note 1. Since \(3 - 4x - 2x^2 = 3 - 2(x^2 + 2x) = 5 - 2(x^2 + 2x + 1) = 5 - 2(x + 1)^2\), the equation may also be written as
\[y = 5 - 2(x + 1)^2;\]
which shows that for all values of \(x\), \(y\) is less than 5, except when \(x = -1\), and in this case \(y = 5\). This is also clear from the curve drawn. Hence, the maximum value of \(y\) (i.e., that of the expression \(3 - 4x - 2x^2\)) is 5.

Note 2. If through the point \((-1, 5)\) a straight line be drawn parallel to the axis of \(y\), it is easy to see that the curve is symmetrical about this straight line.

Note 3. From the figure it is evident that \(y = 0\) when \(x\) is approximately equal to \(-6\) or \(-2-6\). Hence, \(3 - 4x - 2x^2 = 0\) when \(x = -6\) or \(-2-6\) approximately; in other words, the roots of the equation \(3 - 4x - 2x^2 = 0\) are \(-6\) and \(-2-6\) approximately. From this it is clear that the roots of the equation \(3 - 4x - 2x^2 = 0\) are the abscissas of the points where the graph of the expression \(3 - 4x - 2x^2\) cuts the axis of \(x\).

Note 4. The graph of any expression of the form \(ax^2 + bx + c\) is a Parabola.

8. Draw the graph of the equation \(xy = 1\).

It is easy to see that the following points are on the required graph:

\[
\begin{align*}
  x & = -1, \quad x = 2, \quad x = -4, \\
  y & = 10, \quad y = 5, \quad y = 2.5, \\
  x & = -5, \quad x = 8, \quad x = 1, \quad x = 2, \\
  y & = 2, \quad y = 1.25, \quad y = 1, \quad y = 0.5.
\end{align*}
\]

Evidently also the following points are on the required graph:

\[
\begin{align*}
  x & = -1, \quad x = -2, \quad x = -4, \\
  y & = -10, \quad y = -5, \quad y = -2.5, \\
  x & = -5, \quad x = -8, \quad x = -1, \quad x = -2, \\
  y & = -2, \quad y = -1.25, \quad y = -1, \quad y = -0.5.
\end{align*}
\]

Let one inch be the unit for measuring \(x\) and one-tenth of an inch the unit for measuring \(y\).

Let us now plot the points and draw a curve through them free-hand, as in the following diagram:
The curve so drawn is the required graph.
Note 1. As \( x \) diminishes from 1 to zero, \( y \) increases from 1 to infinity; and as \( x \) diminishes from zero to \(-1\), \( y \) increases from negative infinity to \(-1\).

Note 2. As \( x \) increases from 1 to infinity, \( y \) diminishes from 1 to zero; and as \( x \) diminishes from \(-1\) to negative infinity, \( y \) increases from \(-1\) to zero.

Note 3. The graph consists of two branches, one lying between \( OX \) and \( OY \), and the other between \( OX' \) and \( OY' \).

Note 4. The more we move towards the right or left of \( O \), the nearer does the curve approach the axis of \( x \); whilst the more we move upwards or downwards from \( O \), the nearer does the curve approach the axis of \( y \). But in no case does the curve meet the axes except at an infinite distance from \( O \). Hence, each of the axes is said to be an asymptote to the curve.

Note 5. A curve of this kind is called a Rectangular Hyperbola.

Exercise (8.)

Draw the graphs of the following equations:

1. \[ x^2 + y^2 = 81. \]
2. \[ (x - 5)^2 + (y - 6)^2 = 49. \]
3. \[ (x + 6)^2 + (y - 7)^2 = 100. \]
4. \[ x^2 + y^2 - 8x - 14y + 1 = 0. \]
5. \[ x^2 + y^2 + 14x - 16y + 32 = 0. \]
6. \[ x^2 + y^2 + 12x + 18y + 92 = 0. \]
7. \[ x^2 + y^2 - 10x + 16y - 55 = 0. \]
8. \[ x^2 - y^2 = 0. \]
9. \[ 9x^2 - 4y^2 = 0. \]
10. \[ 9y^2 = 16x^2. \]
11. \[ 4x^2 = 9. \]
12. \[ 25y^2 = 16. \]
13. \[ x^2 + 4y^2 = 4. \]
14. \[ 4x^2 + 9y^2 = 1. \]
15. \[ 25x^2 + y^2 = 25. \]
16. \[ 16x^2 + 9y^2 = 1. \]
17. \[ x^2 - 4y^2 = 4. \]
18. \[ y^2 - x^2 = 1. \]
19. \[ 4x^2 - y^2 = 16. \]
20. \[ y^2 - 9x^2 = 9. \]
21. In one and the same diagram draw the graphs of \( 4x^2 - 9y^2 = 0 \) and \( 4x^2 - 9y^2 = 36 \).
22. In one and the same diagram draw the graphs of
$9y^2 - 4x^2 = 0$ and $9y^2 - 4x^2 = 36$.

23. Draw the graph of the equation $5y = x^2 - 10$,

taking the unit for measuring $y$ five times as large as that
for measuring $x$.

24. Draw the graph of the equation $x^2 - 4x + 2y = 0$,

taking the unit for measuring $y$ twice as large as that
for measuring $x$.

25. Draw the graph of the equation $y^2 + x = 0$,

taking the unit for measuring $x$ equal to half that for
measuring $y$.

26. Draw the graph of the equation $3y = x^2$,

taking the same unit for measuring both $x$ and $y$.

27. Find graphically, correct to the first figure after the
decimal point, the square roots of:—

(i) 3 ; (ii) 5 ; (iii) 7.

28. Find graphically the minimum value of the expression
$2x^2 - 6x + 7$.

29. Find graphically the maximum value of the expression
$1 + 2x - 2x^2$.

30. Draw the graph of the equation $xy = 4$. 
CHAPTER III.

Miscellaneous Examples.

Example 1. The salaries of the teachers of a certain school are increased; so that a salary of Rs. 60 is increased to Rs. 70, and one of Rs. 90 to Rs. 104. What then must be the relation between \( x \) and \( y \), if \( x \) rupees be the old salary of a teacher whose new salary is \( y \) rupees? Hence deduce a graphical method of finding the new salary of a teacher whose old salary is given, and vice versa.

(i) Assume that the relation between \( x \) and \( y \) is

\[ y = ax + b, \]

where \( a \) and \( b \) are any two unknown constants.

Then, by hypothesis,

\[ \begin{align*}
70 &= 60a + b \\
104 &= 90a + b
\end{align*} \]

and

whence,

\[ a = \frac{17}{15} \text{ and } b = 2. \]

Hence the required relation is

\[ y = \frac{17}{15} x + 2. \]

(ii) The graph of the above equation is surely a straight line.

Let one-tenth of an inch, measured horizontally, represent one rupee of the old salary, and let one-tenth of an inch, measured vertically, represent one rupee of the new salary. Then the meaning of the figures in the following diagram is clear, the origin being taken as the point whose co-ordinates are 60 and 60.

The points (60, 70) and (90, 104) satisfy the above equation. Hence the straight line joining these two points is the required graph.

The co-ordinates of the point \( P \) on the graph are 75 and 87; this shews that Rs. 87 is the new salary of a teacher whose
old salary was Rs. 75. Again, if we take the point whose abscissa is 80, we find that its ordinate is 92\(\frac{2}{3}\); this shows that if the old salary of a teacher was Rs. 80, his new salary is 92\(\frac{2}{3}\). And so on.
Example 2. There is a group of 30 children before me, of whom some are boys and the rest are girls. If I get 3 rupees from each boy and give 2 rupees to each girl, I gain altogether Rs. 5. Find graphically the number of boys and girls in the group.

Let one-tenth of an inch, measured horizontally, represent one child, and let one-tenth of an inch, measured vertically, represent one rupee.
OX representing the total number of children, the number of boys may be measured from O and the number of girls from X.

Hence, if A be the point (5, 15) from O, the straight line OAB will be the graph of the money received from the boys; and if C be the point (5, 10) from X, the straight line XCD will be the graph of the money given to the girls.

Let KNM be an ordinate cutting the graphs at K and N. Then KM represents the money received from the boys whose number is represented by OM, and NM represents the money given to the girls whose number is represented by XM, and therefore KN represents the gain.

Hence, if the ordinate be so drawn that the portion KN, intercepted between the two graphs, represents 5 rupees, then OM and XM will represent the required numbers of boys and girls respectively.

We have therefore to find K by taking XII to represent 5 rupees and drawing HK parallel to XN. Now drawing the ordinate KM, we find that OM = 13 units of length and XM = 17 units of length.

Hence the number of boys in the group = 13, and the number of girls = 17.

**Note.** If Rs. 2 were to be not given to, but *taken from*, each girl, the graph of the money received from the girls would have to be drawn below the line XO.

**Example 3.** P and Q can separately perform a certain work in 7 hours and 4 hours respectively. After P has worked for an hour and a half, Q joins. Find graphically the time in which they will together finish the work.

Let '2 of an inch, measured horizontally, represent an h.c.m., and let one inch, measured vertically, represent the work.

Let A be the point (7, 1); then the straight line OA is the graph of the work done by P.
If the vertical through $M \ (1\frac{1}{2}, \ 0)$ meet $OA$ in $B$, then $BM$ represents the work done by $P$ in $1\frac{1}{2}$ hours.

Draw $BC$ (horizontally) to represent 4 hours; let the vertical through $C$ meet $OA$ in $D$, and produce $CD$ to $E$ making $DE$ equal to $OK$.

Then $CE$ represents the work done by $P$ and $Q$ in 4 hours. Hence the straight line $BE$ is the graph of the work jointly done by $P$ and $Q$.

Let $BE$ cut the horizontal line through $K$ in $H$. Then, since the co-ordinates of $H$ are found to be $3\frac{1}{2}$ and 1, it is clear that $P$ and $Q$ will together finish the work $3\frac{1}{2}$ hours after $P$ begins or 2 hours after $Q$ joins.

**Example 4.** The temperature of a patient was observed at intervals of 2 hours from 7 A.M. till 9 P.M. on a certain day, and the observations were recorded as follows:—
Exhibit graphically the variation in temperature during the whole interval; and, from the graph drawn, find the probable time at which the temperature was a maximum, the probable maximum temperature and also the probable temperature at 8 P.M.

Let .2 of an inch, measured horizontally, represent an hour, and let one inch, measured vertically, represent a degree of temperature. The numbers 13, 15, 17, 19 and 21 may also be written for 1 P.M., 3 P.M., 5 P.M., 7 P.M., and 9 P.M. respectively. Hence the meaning of the figures along OX and OY is clear, the origin being taken as the point whose co-ordinates are 7 and 100.
Let $P$, $Q$, $R$, $S$, $T$, $U$, $V$ denote the points $(9, 100\cdot7)$, $(11, 101\cdot1)$, $(13, 101\cdot4)$, $(15, 102)$, $(17, 101\cdot8)$, $(19, 101\cdot5)$ and $(21, 100\cdot9)$ respectively.

Let us now plot the points $P$, $Q$, $R$, $S$, $T$, $U$, $V$ and draw through them free hand a continuous curve, starting from $0$.

The curve so drawn is the required graph.

Now, it is evident from the diagram that the point on the curve which has the greatest ordinate is the point $(15\cdot5, 102\cdot1)$. Hence, the temperature reached its maximum at about $3\cdot30$ p.m.; and the maximum temperature was approximately $102\cdot1$.

The point on the curve which corresponds to $8$ p.m. is the point whose abscissa is $20$ and whose ordinate is found to be $101\cdot25$ approximately. Hence the probable temperature at $8$ p.m. was $101\cdot25$.

**Note.** It may be reasonably supposed that the change in temperature is gradual and not sudden; hence the necessity of drawing a continuous curve, that is, one not having sharp turns.

**Example 5.** Solve graphically

$$3 - 4x - 2x^2 = 0.$$

[One method of solution has been referred to in Note 3 to Art. 7 of the last chapter. A second method is given below.]

We have $2x^2 = 3 - 4x$;

\[\therefore x^2 = \frac{3}{2} - 2x.\]

Let us, first of all, draw the graph of the equation $y = x^2$.

Take $\cdot5$ of an inch as the unit for measuring $x$, and $\cdot3$ of an inch as the unit for measuring $y$. Then the graph will be as in the following diagram:
With the same units, now draw the graph of the equation \( y = \frac{3}{2} - 2x \). The points A and B, whose co-ordinates are respectively \((-1, 3\frac{1}{2})\) and \((2, -2\frac{1}{2})\), evidently lie on this graph. Hence the straight line AB is the required graph.

Let this straight line cut the parabola in P and Q. Then, if \( x, y \) denote the co-ordinates of either P or Q, we must have

\[
\begin{align*}
y &= x^2, \\
y &= \frac{3}{2} - 2x
\end{align*}
\]

because each of these two points lies on both the graphs.
Hence, the abscissa of either P or Q will satisfy the equation
\[ x^2 = \frac{3}{2} - 2x. \]

Thus the abscissa of the points P and Q are the roots of the given equation.

These abscissae are found to be respectively equal to \(-2.6\) and \(-6\) approximately, which therefore are the required roots.

Note. The present method has this advantage that the parabola \( y = x^2 \) being once drawn, any equation of the form \( ax^2 + bx + c = 0 \) may be immediately solved by simply drawing the straight line \( y = -\frac{b}{a}x - \frac{c}{a} \).

Example 6. Find graphically the different pairs of values of \( x \) and \( y \) that simultaneously satisfy the following equations:
\[
\begin{align*}
x^2 + y^2 &= 25 \quad (1) \\
x y &= 12 \quad (2)
\end{align*}
\]

Let us draw the graphs of the two equations, as in the following diagram, taking one-tenth of an inch as the unit for measuring \( x \) and \( y \):—
Let the two graphs intersect each other at the points P, Q, R, S. Then the co-ordinates of each of these points will satisfy both the equations.

Now, the co-ordinates of these points are found to be respectively \((3, 4), (4, 3), (-3, -4)\) and \((-4, -3)\).

Hence the required solutions are

\[
x = 3\}
\{ x = 4\}
\{ x = -3\}
\{ x = -4\}
\{ y = 4\}
\{ y = 3\}
\{ y = -4\}
\{ y = -3\}
\]

Exercise (9).

N.B. The graphical method should be used in solving each of the following problems:

1. A can do a piece of work in 6 days, and B can do it in 9 days; if they work together, how many days will they take to finish it?

2. A tap, which would fill a cistern in 4 hours, and a plug, which would empty it in 6 hours, are both opened at the same instant, when the cistern is empty. How long will they take to fill the cistern?

3. A cistern can be filled by a pipe A in 26 minutes, and by a pipe B in 15 minutes, while it can be emptied by a pipe C in 12 minutes; if all three pipes are set running when the cistern is empty, in what time will it be filled?

4. A, B and C can separately do a piece of work in 5, 8, and 3\(\frac{1}{2}\) days respectively; if all three work together, how long will they take to finish the work?

5. At what time between 2 and 3 o'clock are the two hands of a watch (i) together, (ii) 5 minute-divisions apart?

6. Rs. 45 is the price of 20 balls of which some are white and the rest black. If the price of each white ball be Rs. 3 and that of each black ball Rs. 2, how many of the balls are white?

7. A servant makes a contract to get 4 annas for every day that he works, and to pay a fine of 6 pice for every day that he is absent. If he altogether gets Rs. 3 8 as after 25 days, how many days was he absent from work?
8. From the same spot on a circular course, one mile in length, two boys A and B start at the same moment to walk round it, travelling in the same direction. A walks at 4, and B at 3, miles an hour; how often will they meet if they walk for two hours and a half?

9. In the preceding example, if the boys start in opposite directions, and walk respectively at 3 and 2 miles an hour, how often will they meet within half an hour?

10. The highest and lowest marks obtained in an examination are 283 and 110 respectively; the marks are reduced so that 283 becomes 100 and 110 becomes 50. Find approximately the numbers to which 248 and 124 are respectively reduced.

11. The temperature of a room taken at every hour of the day from 9 A.M. until 4 P.M. is given in the following table:

<table>
<thead>
<tr>
<th>Time</th>
<th>9 A.M.</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>1 P.M.</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature</td>
<td>56</td>
<td>57</td>
<td>59</td>
<td>62</td>
<td>62</td>
<td>61</td>
<td>58</td>
<td>57</td>
</tr>
</tbody>
</table>

Construct a graph to show the variation of temperature, and ascertain from the graph the temperature at 10-30 A.M.

12. Find the roots of the following equations, correct to the first figure after the decimal point:
   (i) \( x^2 - 2x = 4 \);
   (ii) \( \frac{x^2}{4} + x - 2 = 0 \);
   (iii) \( 4x^2 - 16x + 9 = 0 \).

13. Find the different pairs of values of \( x \) and \( y \) that satisfy both the equations \( x^2 + y^2 = 100 \) and \( x + y = 14 \).

14. Find the different pairs of values of \( x \) and \( y \) that satisfy both the equations \( x - y = 3 \) and \( xy = 4 \).
CHAPTER IV.

Graphs of Exponential and Logarithmic Functions.

1. Draw the graph of the function $10^x$.

The required graph is the same as that of the equation $y = 10^x$.

It is easy to see that, among many, the following pairs of values of $x$ and $y$ will satisfy the above equation:—

\[
\begin{align*}
  x = 0 & \quad y = 1 \\
  x = \frac{1}{2} & \quad y = 3.16 \\
  x = 1 & \quad y = 10 \\
  x = 2 & \quad y = 100 \\
  x = -\frac{1}{2} & \quad y = 0.32 \\
  x = -1 & \quad y = 0.1 \\
  x = -2 & \quad y = 0.01 \\
\end{align*}
\]

Let A, B, C, D, E denote the points whose co-ordinates are respectively $(-2, 0.01)$, $(-1, 0.1)$, $(-\frac{1}{2}, 0.32)$, $(0, 1)$ and $(\frac{1}{2}, 3.16)$.

Plot these points, taking one inch as the unit of length.

Then the curve, drawn through the points thus plotted, is the required graph, as shewn in the following diagram:—
Note 1. For all real values of $x$, $y$ is positive. Hence, the curve lies entirely above the axis of $x$.

Note 2. As $x$ increases from zero, $y$ increases from 1; but the increment of $y$ is much more rapid than that of $x$, so that when a large value is given to $x$, the corresponding value of $y$ becomes comparatively very much larger. Hence, any straight line, that cuts $OX$ or $OX$ produced, and is also parallel to $OY$, must intersect the curve, although the point of intersection may be at a very great distance from $OX$. 
Note 3. As \( x \) diminishes from zero to negative infinity, \( y \) diminishes from one to zero. Hence it is clear that the left portion of the curve gradually approaches \( OX \) and ultimately meets it at infinity. Hence \( OX \) is an asymptote to the curve.

Note 4. From the equation \( y = 10^x \), we have \( x = \log_{10} y \). That is, if we take any point on the curve, its abscissa will give the logarithm of its ordinate to the base 10. For instance, \( P \) is the point whose ordinate is 2 and abscissa is approximately \( -1 \); hence we at once conclude that \( \log_{10} 2 = -1 \) approximately.

2. Draw the graph of the function \( \log_{10} x \).

The required graph is the same as that of the equation \( y = \log_{10} x \).

Since \( y = \log_{10} x \), we have \( x = 10^y \). Hence, it is clear that, among many, the following pairs of values of \( x \) and \( y \) will satisfy the equation \( y = \log_{10} x \):

\[
\begin{align*}
y &= 0, & x &= 1; \\
y &= \frac{1}{2}, & x &= 3.16; \\
y &= 1, & x &= 10; \\
y &= 2, & x &= 100; \\
y &= -\frac{1}{2}, & x &= 32; \\
y &= -1, & x &= 1; \\
y &= -2, & x &= 0.01.
\end{align*}
\]

Let \( A, B, C, D, E \) denote the points whose coordinates are respectively \( (0.01, -2), (1, -1), (32, -\frac{1}{2}), (1, 0) \) and \( (3.16, \frac{1}{2}) \).

Plot these points, taking one inch as the unit of length.

Then the curve, drawn through the points thus plotted, is the required graph, as shewn in the following diagram:
**Note 1.** As \( x = 10^x \), it is clear that \( y \) may be positive or negative, but \( x \) can never be negative. Hence the curve lies entirely on the right side of the axis of \( y \).

**Note 2.** As \( y \) diminishes from zero to negative infinity, \( x \) diminishes from one to zero. Clearly therefore the lower portion of the curve gradually approaches \( OY' \) and ultimately meets it at infinity. Hence \( OY' \) is an asymptote to the curve.

**Note 3.** As \( y \) increases from zero, \( x \) increases from 1, but the increment of \( x \) is much more rapid than that of \( y \), so that when a large value is given to \( y \), the corresponding value of \( x \) becomes comparatively very much larger. Hence any straight line that cuts \( OY \) or \( OY' \) produced, and is also parallel to \( OX \), must intersect the curve, although the point of intersection may be at a very great distance from \( OY \).
**Note 4.** If any point be taken on the curve, its ordinate will give the logarithm of its abscissa to the base 10. For instance, Q is the point whose abscissa is 2.5 and ordinate is 4 approximately; hence we at once conclude that \( \log_{10} 2.5 \approx 0.4 \) approximately.

**Exercise (10).**

1. Solve graphically the following equations:
   (i) \( 10^x = 10^x \); (ii) \( \log_{10} x = \frac{1}{3} \); (iii) \( 10^{x-1} = 6x - 8 \).
2. Draw the graphs of the following functions, using Logarithmic Tables:
   (i) \( (1 + x)^x \); (ii) \( 10^{\frac{1}{3}x} \).

**Answers to Exercises in the Appendix.**

1. [Page 514.]

2. Take BE equal to AD; by guess let F be the middle point of DE. Then F is very approximately the middle point of AB, the error, if any, being indefinitely small.

7. 2.56, 1.68, 3.79; 2.39, 1.40.

2. [Pages 517, 518]

1. \( \frac{6}{3} \) units of length.
2. \( 7\frac{1}{3} \) feet.
3. \( 7\frac{1}{4} \) yards.
4. 3.5 inches.
5. 3.6 feet.
6. \( \frac{7}{16} \) ft.
7. 5 yards.
8. 65 feet.
9. 17 feet.
10. 28.3 feet.

4. [Pages 522—524]

1. (i) \( (11, 7); (-9, 13); (-5, -7); (8, -10) \).
   (ii) \( (2.2, 1.4); (-1.8, 2.6); (-1, -1.4); (1.6, -2) \).
2. \( (3\frac{2}{3}, 2\frac{1}{3}); (-3, 4\frac{1}{3}); (-1\frac{2}{3}, -2\frac{1}{3}); (2\frac{2}{3}, -3\frac{1}{3}) \).
3. 20.
4. 13.
5. 50.
6. 11; -13.
7. 17.5; 36.
8. 12; 18.

5. [Pages 530, 531]

7. (1) \( 6x - 5y = 0 \);
   (2) \( 5x + 7y = 35 \);
   (3) \( x + y + 2 = 0 \);
   (4) \( 21x - 5y + 121 = 0 \);
   (5) \( 5x + 9y + 55 = 0 \).
6. [Page 532.]

1. \( x = 5, \, y = 4 \). \hspace{1cm} 2. \( x = 7, \, y = -5 \). \hspace{1cm} 3. \( x = 8, \, y = 6 \). \hspace{1cm} 4. \( x = 9, \, y = 11 \). \hspace{1cm} 5. \( x = 10, \, y = 13 \).

6. (Take ten times the side of a small square as the unit of length) \( x = 1\cdot2 \).

7. \( x = 7 \). \hspace{1cm} 8. \( x = 7 \).

7. [Pages 539, 540.]

1. 13 as. 3 pcks; 2 seers 11 chattleks.
2. 1 Re. 9 as. 6 p.; 19. \hspace{1cm} 3. \( 3\frac{1}{2} \) hours; 19 miles.
4. \( 8\frac{1}{2} \) feet; \( 4\frac{1}{2} \) cubits. \hspace{1cm} 5. \( 2\frac{1}{2} \) hours after A starts;
7\frac{1}{2} miles from the place of starting.

6. 4 hours after starting; 12 miles from A.

7. 1 Re. 3 as. 39. \hspace{1cm} 8. \hspace{1cm} 11. At 4-30 p.m.; 13\frac{1}{2} miles from B.

8. [Pages 552, 553.]

27. (i) 1\cdot7; (ii) 2\cdot2; (iii) (2\cdot6).
29. \( \frac{5}{2} \). \hspace{1cm} 29. \( \frac{3}{2} \).

9. [Pages 563, 564.]

1. 3\frac{1}{2} days. \hspace{1cm} 2. 12 hours. \hspace{1cm} 3. 30 minutes.
4. 1\frac{1}{2} days. \hspace{1cm} 5. (i) At 10\cdot9 minutes past 2; (ii) at 5\cdot5 and 16\cdot4 minutes past 2.

6. 5. \hspace{1cm} 7. 8 days. \hspace{1cm} 8. Twice.

9. Twice. \hspace{1cm} 10. 90; 54. \hspace{1cm} 12. (i) 3\cdot2, -1\cdot2; (ii) 1\cdot5,
-5\cdot5; (iii) 3\cdot3, 7. 13. \( x = 8, \, y = 6 \); \( x = 6, \, y = 8 \).

14. \( x = 4, \, y = 1 \); \( x = -1, \, y = -4 \).

10. [Page 569.]

1. (i) 1; (ii) 10; (iii) 1\cdot4, 4\cdot6.