INTRODUCTORY CHAPTER.

Resume of important theorems and examples from the earlier portions of the subject.

1. To prove that $a \times b = b \times a$, i.e., $b$ multiplied by $a$ gives the same result as $a$ multiplied by $b$.

   (i) First let $a$ and $b$ be any two positive integers.

   Place $b$ units in a horizontal row and write down $a$ such rows in such a manner that units in similar positions in the different rows may be in the same vertical column; thus:

   \[
   \begin{array}{ccccccc}
   1 & 1 & 1 & 1 & 1 & \ldots & b \\
   1 & 1 & 1 & 1 & 1 & \ldots & b \\
   1 & 1 & 1 & 1 & 1 & \ldots & b \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
   \end{array}
   \]

   to $a$ lines.

   This being done, evidently it may also be said that we have written down $b$ columns each containing $a$ units.

   Now let us count up the total number of units thus written down.

   Since we have got $a$ rows each containing $b$ units, the total number of units $= (\text{the number in the first row}) + (\text{the number in the second row}) + \ldots + (\text{the number in the } a^{th} \text{ row}) = b + b + b + \ldots \text{to } a \text{ terms} = a \times b$ \hspace{1cm} (1)

   Also, since we have $b$ columns each containing $a$ units, the total number of units $= (\text{the number in the 1st. column}) + (\text{the number in the 2nd. column}) + \ldots + (\text{the number in the } b^{th} \text{ column}) = a + a + a + \ldots \text{to } b \text{ terms} = b \times a$ \hspace{1cm} (2)

   Hence, from (1) and (2), we have $a \times b = b \times a$, i.e., $b$ taken $a$ times $= a$ taken $b$ times.
(ii) Next let \( a \) and \( b \) be two positive fractions; suppose \( a = \frac{m}{n} \) and \( b = \frac{p}{q} \), where \( m, n, p, q \) are positive integers.

Then \( a \times b = \frac{m}{n} \times \frac{p}{q} = m \times \left( \frac{p}{q} \div n \right) \)

\[
= m \times \frac{p}{nq} = \frac{mp}{nq} \quad \ldots \quad (I)
\]

and \( b \times a = \frac{p}{q} \times \frac{m}{n} = p \times \left( \frac{m}{n} \div q \right) \)

\[
= p \times \frac{m}{qn} = \frac{pm}{qn} \quad \ldots \quad (II)
\]

But \( m \) and \( p \) are positive integers, therefore \( mp = pm \), and similarly \( nq = qn \).

Hence, from (I) and (II) we have \( a \times b = b \times a \).

Thus it is established that for all positive values of \( a \) and \( b \) we must have \( a \times b = b \times a \). \( \ldots \quad \ldots \quad (A) \)

**Cor. 1.** We have \( x \times (-y) = -(xy) \), and \( (-y) \times x = -(yx) \); but \( xy = yx \), \( \therefore x \times (-y) = (-y) \times x \). \( \ldots \quad \quad (B) \)

**Cor. 2.** \( (-x) \times (-y) = +xy \), and \( (-y) \times (-x) = +yx \); but \( xy = yx \), \( \therefore (-x) \times (-y) = (-y) \times (-x) \). \( \ldots \quad \quad (C) \)

Hence, from (A), (B) and (C) we conclude that for all values of \( a \) and \( b \), \( a \times b = b \times a \).

2. To prove that \((ab) \times c = a \times (bc) \) or \( b \times (ac) \), \( i.e., \) to multiply \( c \) by the product of \( a \) and \( b \) is the same as to multiply \( c \) first by either of them and then that result by the other.

Place \( b \) brackets in a horizontal row each containing \( c \) units and write down \( a \) such rows in such a manner that the brackets in similar positions in the different rows may be in the same vertical column; thus:

\[
\begin{array}{cccc}
\ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

\( b \) times

\( a \) rows.
This being done, it may also be said that we have written down \( b \) columns each containing \( a \) brackets.

As we have got altogether \( a \times b \) brackets and as each bracket contains \( c \) units, the total number of units = \((ab) \times c\) ...(\(a\))

Again, since we have got \( b \) brackets in a row each containing \( c \) units, the number of units in a row = \( bc \), and as there are \( a \) rows altogether, therefore the total number of units = \( a \times (bc) \) ... ... ... ... \((\beta)\)

Again, since we have got \( a \) brackets in a column each containing \( c \) units, the number of units in a column = \( ac \), and as there are \( b \) columns altogether, therefore the total number of units = \( b \times (ac) \) ... ... ... ... \((\gamma)\)

Hence, from \((a)\), \((\beta)\) and \((\gamma)\) we have

\[(ab) \times c = a \times (bc) = b \times (ac).\]

3. **Formulae to be committed to memory.**

1. \((a + b)^2\) = \(a^2 + 2ab + b^2\).
2. \((a - b)^2\) = \(a^2 - 2ab + b^2\).
3. \((a + b)(a - b)\) = \(a^2 - b^2\).
4. \((a + b)^3\) = \(a^3 + 3a^2b + 3ab^2 + b^3\)
   
   = \(a^3 + b^3 + 3ab(a + b)\).
5. \((a + b + c)^3\) = \(a^3 + b^3 + c^3 + 3a^2(b + c) + 3b^2(a + c) + 3c^2(a + b) + 6abc\).
6. \((a - b)^3\) = \(a^3 - 3a^2b + 3ab^2 - b^3\)
   
   = \(a^3 - b^3 - 3ab(a - b)\).
7. \(a^3 + b^3\) = \((a + b)^3 - 3ab(a + b)\)
   
   = \((a + b)(a^2 - ab + b^2)\).
8. \(a^3 - b^3\) = \((a - b)^3 + 3ab(a - b)\)
   
   = \((a - b)(a^2 + ab + b^2)\).
9. \((x + a)(x + b)\) = \(x^2 + (a + b)x + ab\).
10. \((x - a)(x + b)\) = \(x^2 + (b - a)x - ab\).
11. \((x - a)(x - b)\) = \(x^2 - (a + b)x + ab\).
12. \((x + a)(x + b)(x + c)\) = \(x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc\).
13. \((x - a)(x - b)(x - c)\) = \(x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc\).
14. \((a + b + c)^2\) = \(a^2 + b^2 + c^2 + 2ab + 2ac + 2bc\).
15. \[(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2a(b + c + d) + 2b(c + d) + 2cd.\]
16. \[(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.\]
17. \[(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.\]
18. \[a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc).\]
19. \[a^2(b - c) + b^2(c - a) + c^2(a - b) = (a - b)(a - c)(b - c)\]
\[= -(a - b)(a - c)(c - a).\]
20. \[ab(a - b) + bc(b - c) + ca(c - a) = (a - b)(a - c)(b - c)\]
\[= -(a - b)(b - c)(c - a).\]

4. The expression \(x^n - a^n\) is divisible by \(x - a\) for all positive integral values of \(n\).

Since \[x^n - a^n = x^n - ax^{n-1} + ax^{n-1} - a^n\]
\[= x^{n-1}(x - a) + a(x^{n-1} - a^{n-1}),\]
we have \[\frac{x^n - a^n}{x - a} = x^{n-1} + a\cdot\frac{x^{n-1} - a^{n-1}}{x - a},\]
which shews that \(x - a\) will divide \(x^n - a^n\), if it divides \(x^{n-1} - a^{n-1}\).

Hence, \(x - a\) will divide \(x^4 - a^4\), because we know that it divides \(x^3 - a^3\) (i.e., \(x^4 - a^4 = x^{4-1} - a^{4-1}\)); and since \(x - a\) divides \(x^4 - a^4\) (i.e., \(x^5 - a^5 - a^{n-1}\)), therefore it will divide \(x^5 - a^5\); and so on.

Thus for all positive integral values of \(n\), \(x^n - a^n\) is divisible by \(x - a\).

Again, since \(x^n + a^n = (x^n - a^n) + 2a^n\), of which \(x^n - a^n\) is divisible by \(x - a\) and \(2a^n\) is not, \(:\) \(x^n + a^n\) is not divisible by \(x - a\).

Thus when \(n\) is a positive integer,
\[x - a\text{ always divides } x^n - a^n \]
\[\text{ but never divides } x^n + a^n \]
\[\text{ ... ... (A)}\]

**Cor. 1.** \((x + a)\) divides \(x^n - a^n\) only when \(n\) is an even integer.

For, when \(n\) is even, \((-a)^n = a^n\), and \(\therefore x^n - a^n = x^n - (-a)^n\); \(\therefore\)
when \(n\) is odd, \((-a)^n = -a^n\), and \(\therefore x^n - a^n = x^n + (-a)^n\); \(\therefore\)
also \(x + a = x - (-a).\)
Now, from (A), we know that \(x - (-a)\) divides \(x^n - (-a)^n\) but not \(x^n + (-a)^n\). Hence \(x + a\) divides \(x^n - a^n\) when \(n\) is even, but not when \(n\) is odd; i.e., \(x + a\) divides \(x^n - a^n\) only when \(n\) is an even integer.

**Cor. 2.** \(x + a\) divides \(x^n + a^n\) only when \(n\) is an odd integer.

For, when \(n\) is odd, \((-a)^n = -a^n\), and \(\therefore \ x^n + a^n = x^n - (-a)^n\).

When \(n\) is even, \((-a)^n = a^n\), and \(\therefore \ x^n + a^n = x^n + (-a)^n\); also \(x + a = x - (-a)\).

Now, from (A), we know that \(x - (-a)\) divides \(x^n - (-a)^n\) but not \(x^n + (-a)^n\). Hence, \(x + a\) divides \(x^n + a^n\) when \(n\) is odd, but not when \(n\) is even; i.e., \(x + a\) divides \(x^n + a^n\) only when \(n\) is an odd integer.

Thus we have obtained the following results:

\[
\begin{align*}
& x - a \text{ divides } x^n - a^n \text{ always, } \\
& x^n + a^n \text{ never. } \\
& x + a \text{ divides } x^n - a^n \text{ only when } n \text{ is even, } \\
& x^n + a^n \text{ only when } n \text{ is odd. }
\end{align*}
\]

5. **Examples in Factorisation.**

**Example 1.** Resolve into factors \(a^4 + a^2b^2 + b^4\).

\[
a^4 + a^2b^2 + b^4 = (a^4 + 2a^2b^2 + b^4) - a^2b^2 = (a^2 + b^2)^2 - (ab)^2 = \{(a^2 + b^2) + ab\}/\{(a^2 + b^2) - ab\} = (a^2 + ab + b^2)(a^2 - ab + b^2).
\]

**Example 2.** Resolve into factors \(x^4 + 4\).

\[
x^4 + 4 = (x^4 + 4x^2 + 4) - 4x^2 = (x^2 + 2)^2 - (2x)^2 = \{(x^2 + 2) + 2x\}/\{(x^2 + 2) - 2x\} = (x^2 + 2x + 2)(x^2 - 2x + 2).
\]

**Example 3.** Resolve into factors \(x^2 + 2xy - 8y^2 - 4z^2 + 12yz\).

The given expression

\[
= (x^2 + 2xy + y^2) - (9y^2 + 4z^2 - 12yz) = (x + y)^2 - (3y - 2z)^2 = \{(x + y) + (3y - 2z)\}/\{(x + y) - (3y - 2z)\} = (x + 4y - 2z)(x - 2y + 2z).
\]
Example 4. Resolve into factors \(2a^2 + 5ab - 12b^2\).

\[
2a^2 + 5ab - 12b^2 = 2 \left( a^2 + \frac{5}{2}ab - 6b^2 \right)
\]
\[
= 2 \left\{ \left( a + \frac{5}{4}b \right)^2 - \left( \frac{15}{16}b \right)^2 \right\}
\]
\[
= 2 \left\{ \left( a + \frac{5}{4}b \right) + \frac{11}{4}b \right\} \left\{ a + \frac{5}{4}b - \frac{11}{4}b \right\}
\]
\[
= 2(a + 4b)(a - \frac{3}{2}b)
\]
\[
= (a + 4b)(2a - 3b).
\]

Example 5. Resolve into factors

\[
4(x^2 + 2x + 5)^2 + 17(x^2 + 2x + 5)(x^2 + 6x) + 4(x^2 + 6x)^2.
\]

Putting \(a\) for \(x^2 + 2x + 5\) and \(b\) for \(x^2 + 6x\), the given expression becomes \(4a^2 + 17ab + 4b^2\); and it is easy to see that \(4a^2 + 17ab + 4b^2 = (a + 4b)(4a + b)\).

Hence, the given expression

\[
= \{ (x^2 + 2x + 5) + 4(x^2 + 6x) \} \{ 4(x^2 + 2x + 5) + (x^2 + 6x) \}
\]
\[
= (5x^2 + 26x + 5)(5x^2 + 14x + 20)
\]
\[
= (x + 5)(5x + 1)(5x^2 + 14x + 20).
\]

Example 6. Resolve into factors

\[
6x^2 + 17xy + 7y^2 - 2x - 23y - 20.
\]

The given expression arranged according to descending powers of \(x\)

\[
= 6x^2 + (17y - 2)x + (7y^2 - 23y - 20)
\]
\[
= 6 \left\{ x^2 + \frac{1}{6}(17y - 2)x + \frac{1}{6}\left(7y^2 - 23y - 20\right) \right\}
\]
\[
= 6 \left\{ x^2 + \frac{1}{6}(17y - 2)x + \frac{1}{144}\left(17y - 2\right)^2
\]
\[
- \frac{1}{144}\left(17y - 2\right)^2 + \frac{1}{6}\left(7y^2 - 23y - 20\right) \right\}
\]
\[
= 6 \left[ \left\{ x + \frac{1}{13}(17y - 2) \right\}^2 - \frac{1}{144}\left(289y^2 - 68y + 4 \right)
\]
\[
- 24(7y^2 - 23y - 20) \right\}
\]
\[
= 6 \left[ \left\{ x + \frac{1}{13}(17y - 2) \right\}^2 - \frac{131}{144}(y + 2)^2 \right]\]
\[ = 6\left[ \left\{ x + \frac{1}{12}(17y - 2) \right\} + \frac{11}{12}(y + 2) \right] \\
\times \left[ \left\{ x + \frac{1}{12}(17y - 2) \right\} - \frac{11}{12}(y + 2) \right] \]
\[ = 6\left( x + \frac{7}{3}y + \frac{5}{3} \right)\left( x + \frac{1}{2}y - 2 \right) \]
\[ = 3\left( x + \frac{7}{3}y + \frac{5}{3} \right) \times 2\left( x + \frac{1}{2}y - 2 \right) \]
\[ = (3x + 7y + 5)(2x + y - 4). \]

**Example 7.** Resolve into factors \( x^3 + 7x^2 - 21x - 27. \)

The given expression
\[ = (x^3 - 27) + (7x^2 - 21x) \]
\[ = (x - 3)(x^2 + 3x + 9) + 7x(x - 3) \]
\[ = (x - 3)(x^2 + 3x + 9) + 7x \]
\[ = (x - 3)(x^2 + 10x + 9) \]
\[ = (x - 3)(x + 9)(x + 1). \]

**Example 8.** Resolve into factors
\((a + b + c)(ab + bc + ca) - abc.\)

Arranging the expression within brackets according to powers of \(a\) we have the given expression
\[ = \{a + (b + c)\}\{a(b + c) + bc\} - abc \]
\[ = a^2(b + c) + a(b + c) + bc(b + c) \]
\[ = (b + c)\{a^2 + a(b + c) + bc\} \]
\[ = (b + c)(a + b)(a + c). \]

**Example 9.** Resolve into factors
\[ a^3(b - c) + b^3(c - a) + c^3(a - b). \]
\[ a^3(b - c) + b^3(c - a) + c^3(a - b) \]
\[ = a^2(b - c) - a(b^3 - c^3) + bc(b^2 - c^2) \quad \text{[arranged according to powers of } a]\]
\[ = (b - c)\{a^3 - a(b^2 + bc + c^2) + bc(b + c)\} \]
\[ = (b - c)\{-b^2(a - c) - bc(a - c) + a(a^2 - c^2)\} \quad \text{[arranged according to powers of } b]\]
\[ = (b - c)(a - c)\{ -b^2 - bc + a(a + c)\} \]
\[ = (b - c)(a - c)\{c(a - b) + (a^2 - b^2)\} \quad \text{[arranged according to powers of } c]\]
\[ = (b - c)(a - c)(a - b)(c + a + b). \]
Example 10. Resolve into factors
\[ a^3(b^2 - c^2) + b^3(c^2 - a^2) + c^3(a^2 - b^2). \]

In this expression also the letters occur in cyclic order and we can at once proceed as in the last example.

\[
\begin{align*}
&= a^3(b^2 - c^2) - a^2(b^3 - c^3) + b^2c^2(b - c) \\
&\quad [\text{arranged according to powers of } a] \\
&= (b - c)[a^3(b + c) - a^2(b^2 + bc + c^2) + b^2c^2] \\
&= (b - c)[a^3(b + c) - b^2(a^2 - c^2) + ba^2(a - c) + a^2c(a - c)] \\
&\quad [\text{arranged according to powers of } b] \\
&= (b - c)(a - c)(a + b)[c(a + b) + ab(a - b)] \\
&\quad [\text{arranged according to powers of } c] \\
&= (b - c)(a - c)(a + b)(ab + bc + ca).
\end{align*}
\]

Example 11. Find the quotient of \(a^3 + b^3 + c^3 - 3abc\) by \(a + b + c\).

Since \(b^3 + c^3 = (b + c)^3 - 3bc(b + c)\), we have

\[
\begin{align*}
&= a^3 + b^3 + c^3 - 3abc \\
&\quad [\text{arranged according to powers of } a + b + c].
\end{align*}
\]

Hence, the required quotient \(a^2 + b^2 + c^2 - ab - ac - bc\).

Example 12. Resolve into factors \(x^3 + 7x^2 + 14x + 8\).

On inspection it is observed that we can split up the given expression into parts each of which is divisible by \(x + 2\) in either of the two following ways:

(i) \((x^3 + 8) + 7x(x + 2) ;\)

(ii) \(x^2(x + 2) + 5x(x + 2) + 4(x + 2)\).

Hence, choosing the latter way, we have

\[
\begin{align*}
x^3 + 7x^2 + 14x + 8 &= (x + 2)(x^2 + 5x + 4) \\
&= (x + 2)(x + 1)(x + 4).
\end{align*}
\]
**Example 13.** Resolve into factors $8x^3 + 16x - 9$.

We find that the given expression can be split up into parts divisible by $2x - 1$ in either of the two following ways:

(i) $$(8x^3 - 1) + 8(2x - 1);$$
(ii) $$2x(4x^2 - 1) + 9(2x - 1).$$

Hence, choosing the former way, we have

$$8x^3 + 16x - 9 = (8x^3 - 1) + 8(2x - 1)$$
$$= (2x - 1)(4x^2 + 2x + 1) + 8$$
$$= (2x - 1)(4x^2 + 2x + 9).$$

**Example 14.** Resolve into factors $a^3 + 7ab^2 - 22b^3$.

We find that the expression can be split up into parts each of which is divisible by $a - 2b$ in either of the two following ways:

(i) $$a^3 - 8b^3 + 7b^2(a - 2b);$$
(ii) $$a(a^2 - 4b^2) + 11b^2(a - 2b).$$

Hence, choosing the former way, we have

$$a^3 + 7ab^2 - 22b^3 = (a^3 - 8b^3) + 7b^2(a - 2b)$$
$$= (a - 2b)(a^2 + 2ab + 4b^2) + 7b^2$$
$$= (a - 2b)(a + 2b + 11b^2).$$

**Example 15.** Resolve into factors $(a^2 - b^2)(x^2 - y^2) + 4abxy$.

The given expression

$$= a^2x^2 - a^2y^2 - b^2x^2 + b^2y^2 + 4abxy$$
$$= (a^2x^2 + b^2y^2 + 2abxy) - (a^2y^2 + b^2x^2 - 2abxy)$$
$$= (ax + by)^2 - (ay - bx)^2$$
$$= (ax + by + (ay - bx))(ax + by) - (ay - bx))$$
$$= (a - b)x + (a + b)y\{(a + b)x - (a - b)y\}}.$$}

**Example 16.** Resolve into factors $x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4$.

The given expression

$$= (x^4 + 2x^3y + y^4) + x^2y^2 + (2x^3y + 2xy^3)$$
$$= (x^2 + y^2)^2 + (xy)^2 + 2(xy)(x^2 + y^2)$$
$$= \{(x^2 + y^2) + xy\}^2$$
$$= (x^2 + xy + y^2)^2.$$
Example 17. Resolve into factors
\[(x - 1)(x - 2)(x + 3)(x + 4) + 4.\]
\[(x - 1)(x - 2)(x + 3)(x + 4)
= \{(x - 1)(x + 3)\}(x - 2)(x + 4)
= (x^2 + 2x - 3)(x^2 + 2x - 8).
Hence, putting \(z\) for \(x^2 + 2x\), the given expression
\[= (z - 3)(z - 8) + 4\]
\[= z^2 - 11z + 28\]
\[= (z - 4)(z - 7)\]
\[= (x^2 + 2x - 4)(x^2 + 2x - 7).\]

Example 18. If \(x + y = a\) and \(xy = b^2\), find the value of
\[(i)\] \(x^4 + y^4\) and \((ii)\) \(x^3 - x^2y - xy^2 + y^3\) in terms of \(a\) and \(b\).
\[(i)\] \(x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2\)
\[= \{(x + y)^2 - 2xy\}^2 - 2x^2y^2,\]
and \(\therefore\) the required value
\[= (a^2 - 2b^2)^2 - 2b^4\]
\[= a^4 - 4a^2b^2 + 2b^4.\]
\[(ii)\] \(x^3 - x^2y - xy^2 + y^3\)
\[= x^2(x - y) - y^2(x - y)\]
\[= (x - y)(x^2 - y^2)\]
\[= (x - y)^2(x + y)\]
\[= \{(x + y)^2 - 4xy\}(x + y)\]
\[= (a^2 - 4b^2)a.\]

Example 19. Find the value of \(x^4 - x^3 + x^2 + 2\), when \(x^2 + 2 = 2x.\)
\[x^4 - x^3 + x^2 + 2 = (x^4 + x^3 + x^2) - 2(x^3 - 1)\]
\[= x^2(x^2 + x + 1) - 2(x - 1)(x^2 + x + 1)\]
\[= (x^3 + x + 1)(x^2 - 2(x - 1))\]
\[= (x^3 + x + 1)(x^2 - 2x + 2),\]
and \(\therefore\) the required value
\[= (x^3 + x + 1) \times 0 = 0.\]

Example 20. Find the value of
\[a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2,\] when \(a + b = c.\)
The given expression
\[ \begin{align*}
&= a^4 - 2(b^2 + c^2)a^2 + b^4 + c^4 - 2b^2c^2 \\
&= \{a^4 - 2(b^2 + c^2)a^2 + (b^2 + c^2)^2\} \\
&\quad - (b^2 + c^2)^2 + b^4 + c^4 - 2b^2c^2 \\
&= \{a^2 - (b^2 + c^2)^2\}^2 - 4b^2c^2 \\
&= \{(a^2 - b^2 - c^2) + 2bc\}\{(a^2 - b^2 - c^2) - 2bc\} \\
&= \{a^2 - (b - c)^2\}\{a^2 - (b + c)^2\} \\
&= (a + b - c)(a - b + c)(a + b + c)(a - b - c),
\end{align*}\]
and \(\therefore\) \(= 0\), when \(a + b = c\).

6. The ordinary method of finding the H.C.F. of two multinomial expressions which have no monomial factors.

Let \(A\) and \(B\) stand for two such expressions both arranged according to descending powers of some common letter, and let the index of the highest power of that letter in \(A\) be not less than the index of the highest power of that letter in \(B\).

Divide \(A\) by \(B\), and let \(Q\) be the quotient and \(C\) the remainder.

Then we must have \(C = A - BQ \ldots \ldots (1)\)
or,
\[A = BQ + C \ldots \ldots (2)\]

From \((1)\) it is clear that every common factor of \(A\) and \(B\) is a factor of \(C\) [for if \(A = pmA\) and \(B = pQB\), we have \(C = p(a - bQ)\)]. Hence, if \(H\) denote the H.C.F. of \(A\) and \(B\), \(H\) also is a factor of \(C\), and is therefore a common factor of \(B\) and \(C\).

It is clear therefore that the H.C.F. of \(B\) and \(C\) is either \(H\) or an expression of higher dimensions than \(H\) \ldots \ldots \((a)\)

Now, from \((2)\) it is evident that every common factor of \(B\) and \(C\) is a factor of \(A\) and is therefore a common factor of \(B\) and \(A\). Hence, the H.C.F. of \(B\) and \(C\) also is a common factor of \(B\) and \(A\), and therefore cannot be of higher dimensions than \(H\).

Hence, from \((a)\), the H.C.F. of \(B\) and \(C\) is \(H\).

Thus the H.C.F. of \(B\) and \(C\) is the H.C.F. required.

Similarly, if \(B\) be divided by \(C\), and \(D\) be the new remainder, the H.C.F. of \(C\) and \(D\) is the same as the H.C.F. of \(B\) and \(C\) and is therefore the H.C.F. required.

Now, divide \(C\) by \(D\) and let there be no remainder. Then \(D\) is the H.C.F. of \(C\) and \(D\) and is therefore the H.C.F. required.
Cor. 1. As the H. C. F. of any divisor and the corresponding dividend is the H. C. F. required, it is clear that, for the sake of convenience, either of them may be multiplied or divided by any monomial expression which is not a factor of the other.

Cor. 2. In dividing A by B if we stop before the complete quotient is obtained so that q is the partial quotient and C' the corresponding remainder, then the H. C. F. of B and C', just as the H. C. F. of B and C, is the H. C. F. required. Hence, by Cor. 1. in dividing C' by B (or if convenient, B by C' when C' is not of higher degree than B) we can multiply or divide either of them, if necessary, by any monomial expression which is not a factor of the other.

Hence, we have the following rule:

Arrange the two expressions according to descending powers of some common letter; divide the expression which is of higher degree in that letter by the other, or if they be of the same degree, either of them by the other; if there be any remainder take it for a new divisor and the preceding divisor for the dividend, and continue the process till there is no remainder. The last divisor will be the H. C. F. required. Of any divisor and the corresponding dividend either may be multiplied or divided by any number which is not a factor of the other.

Example. Find the H. C. F. of

\[4x^4 + 11x^3 + 27x^2 + 17x + 5 \text{ and } 6x^4 + 14x^3 + 36x^2 + 14x + 10\]

The 2nd expression = \(2(3x^4 + 7x^3 + 18x^2 + 7x + 5)\), but 2 is not a factor of the 1st expression. Hence, the H. C. F. required is the H. C. F. of the 1st expression and \(3x^4 + 7x^3 + 18x^2 + 7x + 5\).

\[
\frac{4x^4 + 11x^3 + 27x^2 + 17x + 5}{3} \]

\[
3x^4 + 7x^3 + 18x^2 + 7x + 5 \quad \frac{12x^4 + 33x^3 + 81x^2 + 51x + 15}{4} \]

\[
12x^4 + 28x^3 + 72x^2 + 28x + 20 \]

\[
5x^3 + 9x^2 + 23x - 5
\]
3x^4 + 7x^3 + 18x^2 + 7x + 5
5

(15x^4 + 35x^3 + 90x^2 + 35x + 25) (3x)
15x^4 + 27x^3 + 69x^2 - 15x


8x^3 + 21x^2 + 50x + 25
5

(40x^3 + 105x^2 + 250x + 125) (8)
40x^3 + 72x^2 + 184x - 40

33(33x^2 + 66x + 165)
x^2 + 2x + 5

x^2 + 2x + 5 \div 5x^3 + 9x^2 + 23x - 5 (5x - 1)

\begin{align*}
5x^3 + 10x^2 + 25x & - x^2 - 2x - 5 \\
x^2 + 2x + 5 & - x^2 - 2x - 5
\end{align*}

Thus the required H. C. F. = x^2 + 2x + 5.

7. An important principle.

If A and B denote two expressions having no monomial factors and if m, n, p, q be any four numerical quantities such that mq - np is not equal to zero, then the H. C. F. of A and B is the same as the H. C. F. of mA + nB and pA + qB, numerical common factors, if any, being left out. This may be proved as follows:

Let H denote the H. C. F. of A and B, and H' the H. C. F. of mA + nB and pA + qB after removal from them of any numerical common factor that may occur.

Now, since every common factor of A and B is a factor of mA + nB and also of pA + qB, therefore H is a common factor of mA + nB and pA + qB.

Hence, H' is either equal to H or is an expression of higher dimensions than H ... ... ... (a)

Again, since q(mA + nB) - n(pA + qB) = (mq - np)A, and m(pA + qB) - p(mA + nB) = (mq - np)B, it is clear that every common factor of mA + nB and pA + qB is a factor of (mq - np)A, and also of (mq - np)B. Hence, as mq - np is only a numerical quantity, every common factor of those
expressions other than numerical must be a factor of \( A \) as well as of \( B \). Hence \( H' \) is a common factor of \( A \) and \( B \) and therefore cannot be of higher dimensions than \( H \).

Hence, by (a), \( H' = H \), which proves the proposition.

**Cor. 1.** The H. C. F. of \( A \) and \( B \) is the same as the H. C. F. of \( A + B \) and \( A - B \). Here \( m = 1 \), \( n = 1 \), \( p = 1 \) and \( q = -1 \).

**Cor. 2.** The H. C. F. of \( A \) and \( B \) is the same as the H. C. F. of \( A \pm B \) and \( B \); here \( m = 1 \), \( n = \pm 1 \), \( p = 0 \) and \( q = 1 \). Similarly, it is the same as the H. C. F. of \( A \pm B \) and \( A \).

**Example 1.** Find the H. C. F. of
\[ x^4 + x^3 - 5x^2 - 3x + 2 \text{ and } x^4 - 3x^3 + x^2 + 3x - 2. \]
Let \( A = x^4 + x^3 - 5x^2 - 3x + 2 \),
and \( B = x^4 - 3x^3 + x^2 + 3x - 2 \).
Then
\[
A + B = 2x^4 - 2x^3 - 4x^2 \\
= 2x^2(x^2 - x - 2),
\]
and
\[
A - B = 4x^3 - 6x^2 - 6x + 4 \\
= 2(2x^3 - 3x^2 - 3x + 2).
\]
Hence, by Cor. 1, the required H. C. F. is the H. C. F. of \( x^2(x^2 - x - 2) \) and \( 2x^3 - 3x^2 - 3x + 2 \), and therefore of \( x^2 - x - 2 \) and \( 2x^3 - 3x^2 - 3x + 2 \).
Let \( A' = x^2 - x - 2 \),
and \( B' = 2x^3 - 3x^2 - 3x + 2 \).
Then
\[
A' + B' = 2x^3 - 2x^2 - 4x \\
= 2x(x^2 - x - 2).
\]
Hence, the required H. C. F.
\[
= \text{the H. C. F. of } A' \text{ and } A' + B' \quad [\text{Cor. 2.}]
\]
\[
= x^2 - x - 2.
\]

**Example 2.** Find the H. C. F. of
\[ 4x^4 + 11x^3 + 27x^2 + 17x + 5 \text{ and } 3x^4 + 7x^3 + 18x^2 + 7x + 5. \]
Let \( A = 4x^4 + 11x^3 + 27x^2 + 17x + 5 \),
and \( B = 3x^4 + 7x^3 + 18x^2 + 7x + 5 \).
Then
\[
A - B = -x^4 + 4x^3 + 9x^2 + 10x \\
= x(3x^3 + 4x^2 + 9x + 10),
\]
and \(3A - 4B = 5x^3 + 9x^2 + 23x - 5\).

Hence, the H. C. F. of \(x^3 + 4x^2 + 9x + 10\) and \(5x^3 + 9x^2 + 23x - 5\) is the H. C. F. required.

Let \(A' = x^3 + 4x^2 + 9x + 10\),

and \(B' = 5x^3 + 9x^2 + 23x - 5\).

Then \(A' + 2B' = 11x^3 + 22x^2 + 55x\)

\[= 11x(x^2 + 2x + 5),\]

and \(5A' - B' = 11x^2 + 22x + 55\)

\[= 11(x^2 + 2x + 5).\]

Hence, the H. C. F. required is the H. C. F. of \(x(x^2 + 2x + 5)\)
and \(x^2 + 2x + 5\), and is therefore \(x^2 + 2x + 5\).

8. The H. C. F. of three or more expressions whose factors cannot be easily found.

Let \(A, B, C\) stand for any three expressions of which the H. C. F. is to be found.

Let \(G\) denote the H. C. F. of \(A\) and \(B\) and \(H\) that of \(G\) and \(C\).

Then \(G\) being the product of all the elementary common factors of \(A\) and \(B\), every factor of \(G\) is a common factor of \(A\) and \(B\), and therefore every common factor of \(G\) and \(C\) is a common factor of \(A, B\) and \(C\).

Hence, \(H\) also is a common factor of \(A, B\) and \(C\). Therefore the H. C. F. required is either \(H\) or an expression of higher dimensions than \(H \ldots \ldots \ldots \ldots \ldots \) (\(\beta\))

But, since every common factor of \(A\) and \(B\) is a factor of \(G\), every common factor of \(A, B\) and \(C\) is a common factor of \(G\) and \(C\). Hence, the H. C. F. required is a common factor of \(G\) and \(C\) and therefore cannot be of higher dimensions than \(H\).

Hence, by (\(\beta\)), the H. C. F. required = \(H\).

By a similar reasoning it follows that if \(D\) be a fourth expression, then the H. C. F. of \(H\) and \(D\) is the H. C. F. of \(A, B, C\) and \(D\).

Thus we have the following rule:—To find the H. C. F. of any number of expressions \(A, B, C, D, \&c.,\) first find the H. C. F. of \(A\) and \(B\), then the H. C. F. of this result and \(C\), and so on; the result obtained last of all is the H. C. F. required.
9. L. C. M. of two expressions whose factors are not obvious by inspection.

Let A and B stand for two such expressions, and suppose their H. C. F. is found to be \( H \).

Divide A and B by \( H \) and let the respective quotients be \( a \) and \( b \). Then we have

\[
\begin{align*}
A &= aH \\
B &= bH
\end{align*}
\]

Hence, since \( a \) and \( b \) have no common factor, every common multiple of \( A \) and \( B \) must necessarily contain \( a \times H \times b \) as a factor.

Hence, the L. C. M. required = \( aHb \).

But

\[
\begin{align*}
aHb &= a(Hb) \\
&= \frac{A}{H} \times B
\end{align*}
\]

or

\[
\begin{align*}
(aH)b &= A \times \frac{B}{H}
\end{align*}
\]

Hence, the required L. C. M. = \( \frac{A}{H} \times B \), or = \( A \times \frac{B}{H} \).

Thus, to find the L. C. M. of any two expressions we have to divide one of them by their H. C. F. and multiply the quotient by the other.

**Cor.** If \( L \) denote the L. C. M. of \( A \) and \( B \) we have \( L \times H = A \times B \); that is, the product of the L. C. M. and H. C. F. of any two expressions is equal to the product of those expressions.

*Note.* If any two expressions have no common factor, their L. C. M. is evidently equal to their product.

10. L. C. M. of three or more expressions whose factors are not obvious by inspection.

Let \( A, B, C \) stand for three such expressions; to find their L. C. M.

Let \( L \) denote the L. C. M. of \( A \) and \( B \), and \( M \) that of \( L \) and \( C \).

Then evidently every common multiple of \( L \) and \( C \) is a common multiple of \( A, B, C \); ... ... ... \( (1) \)

also every common multiple of \( A, B, C \) is a common multiple of \( L \) and \( C \). ... ... ... \( (2) \)
From (1) $M$ is a common multiple of $A$, $B$, $C$. Hence, either $M$ or an expression of a lower degree than $M$ is the L.C.M. of $A$, $B$, $C$.

But an expression of a lower degree than $M$ cannot be the L.C.M. of $A$, $B$, $C$; because, from (2), the L.C.M. of $A$, $B$, $C$ is a common multiple of $L$ and $C$.

Hence, the required L.C.M. = $M$.

Thus, to find the L.C.M. of any number of expressions $A$, $B$, $C$, $D$, etc., we have first to find the L.C.M. of $A$ and $B$, then the L.C.M. of this result and $C$, and so on; the last result thus obtained is the L.C.M. required.

**Example.** If $h_1$, $h_2$, $h_3$ be the highest common factors, and $l_1$, $l_2$, $l_3$ the lowest common multiples of $B$ and $C$, $C$ and $A$, $A$ and $B$ respectively; and if $H$ be the highest common factor, and $L$ the lowest common multiple of $A$, $B$ and $C$; prove that

(i) \[ h_1 h_2 h_3 l_1 l_2 l_3 = (ABC)^2, \]

(ii) \[ \frac{L}{H} = \frac{ABC}{h_1 h_2 h_3}. \]

Since $h_1$ is the H.C.F., and $l_1$ the L.C.M., of $B$ and $C$, we must have

\[ h_1 l_1 = BC. \quad [\text{Cor. Art. 9.}] \]

Similarly \[ h_2 l_2 = CA, \] and \[ h_3 l_3 = AB. \]

Hence, \[ h_1 h_2 h_3 l_1 l_2 l_3 = (ABC)^2. \]

Again, since $H$ is the H.C.F. of $A$, $B$ and $C$, it is a factor common to $B$ and $C$ and is therefore a factor of $h_1$. Hence we must have

\[ h_1 = H f_1 \quad \ldots \ldots \ldots \ldots \ldots \quad (1) \]

where $f_1$ is the quotient of $h_1$ by $H$.

Similarly we have

\[ h_2 = H f_2 \quad \ldots \ldots \ldots \ldots \ldots \quad (2) \]

and \[ h_3 = H f_3 \quad \ldots \ldots \ldots \ldots \ldots \quad (3) \]

It is clear that no two of the three quantities $f_1$, $f_2$, $f_3$ have a common factor; for instance if $f_2$ and $f_3$ had a common factor, say $k$, then $H k$ would be a factor of $h_2$ as well as of $h_3$ and would therefore be a common factor of $C$, $A$, $B$, which is impossible.
Now, from hypothesis, it is evident that both $h_1$ and $h_2$ are factors of $C$; hence from (1) and (2) we must have

$$C = Hf_1f_2.p.$$  
(4)

where $p$ is the quotient of $C$ by $Hf_1f_2$.

Similarly we must have

$$A = Hf_2f_3.q, \quad \text{ . . . . . . . (5)}$$

and

$$B = Hf_3f_1.r \quad \text{ . . . . . . . (6)}$$

From these relations it is easy to see that the following pairs of quantities cannot have a common factor:—$(p, q), (p, r),$ $(q, r), (p f_3), (q f_1)$ and $(r f_2)$. For instance if $p$ and $f_3$ had a common factor, say $k'$, then $Hf_2k'$, i.e., $h_2k'$ would be, as evident from (4) and (5), a common factor of $C$ and $A$, which is impossible.

Now, from (4), (5) and (6) it is clear that

$$L = Hf_1f_2f_3.pqr;$$

and we have also

$$ABC = H^3.(f_1f_2f_3)^2.pqr.$$  

Hence,

$$\frac{L}{ABC} = \frac{Hf_1f_2f_3.pqr}{H^3.(f_1f_2f_3)^2.pqr}$$

$$= \frac{H}{H^3f_1f_2f_3}$$

$$= \frac{H}{(Hf_1).(Hf_2).(Hf_3)}$$

$$= \frac{H}{h_1h_2h_3}$$

and

$$\therefore \quad \frac{L}{H} = \frac{ABC}{h_1h_2h_3}.$$  

11. A few examples in fractions.

**Example 1.** Reduce to its simplest form

$$\frac{bc}{(a-b)(a-c)} + \frac{ac}{(b-a)(b-c)} + \frac{ab}{(c-a)(c-b)}.$$  

Since $b-a = -(a-b)$, and $(c-a)(c-b) = [-(a-c)] \times [-(b-c)] = (a-c)(b-c)$, the given expression

$$= \frac{bc}{(a-b)(a-c)} + \frac{ac}{(a-b)(b-c)} + \frac{ab}{(a-c)(b-c)}$$

$$= \frac{bc(a+b)(b-c) - ac(b+d)(a-c) + ab(c+d)(a-b)}{(a-b)(a-c)(b-c)}.$$  

Now, the numerator \( = abc((b-c)-(a-c)+(a-b)) \)
\[ + d\{bc(b-c)-ac(a-c)+ab(a-b)\} \]
\[ = d\{bc(b-c)-ac(a-c)+ab(a-b)\} \]
\[ = d\{a^2(b-c)+b^2(c-a)+c^2(a-b)\} \]
\[ = d(a-b)(a-c)(b-c). \]

Hence, the given expression = \( d \).

**Example 2.** Simplify
\[
\frac{a^2}{(a-b)(a-c)(x+a)} + \frac{b^3}{(b-a)(b-c)(x+b)} + \frac{c^3}{(c-a)(c-b)(x+c)}
\]
The given expression
\[
\frac{a^2}{(a-b)(a-c)(x+a)} + \frac{b^3}{(b-a)(b-c)(x+b)} + \frac{c^3}{(c-a)(c-b)(x+c)}
\]
\[
= \frac{x^2(a^2(b-c)+b^2(c-a)+c^2(a-b))}{(a-b)(a-c)(b-c)(x+a)(x+b)(x+c)} + \frac{x^2(a^2(b-c)+b^2(c-a)+c^2(a-b))}{(a-b)(a-c)(b-c)(x+a)(x+b)(x+c)} + \frac{x^2(a^2(b-c)+b^2(c-a)+c^2(a-b))}{(a-b)(a-c)(b-c)(x+a)(x+b)(x+c)}
\]
Now, the numerator
\[
= a^2(b-c)x^2 + x^2(b+c) + bc + b^3(c-a)x^2 + x^2(c+a) + ca \]
\[ + c^2(a-b)x^2 + x^2(a+b) + ab \]
\[ = a^2(b-c)x^2 + b^2(c-a)x^2 + c^2(a-b)x^2 \]
\[ + x^2(a^2(b-c)+b^2(c-a)+c^2(a-b)) \]
\[ + abc(a(b-c)+b(c-a)+c(a-b)) \]
\[ = x^2(a^2(b-c)+b^2(c-a)+c^2(a-b)) \]
\[ = x^2(a-b)(a-c)(b-c). \]

Hence, the given expression = \( x^2 \)
\[
\frac{1}{(x+a)(x+b)(x+c)}
\]

**Example 3.** Simplify
\[
\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)}
\]
The given expression
\[
= \frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(a-c)(b-c)} \]
\[ = a^3(b-c)-b^3(a-c)+c^3(a-b) \]
\[ = (a-b)(a-c)(b-c)(a+b+c). \]

Hence, the given expression = \( a+b+c \).
12. Solutions of some equations.

Example 1. Solve \( \frac{3}{x-2} + \frac{5}{x-6} = \frac{8}{x+3} \).

Since \( \frac{8}{x+3} = \frac{3}{x+3} + \frac{5}{x+3} \)
we have \( \frac{3}{x-2} + \frac{5}{x-6} = \frac{3}{x+3} + \frac{5}{x+3} \)

Hence, by transposition,
\[
\frac{3}{x-2} - \frac{3}{x+3} = \frac{5}{x+3} - \frac{5}{x-6}
\]
or, \((x-2)(x+6) = (x+3)(x-6)\).

Multiplying both sides by \(x+3\), and dividing by 15,
we have \( \frac{1}{x-2} = \frac{-3}{x-6} \).
Hence, \( x-6 = -3(x-2) \);
\[\therefore \quad 4x = 12;\]
\[\therefore \quad x = 3.\]

Example 2. Solve \( \frac{8}{2x-1} + \frac{9}{3x-1} = \frac{7}{x+1} \).

We have \( \frac{8}{2x-1} + \frac{9}{3x-1} = \frac{4}{x+1} + \frac{3}{x+1} \).

Hence, \( \left\{ \frac{8}{2x-1} - \frac{4}{x+1} \right\} + \left\{ \frac{9}{3x-1} - \frac{3}{x+1} \right\} = 0 \)
or, \( \frac{12}{2x-1} + \frac{12}{3x-1} = 0 \).
Hence, \( \frac{1}{2x-1} + \frac{1}{3x-1} = 0 \).

Multiplying both sides by \((2x-1)(3x-1)\),
we have \( (3x-1) + (2x-1) = 0 \).
Therefore \( 5x = 2 \), or, \( x = \frac{2}{5} \).

Example 3. Solve \( \frac{a-c}{2b+x} + \frac{b-c}{2a+x} = \frac{a+b-2c}{a+b+x} \).
We have \[
\frac{a-c}{2b+x} + \frac{b-c}{2a+x} = \frac{(a-c)+(b-c)}{a+b+x}
\]
\[= \frac{a-c}{a+b+x} + \frac{b-c}{a+b+x}.
\]
Hence, by transposition,
\[(a-c)\left(\frac{1}{2b+x} - \frac{1}{a+b+x}\right) = (b-c)\left(\frac{1}{a+b+x} - \frac{1}{2a+x}\right),
\]
or,
\[(a-c)\frac{a-b}{(2b+x)(a+b+x)} = (b-c)\frac{a-b}{(a+b+x)(2a+x)}.
\]
Hence,
\[\frac{a-c}{2b+x} = \frac{b-c}{2a+x};
\]
\[\therefore (a-c)(2a+x) = (b-c)(2b+x),
\]
\[\therefore x[(a-c)-(b-c)] = 2b(b-c) - 2a(a-c)
\]
or,
\[x(a-b) = 2(b^2 - a^2) - 2c(b-a)
\]
\[= 2(b-a)(b+a-c)
\]
\[= 2(a-b)(c-a-b),
\]
\[\therefore x = 2(c-a-b).
\]

**Example 4.** Solve \[\frac{7x-55}{x-8} + \frac{2x-17}{x-9} = \frac{6x-71}{x-12} + \frac{3x-14}{x-5}.
\]

We have
\[\frac{7(x-8)+1}{x-8} + \frac{2(x-9)+1}{x-9} = \frac{6(x-12)+1}{x-12} + \frac{3(x-5)+1}{x-5};
\]
or,
\[\left(7 + \frac{1}{x-8}\right) + \left(2 + \frac{1}{x-9}\right) = \left(6 + \frac{1}{x-12}\right) + \left(3 + \frac{1}{x-5}\right);
\]
\[\therefore \frac{1}{x-8} + \frac{1}{x-9} = \frac{1}{x-12} + \frac{1}{x-5}.
\]
Hence, by transposition,
\[\frac{1}{x-8} - \frac{1}{x-5} = \frac{1}{x-12} - \frac{1}{x-9},
\]
or,
\[\frac{3}{(x-8)(x-5)} = \frac{3}{(x-12)(x-9)}
\]
\[\therefore (x-8)(x-5) = (x-12)(x-9)
\]
or,
\[x^2 - 13x + 40 = x^2 - 21x + 108,
\]
\[\therefore 8x = 68, \therefore x = 8\frac{1}{2}.
\]
Example 5. Solve \((x - 2a)^3 + (x - 2b)^3 = 2(x - a - b)^3\).

By transposition, we have

\((x - 2a)^3 - (x - a - b)^3 = (x - a - b)^3 - (x - 2b)^3\).

Putting \(X\) for \(x - 2a\), \(Y\) for \(x - 2b\), and \(Z\) for \((x - a - b)\), we have

\[X^3 - Z^3 = Z^3 - Y^3,\]
or,

\[(X - Z)(X^2 + XZ + Z^2) = (Z - Y)(Z^2 + ZY + Y^2)\].

But \(X - Z = Z - Y\), because each of them = \(b - a\).

\[
\therefore \quad X^2 + XZ + Z^2 = Z^2 + ZY + Y^2.\]

Hence, by transposition,

\[X^2 - Y^2 = Z(Y - X)\]

Removing the common factor \(X - Y\), which = \(2b - 2a\), we have

\[X + Y = -Z,\]
i.e., \((x - 2a) + (x - 2b) = -(x - a - b);\)

\[
\therefore \quad 3x = 3(a + b), \text{ and } \therefore \quad x = a + b.
\]

Example 6. Solve \[
\begin{align*}
4x - 3y + 2z &= 40 \quad (1) \\
5x + 9y - 7z &= 47 \quad (2) \\
9x + 8y - 3z &= 97 \quad (3)
\end{align*}
\]

Multiplying \((1)\) by 7, and \((2)\) by 2, we have

\[
\begin{align*}
28x - 21y + 14z &= 280 \\
and \quad 10x + 18y - 14z &= 94
\end{align*}
\]

Hence, by addition, \(38x - 3y = 374 \quad (4)\).

Again, multiplying \((1)\) by 3, and \((3)\) by 2, we have

\[
\begin{align*}
12x - 9y + 6z &= 120 \\
and \quad 18x + 16y - 6z &= 194
\end{align*}
\]

Hence, by addition, \(30x + 7y = 314 \quad (5)\).

Now, from \((4)\) and \((5)\), we have

\[
\begin{align*}
38x - 3y - 374 &= 0 \\
and \quad 30x + 7y - 314 &= 0
\end{align*}
\]

Hence, by cross multiplication,

\[
x = \frac{y}{3 \times 314 - 7(-374)} = \frac{(374)30 - (-314)38}{38 \times 7 - 30(-3)}
\]
or, \[ \frac{x}{942 + 2618} = \frac{y}{-11220 + 11932} = \frac{1}{266 + 90} \]

or, \[ \frac{x}{3560} = \frac{y}{712} = \frac{1}{356} \]

Therefore \( x = 10 \), and \( y = 2 \).

Substituting these values of \( x \) and \( y \) in (1), we have
\[ 40 - 6 + 2z = 40, \text{ whence } z = 3. \]

Thus we have \( x = 10, \ y = 2, \ z = 3 \).

**Example 7.** Solve \( \begin{cases} x + y + z = 0 \\ (b + c)x + (c + a)y + (a + b)z = 0 \\ bcx + cay + abz = 1 \end{cases} \)

Since \( (b + c)x + (c + a)y + (a + b)z = 0 \)
and \( x + y + z = 0 \)
therefore, by cross multiplication,
\[
\frac{x}{(c + a) - (a + b)} = \frac{y}{(a + b) - (b + c)} = \frac{z}{(b + c) - (c + a)}
\]

or, \( \frac{x}{c - b} = \frac{y}{a - c} = \frac{z}{b - a} \).

Supposing each of these fractions = \( k \), we have
\( x = k(c - b), \ y = k(a - c), \ z = k(b - a) \).

Substituting these values of \( x, y, z \) in the 3rd equation, we have \( kbc(c - b) + ca(a - c) + ab(b - a) = 1 \).

But \( bc(c - b) + ca(a - c) + ab(b - a) = bc(c - b) + a^2(c - b) - a(c^2 - b^2) = (c - b)(bc + a^2 - a(c + b)) = (c - b)(a - c)(a - b) \).

Thus, \( k(c - b)(a - c)(a - b) = 1 \);
\[ \therefore \ k = \frac{1}{(c - b)(a - c)(a - b)} \]

Hence, \( x = k(c - b) = \frac{1}{(a - c)(a - b)} \);
\( y = k(a - c) = \frac{1}{(c - b)(a - b)} \);
\( z = k(b - a) = \frac{1}{(c - b)(c - a)} \).
CHAPTER I.

SQUARE AND CUBE ROOTS.

1. Extraction of square roots by the application of the formula \(a^2 ± 2ab + b^2 = (a ± b)^2\).

Example 1. Find the square root of

\[
4 - 4c + 2b + c^2 - bc + \frac{b^2}{4}
\]

The given expression, arranged according to powers of \(b\),

\[
= \frac{b^2}{4} - b(c - 2) + (c^2 - 4c + 4)
\]

\[
= \left(\frac{b}{2}\right)^2 - 2\left(\frac{b}{2}(c - 2)\right) + (c - 2)^2
\]

\[
= \left(\frac{b}{2} - (c - 2)\right)^2
\]

Therefore the required root = \(\frac{b}{2} - c + 2\).

Example 2. Extract the square root of

\[(ab + ac + bc)^2 - 4abc(a + c)\]

The given expression

\[
= \{b(a + c) + ac\}^2 - 4abc(a + c)
\]

\[
= b^2(a + c)^2 + a^2c^2 - 2abc(a + c)
\]

\[
= \{b(a + c) - ac\}^2 = (ab - ac + bc)^2
\]

Therefore the required root = \(ab - ac + bc\).

Example 3. Extract the square root of

\[a^4 + b^4 + c^4 + d^4 - 2(a^2 + c^2)(b^2 + d^2) + 2a^2c^2 + 2b^2d^2\]

Arranging the given expression according to descending powers of \(a\), we have

\[a^4 - 2a^2(b^2 + d^2 - c^2) + \{b^4 + c^4 + d^4 - 2c^2(b^2 + d^2) + 2b^2d^2\}\]
and the expression within the braces, arranged according to descending powers of \( b \),
\[
\begin{align*}
&= b^4 - 2b^2(c^2 - d^2) + (c^4 + d^4 - 2c^2d^2) \\
&= b^4 - 2b^2(c^2 - d^2) + (c^2 - d^2)^2 \\
&= (b^2 - (c^2 - d^2))^2.
\end{align*}
\]

Hence, the given expression
\[
\begin{align*}
&= a^4 - 2a^2(b^2 - c^2 + d^2) + (b^2 - c^2 + d^2)^2 \\
&= (a^2 - (b^2 - c^2 + d^2))^2 \\
&= (a^2 - b^2 + c^2 - d^2)^2.
\end{align*}
\]

Therefore the required root \( = a^2 - b^2 + c^2 - d^2 \).

**Example 4.** Extract the square root of
\[
\frac{(a^2 + b^2)^2}{a^4 + b^4 - 2a^2b^2} + \frac{4ab}{a + b} \times \frac{a}{a - b}.
\]

The given expression
\[
\begin{align*}
&= \frac{(a^2 + b^2)^2}{(a^2 - b^2)^2} + \frac{4ab}{a^2 - b^2} \\
&= (a^2 + b^2)^2 + \frac{4ab(a^2 - b^2)}{(a^2 - b^2)^2},
\end{align*}
\]

of which the numerator
\[
\begin{align*}
&= [(a^2 - b^2)^2 + 4a^2b^2] + 4ab(a^2 - b^2) \\
&= (a^2 - b^2)^2 + 4ab(a^2 - b^2) + 4a^2b^2 \\
&= [(a^2 - b^2) + 2ab]^2;
\]

\[
\therefore \quad \text{the given expression} \quad = \frac{(a^2 + 2ab - b^2)^2}{(a^2 - b^2)^2}.
\]

Therefore the required root \( = \frac{a^2 + 2ab - b^2}{a^2 - b^2} \).

**Exercise (I).**

Find the square root of:

1. \( 49a^2x^4 - 42ab^2x^2 + 9b^4 \).
2. \( a^2 + b^2 + c^2 - 2ab + 2ac - 2bc \).
3. \( 4a^2 + b^2 + 9c^2 + 6bc - 12ac - 4a\).\)
4. \( a^4 + 4b^4 + 9c^4 + 4a^2b^2 - 6a^2c^2 - 12b^2c^2 \).
5. \( x^2 + \frac{a^2}{9} - bx + \frac{b^2}{4} - \frac{ab}{3} + 2ax \).

6. \( (x + \frac{1}{x})^2 - 4\left(x - \frac{1}{x}\right) \).

7. \( x^4 + \frac{1}{x^4} + 2\left(x^2 + \frac{1}{x^2}\right) + 3. \)

8. \( \frac{x^2}{y^2} + \frac{y^2}{x^2} - \left(\frac{x}{y} + \frac{y}{x}\right) \sqrt{2 + 2\frac{1}{2}}. \)

9. \( a^2 + b^2 + c^2 + d^2 - 2a(b - c + d) - 2b(c - d) - 2cd. \)

10. \( (a - b)^4 - 2(a^2 + b^2)(a - b)^2 + 2(a^4 + b^4). \)

11. \( a^4 + b^4 + c^4 + d^4 - 2a^2(b^2 + d^2) - 2b^2(c^2 - d^2) + 2c^2(a^2 - d^2). \)

12. \( 2a^2(b + c)^2 + 2b^2(c + a)^2 + 2c^2(a + b)^2 + 4abc(a + b + c). \)

2. The ordinary method of finding the square root of a compound algebraical expression.

From our previous knowledge of formulae the following results are obvious:

\[ (a + b)^2 = a^2 + (2a + b)b; \]
\[ (a + b + c)^2 = a^2 + (2a + b)b + (2a + 2b + c)c; \]
\[ (a + b + c + d)^2 = a^2 + (2a + b)b + (2a + 2b + c)c + (2a + 2b + 2c + d)d; \]

and so on.

Clearly therefore we must have

\[ (ax^2 + bx + c)^2 = a^2x^4 + (2ax^2 + bx)bx + (2ax^2 + 2bx + c)c, \]

and this latter, when arranged according to descending powers of \( x \), \( = a^2x^4 + 2abx^3 + (b^2 + 2ac)x^2 + 2bcx + c^2. \)

Now, if it is proposed to find the square root of the above expression, let us see what means we have of discovering successively the several terms of the root:

The first term of the root, viz., \( ax^2 \), is evidently the square root of the first term of the given expression, which is \( a^2x^4 \);

if we subtract \( a^2x^4 \) from the given expression, the remainder is \( (2ax^2 + bx)bx + (2ax^2 + 2bx + c)c \), in which the term containing the highest power of \( x = 2ax^2 \times bx \), i.e., \( = \) twice the first term of the root into the second term; this enables us to get the 2nd term after having obtained the first;
if now from the above remainder we subtract \((2ax^2 + bx)bx\), the second remainder is \((2ax^2 + 2ax + c)c\), in which the term containing the highest power of \(x = 2ax^2 \times c\), i.e., = twice the first term of the root into the third; this shows how to get the 3rd term after having obtained the 1st and the 2nd.

Thus we are furnished with a clue for successively discovering the terms of the expression \(ax^2 + bx + c\) when its square is given.

The operation may be performed as follows:

\[
\begin{align*}
\frac{a^2x^4 + 2abx^3 + (b^2 + 2ac)x^2 + 2bcx + c^2}{a^2x^4 + 2abx^3 + (b^2 + 2ac)x^2 + 2bcx + c^2} & \quad \frac{2ax^2 + bx + c}{2ax^2 + bx + c} \\
2ax^2 + bx & \quad 2ax^2 + bx \\
2abx^3 + (b^2 + 2ac)x^2 + 2bcx + c^2 & \quad 2abx^3 + b^2x^2 \\
2acx^2 + 2bcx + c^2 & \quad 2acx^2 + 2bcx + c^2
\end{align*}
\]

1. Find the square root of \(a^2x^4\), the first term of the proposed expression, and set it down as the first term of the required root;

2. Subtract \(a^2x^4\) from the given expression and bring down the remainder \(2abx^3 + (b^2 + 2ac)x^2 + 2bcx + c^2\);

3. Set down \(2ax^2\), i.e., twice the 1st term of the root, on the left of the above remainder as the first term of a divisor;

4. Divide the first term of the remainder by \(2ax^2\), and set down the quotient, \(bx\), as the second term of the root and also as the second term of the divisor;

5. Multiply the divisor thus obtained by the second term of the root, and subtract the product from the first remainder;

6. Bring down the second remainder \(2acx^2 + 2bcx + c^2\) and put \(2ax^2 + 2bx\) (i.e., twice the sum of the two terms of the root already obtained) on the left of the remainder for the first two terms of a divisor;

7. Divide the first term of the new remainder by the first term of the new divisor, and set down the quotient, \(c\), as the third term of the root and also as the third term of the divisor;
(8) Multiply the complete divisor thus obtained by the third term of the root, and subtract the product from the second remainder.

After this nothing remains, and we obtain \( ax^2 + bx + c \) for the required root.

Note. The expression considered above stands arranged according to descending powers of \( x \). Similarly every expression of which the square root is sought must be arranged according to the descending or ascending order of the powers of some letter.

Example 1. Extract the square root of

\[
\frac{x^4 + 4y^4}{4y^4} + \frac{x^2}{x^2} + \frac{4y^2}{y^2} + \frac{x^4}{x^2} + 3.
\]

The expression when arranged according to descending powers of \( x \) stands thus:

\[
\frac{x^4}{4y^4} + \frac{x^2}{y^2} + 3 + \frac{4y^2}{x^2} + \frac{4y^4}{x^4},
\]

for, now the indices of the powers of \( x \) in the successive terms are respectively 4, 2, 0, -2 and -4, which numbers evidently are in descending order of magnitude. Hence we proceed as follows:

\[
\frac{x^4}{4y^4} + \frac{x^2}{y^2} + 3 + \frac{4y^2}{x^2} + \frac{4y^4}{x^4} \left( \frac{x^2}{2y^2} + 1 + \frac{2y^2}{x^2} \right)
\]

\[
\frac{x^2}{y^2} + 1 \left( \frac{x^2}{y^2} + 3 \right)
\]

\[
\frac{x^2 (x^2 + 2 + \frac{2y^2}{x^2})}{y^2} \left( 2 + \frac{4y^2}{x^2} + \frac{4y^4}{x^4} \right)
\]

\[
2 + \frac{4y^2}{x^2} + \frac{4y^4}{x^4}
\]

Thus the required root = \( \frac{x^2}{2y^2} + 1 + \frac{2y^2}{x^2} \).
Example 2. Extract the square root of
\[ x^\frac{8}{5} - 2a^{\frac{3}{5}}x^{\frac{11}{5}} + 2a^{\frac{4}{5}}x^8 + a^{\frac{5}{5}}x^{\frac{14}{5}} - 2a^{\frac{6}{5}}x^2 + a^8. \]

Let us proceed by arranging the expression according to descending powers of \( x \), thus:
\[
2a^{-\frac{3}{5}}x^{\frac{7}{5}} - x^2)
- 2a^{-\frac{3}{5}}x^{\frac{11}{5}} + x^8
- 2a^{-\frac{3}{5}}x^3 + x^8.
\]
\[
2a^{-\frac{3}{5}}x^3 - 2x^5 - a^\frac{4}{5}
- 2a^{-\frac{3}{5}}x^3 + x^8 + a^\frac{6}{5}.
\]

Thus the required root = \( a^{-\frac{3}{5}}x^\frac{7}{5} - x^2 - a^\frac{4}{5} \).

Exercise (2).

Find the square root of:

1. \( x^4 - 4x^3 + 10x^2 - 12x + 9. \)
2. \( x^6 - 2x^4 + 2x^3 + x^2 - 2x + 1. \)
3. \( 4x^4 + 8ax^3 + 4a^2x^2 + 16a^2x^2 + 16abx + 16b^2. \)
4. \( \frac{1051x^2}{25} - 6x - \frac{14x^3}{5} + 49x^4 + 9. \)
5. \( \frac{x^4}{4} + 4x^3 + \frac{ax^2}{3} + \frac{a^2}{9} - 2x^3 - \frac{4ax}{3}. \)
6. \( \frac{y^2}{x^2} + \frac{y}{x} - \frac{x}{y} - \frac{1}{4}. \)
7. \( \frac{3}{25} - \frac{20x}{7} + 9y^2 - \frac{15y}{2} + \frac{4x^2}{49y^2}. \)
8. \( x^\frac{3}{5} - 4x^\frac{4}{3} + 2x + 4x^\frac{9}{5} + x^\frac{4}{3}. \)
9. \(a^2x^{-2} + 2ax^{-1} + a^{-2}x^2 + 3 + 2a^{-1}x.\)

10. \(x^3 + xy^{-\frac{1}{2}} - 2x^2y^{-\frac{1}{2}} - 2x^\frac{1}{2}y^\frac{1}{2} + 2x^\frac{3}{2}y^\frac{1}{2} + y.\)

11. \(\frac{9x^3}{4} - 5x^\frac{5}{3}y^\frac{1}{5} + \frac{179x^2y}{45} - 4x^\frac{5}{3}y^\frac{1}{3} + \frac{4xy^2}{25}.\)

12. \(x^4 + 2(y+z)x^3 + (3y^2 + 2yz + 3z^2)x^2 + 2(y^2 + y^2z + yz^2 + z^3)x + y^4 + 2y^2x^2 + z^4.\)

3. Square roots of numbers.

Example 1. Given that the greatest integer whose square is contained in 140925 is 375, find the square root of 140925,16.

Since 140925 > (375)^2
and <(376)^2,

.: 140925,16 > (3750)^2
and <(3760)^2.

Evidently therefore the required root must lie between 3750 and 3760; let it therefore be represented by 3750 + x, where x is some integer less than 10.

Hence, we must have

\[140925,16 = (3750 + x)^2\]
\[= 14062500 + 7500x + x^2,\]

or, \[30016 = (7500 + x)x.\]

By trial we find that \(x = 4\) satisfies this equation.

Hence, the required root = 3750 + 4 = 3754.

Note. The operation may be performed briefly, as in finding the square root of a compound algebraical expression as follows:

\[
\frac{14092516}{3750 + 4} \quad \frac{7504}{30016}
\]

or, more briefly thus:

\[
(375)^2 = \frac{14092516}{140625} \quad \frac{7504}{30016}
\]
I.]

SQUARE AND CUBE ROOTS. 31

It may be observed that the figure 4 of the root is the same as the quotient obtained by dividing 3001 by 750.

Cor. Hence we can easily find the square roots of the following numbers:—784, 2116, and 5625, for the greatest integers whose squares are respectively contained in 7, 21 and 56 are known.

Example 2. Given that the greatest integer whose square is contained in 65739 is 256, find the greatest integer whose square is contained in 65739,82.

Since \( 65739 > (256)^2 \)

and \(< (257)^2 \),

\[ \therefore 65739,82 > (2560)^2 \]

and \(< (2570)^2 \).

Evidently therefore the required number must lie between 2560 and 2570; let it therefore be represented by \( 2560 + x \), where \( x \) is some integer less than 10.

Hence we must have

\[ 65739,82 > (2560 + x)^2 \]

but \(< \{2560 + (x + 1)^2 \},

i.e.,

\[ > 6553600 + (5120 + x)x \]

but \(< 6553600 + \{5120 + (x + 1)\}(x + 1),

and \[ \therefore 20382 > (5120 + x)x \]

but \(< \{5120 + (x + 1)\}(x + 1). \]

By trial we find that \( x = 3 \) satisfies these conditions.

Hence the required number = 2560 + 3 = 2563.

Note. The operation may be performed briefly as follows:—

\[
\begin{array}{c}
(2560)^2 = 65536,00 \\
\hline
5120+3 \overline{20382} \\
\hline
15369 \overline{5013}
\end{array}
\]

or, more briefly thus:—

\[
(256)^3 = 65536 \\
\hline
\overline{5123} \overline{20382} \\
\hline
\overline{5013}
\]

or, more briefly thus:—

\[
(256)^3 = 65536 \\
\hline
\overline{5123} \overline{20382} \\
\hline
\overline{5013}
\]

or, more briefly thus:—

\[
(256)^3 = 65536 \\
\hline
\overline{5123} \overline{20382} \\
\hline
\overline{5013}
\]

or, more briefly thus:—

\[
(256)^3 = 65536 \\
\hline
\overline{5123} \overline{20382} \\
\hline
\overline{5013}
\]
Thus the process is just the same in this example as in the last, with this difference only that there is a remainder in this example whilst there was none in the last.

Here also it may be observed that the figure, 3 obtained by trial is the same as the quotient of 2038 by 512.

**Cor.** We are thus furnished with a clue for determining the square root of any given number. For instance, the determination of the square root of 14175225 depends, as evident from example 1, upon finding the greatest integer whose square is contained in 141752;

and this latter, by the present example, depends upon finding the greatest integer whose square is contained in 1417;

which again, in like manner, depends upon finding the greatest integer whose square is contained in 14, which is known. Hence it is easy to see that the successive stages in arriving at the required root are:

1. To find the greatest integer whose square is contained in 14;
2. Thence to find the greatest integer whose square is contained in 1417;
3. Thence to find the greatest integer whose square is contained in 141752;
4. Thence to find the required number.

**Example 3.** Find the square root of 3976185249.

Putting a comma before every second figure from the right we have

\[ 39,76,18,52,49. \]

Now, by the corollary to the last example the successive stages of the required operation will be as follows:

1. The greatest integer whose square is contained in 39 is 6.
2. Hence let us find the greatest integer whose square is contained in 39,76:

\[
\begin{array}{c}
6^2 = 36 \\
123,76 \\
\hline
369 \\
7
\end{array}
\]

Thus the greatest integer whose square is contained in 3976 is 63.
(3) Hence let us find the greatest integer whose square is contained in 3976,18:—

\[
(63)^2 = \frac{3976,18}{3969} \left( \begin{array}{c}
1260 \\
718 \\
718
\end{array} \right)
\]

Thus the greatest integer whose square is contained in 337618 is 630.

(4) Hence let us find the greatest integer whose square is contained in 397618,52:—

\[
(630)^2 = \frac{397618,52}{396900} \left( \begin{array}{c}
12605 \\
71852 \\
63025 \\
8827
\end{array} \right)
\]

Thus the greatest integer whose square is contained in 39761852 is 6305.

(5) Hence let us find the square root of 39761852,49:—

\[
(6305)^2 = \frac{39761852,49}{39753025} \left( \begin{array}{c}
126107 \\
882749 \\
882749
\end{array} \right)
\]

Thus the required root is 63057.

**Note.** 1. It may be observed that the remainder in (2), viz., 7, is the same as the remainder left by subtracting (63)^2 from 3976 in (3); that the remainder in (3), viz., 718, is the same as the remainder left by subtracting (630)^2 from 397618 in (4); and that the remainder in (4), viz., 8827, is the same as the remainder left by subtracting (6305)^2 from 39761852 in (5). Hence the five stages of the process shown above may be most conveniently combined into one as follows:—

\[
\begin{array}{c}
3976,18,52,49/63057 \\
36
\end{array}
\left( \begin{array}{c}
376 \\
369 \\
718 \\
71832 \\
63025 \\
882749 \\
882749
\end{array} \right)
\]

2—3
Note. 2. The groups of figures separated by commas are called *periods* thus the given number in the present example has been divided into five periods altogether, namely, 39, 76, 18, 52 and 49. Hence the operation performed above amounts to the carrying out of the following directions:

1. Find the greatest integer whose square is contained in the first period, *viz.* 39. This is found to be 6.
2. Put down 6 as the first figure of the root and subtract its square from the first period.
3. To the remainder, 3, bring down the second period, *viz.* 76, thus getting 376.
   1. On the left of 376 place 12, *i.e.*, twice the first figure of the root.
   5. Divide 37, *i.e.*, the number obtained by omitting the last figure of 376, by 12. The quotient is found to be 3.
   6. Annex this 3 to 12, thus getting 123 on the left of 376 and also annex it to 6 as the second figure of the root.
   7. Multiply 123 by 3 and subtract the product from 376.
   8. To the remainder, 7, bring down the next period, *viz.* 18, thus getting 718.
   9. On the left of 718 place 126, *i.e.*, twice the part of the root already found.
   10. Divide 71, *i.e.*, the number obtained by omitting the last figure of 718, by 126. The quotient is found to be zero.
   11. Annex this zero to 126, thus getting 1260 on the left of 718, and also annex it to 63 as the third figure of the root.
   12. Multiply 1260 by zero and subtract the product from 718.
   13. To the remainder, 718, bring down the next period, *viz.* 52, thus getting 71852.
   14. On the left of 71852 place 1260, *i.e.*, twice the part of the root already found.
   15. Divide 7185, *i.e.*, the number obtained by omitting the last figure of 71852, by 1260. The quotient is found to be 5.
   16. Annex this 5 to 1260, thus getting 12605 on the left of 71852, and also annex it to 630 as the fourth figure of the root.
   17. Multiply 12605 by 5 and subtract the product from 71852.
   18. To the remainder, 3827, bring down the next period (which is also the last), *viz.* 49, thus getting 382749.
   19. On the left of 382749 place 12610, *i.e.*, twice the part of the root already found.
   20. Divide 38274, *i.e.*, the number obtained by omitting the last figure of 382749, by 12610. The quotient is found to be 7.
   21. Annex this 7 to 12610, thus getting 126107 on the left of 382749, and also annex it to 6305 as the fifth figure of the root.
   22. Multiply 126107 by 7 and subtract the product from 382749.
   23. As no remainder is left the operation ceases and we obtain the required root = 63057.

Note 3. The first period may consist of one figure only, as in 8, 53, 64. It may also be observed with profit that the number of figures in the root is the same as the number of periods into which the given number is divided.
Example 4. Find the square root of 564,345, to three places of decimals.

We have \(\sqrt{564,345} = \frac{564,345,000}{10^6}\).

Hence \(\sqrt{564,345} = \frac{\sqrt{564,345,000}}{10^3}\).

Now let us find the square root of 564,345,000.

\[
\begin{array}{c|c|c}
43 & 1 & 64 \\
& 1 & 29 \\
467 & 35 & 34 \\
& 32 & 69 \\
4745 & 2 & 65 & 50 \\
& 2 & 37 & 25 \\
4750 & 28 & 25 & 00 \\
& 23 & 75 & 25 \\
& 4 & 49 & 75 \\
\end{array}
\]

This shows that the square root can not be exactly found, and all that we have found is the greatest integer whose square is contained in the number.

Hence the required root \(= \frac{23,755}{10^3} = 23.755\).

Note. 1. If we were asked to find the root to 4 decimal places we would have to multiply and divide the number by \(10^4\); if to 5 decimal places, by \(10^5\); and so on.

Note. 2. From the process shown above it is clear that we might with much advantage proceed as follows:—

1. Make the decimal places six in number by affixing three zeroes to the right of the number; thus, 564,345,000.

2. On the left of the decimal point put a comma before every second figure, and on the right of the point put a comma after every second figure, thus, 5,643,45,00. The whole number is thus divided into five periods altogether, of which two are on the left, and three on the right, of the decimal point.

3. Now extract the square root of this number as if it were an integer and put the decimal point after the first two figures in the root thus found.

Note. 3. It should be observed with care that the number of figures on the left of the decimal point in the root is the same as the number of periods on the left of the decimal point in the proposed number. It should also be observed that if we were asked to find the root to four or five decimal places, we would have to make the decimal places respectively eight or ten in number by affixing zeroes.
Example 5. Find the square root of 3 to five places of decimals.

\[
3 \overline{00,00,00,00,00} \left(1.73205\right)
\]

\[
\begin{array}{r}
27 & 200 \\
189 & 1100 \\
1029 & 7100 \\
6924 & 6924 \\
1760000 & 1732025 \\
27975 &
\end{array}
\]

Thus the required root = 1.73205.

Exercise (3).

Find the square root of:

1. 37249.  
2. 824464.  
3. 2819041.
4. 23071204.  
5. 97535376.  
6. 550183936.
7. 28.8369.  
8. 0.000729.  
9. 236.144689.
10. 0.00139876.

Find to four places of decimals the square root of:

11. 16.245.  
12. 2.  
13. 0.0064.  
14. 5.  
15. 20.

4. If the square root of a number consists of \(2n+1\) figures, the first \(n+1\) of them being found by the ordinary method the remaining \(n\) may be obtained by division only.

[Let us take a particular case. Suppose we have to find the square root of 4556235397156. This number when divided into periods stands thus: \(-4, 55, 62, 35, 39, 71, 56\). Hence evidently the square root will consist of 7 figures, and hence in this case \(n = 3\).

If we proceed by the ordinary method the square root will be found to be 2134534.

We have to shew then that after the first four figures, 2134, have been obtained by the ordinary method, the remaining three, 534, may be obtained by a simple operation of division.]
The given number being 4,55,62,35,39,71,56, by example 3 we know that 2134 is the greatest integer whose square is contained in 4,55,62,35. Hence the remainder at this stage of the process

\[
\begin{align*}
&4556235 \equiv 4556235 \\
- (2134)^2 &\equiv -4553956 \\
= &2279
\end{align*}
\]

to which the next period, viz., 39, would be annexed if the ordinary method were continued.

The divisor at this stage = 2 \times 2134 = 4268. We have to show now that if instead of annexing to 2279 the next period only, viz., 39, we annex to it all the remaining three periods and divide the number thus formed by the number formed by annexing three zeroes to 4268, i.e., divide 2279397156 by 4268000, the quotient will be equal to 534 (the remaining three figures of the root).

By actual division we find this to be the case and we are inclined to accept the proposition as true.

It may be observed that

\[
\begin{align*}
2279397156 &= 4556235397156 \\
- 4553956000000 &= 4556235397156 - (2134)^2 10^6.
\end{align*}
\]

Hence, it is clear that the remaining three figures of the root are given by the quotient of

\[
4556235397156 - \{(2134)10^3\}^2 \text{ by } 2 \times (2134)10^5.
\]

Now we shall give an algebraical proof of the proposition.

Let \( N \) represent the given number, \( a \) the number formed by the first \( n + 1 \) figures of the root (found by the ordinary method), and \( x \) the number formed by the remaining \( n \) figures of the root.

Then we must have

\[
\sqrt{N} = a \cdot 10^n + x;
\]

\[
\therefore N = a^2 \cdot 10^{2n} + 2a \cdot 10^n \cdot x + x^2.
\]

\[
\therefore \frac{N - a^2 \cdot 10^{2n}}{2a \cdot 10^n} = x + \frac{x^2}{2a \cdot 10^n}.
\]

Now \( N - a^2 \cdot 10^{2n} \) is the number formed by annexing all the remaining periods of \( N \) to the remainder after the group of root figures represented by \( a \) has been found.
Thus we find that if the above number be divided by $2a.10^n$, the quotient is equal to $x$ together with $\frac{x^2}{2a.10^n}$.

Now since $x$ consists of $n$ digits, $x^2$ consists of $2n$ digits at most; also since $a.10^n$ consists of $2n+1$ digits, $2a.10^n$ consists of $2n+1$ digits, at least. Hence, $\frac{x^2}{2a.10^n}$ is a proper fraction and therefore it forms no part of the arithmetical quotient.

Thus it is clear that when $N - a^2.10^n$ is divided by $2a.10^n$, the arithmetical quotient is the remaining part of the root; which proves the proposition.

Note. The proposition evidently applies to integers which are perfect squares. It may also be shown to be true (with some exceptions) for integers which are not perfect squares.

Suppose $N$ is an integer which is not a perfect square and that it is divided into $2n+1$ periods. Suppose also that $a$ represents the number formed by the first $n+1$ figures of its square root and that $x$ represents the number formed by the next $n$ figures. If now $N - a^2.10^n$ be divided by $2a.10^n$ will $x$ be the quotient?

Let $N'$ be the perfect square next below $N$. Then by the present article we must have

$$\frac{N' - a^2.10^n}{2a.10^n} = x + \frac{x^2}{2a.10^n},$$

and $\therefore$ we must have

$$\frac{N - a^2.10^n}{2a.10^n} = x + \frac{x^2}{2a.10^n} + \frac{N - N'}{2a.10^n}$$

$$= x + \frac{x^2 + (N - N')}{2a.10^n}.$$

Hence it is clear that our answer to the above query can be affirmative only when $x^2 + (N - N')$ is less than $2a.10^n$.

**Example.** Find the square root of 7 to six places of decimals.

The answer will evidently be found by extracting the square root of

$$7,00,00,00,00,00,00$$

and putting a decimal point after the first figure of the root.

Now the square root of the above number will evidently consist of 7 digits, and hence if four of them be found by the
ordinary method the remaining three can be obtained by division. \[ \begin{array}{r}
46 \quad 300 \\
276 \\
\hline
524 \quad 2400 \\
2096 \\
\hline
5285 \quad 30400 \\
26425 \\
\hline
5290 \quad 3975000 \\
751 \\
\hline
5290 \quad 3975000 \\
37030 \\
\hline
27200 \\
26450 \\
\hline
7500 \\
5290 \\
\hline
2210
\end{array} \]

Now that we have found the first four figures of the root, the remaining three will be obtained by dividing 3975,00,00,00 by 5290,000, i.e., by dividing 3975000 by 5290.

\[ \begin{array}{r}
5290 \quad 3975000 \\
37030 \\
\hline
27200 \\
26450 \\
\hline
7500 \\
5290 \\
\hline
2210
\end{array} \]

Thus the square root of the above number = 2645751 and \[ \therefore \] the required root = 2.645751

5. The ordinary method of finding the cube root of a compound algebraical expression.

Evidently we have \((ax^2 + bx + c)^3\)
\[= (ax^2 + bx)^3 + 3(ax^2 + bx)^2c + 3(ax^2 + bx)c^2 + c^3\]
\[= a^3x^6 + 3(a^2x^4)(bx) + 3(ax^2)(bx)^2 + (bx)^3 + 3(ax^2 + bx)^2c + 3(ax^2 + bx)c^2 + c^3.\]

Hence if we are asked to find the cube root of the above expression we see that we have the following means of discovering successively the several terms of the root:

The first term of the root, viz., \(ax^2\) is evidently the cube root of the first term of the given expression, which is \(a^3x^6\).
If we subtract \( a^3x^6 \) from the given expression the term containing the highest power of \( x \) in the remainder is \( 3(a^2x^4)(bx) \), i.e., equal to three times the square of the first term of the root into the second term; the second term is therefore discovered.

If from the above remainder we now subtract \( \{3(a^2x^4) + 3(ax^2)(bx) + (bx)^2\}(bx) \), the second remainder is \( 3(ax^2 + bx)x + 3(ax^2 + bx)c^2 + c^3 \); the term containing the highest power of \( x \) in this remainder is \( 3a^2x^4c \), i.e., equal to three times the square of the first term of the root into the third.

Hence the third term is discovered.

If from the second remainder we now subtract \( \{3(ax^2 + bx)^2 + 3(ax^2 + bx)c + c^2\}c \), nothing is left and we obtain the required root = \( ax^2 + bx + c \).

Let us illustrate the process by an example.

**Example.** Find the cube root of
\[
x^6 - 6x^5y + 24x^4y^2 - 56x^3y^3 + 96x^2y^4 - 96xy^5 + 64y^6.
\]

The given expression stands arranged according to descending powers of \( x \); we need not therefore change the order of the terms.

The second term of the root, viz., \( -2xy \), as shown on the next page, is obtained by dividing \( -6x^2y \), by \( 3x^4 \) (i.e., three times the square of the first term).

Then the divisor, \( 3x^4 - 6x^3y + 4x^2y^2 \), is formed as shown on the next page.

The product of this divisor by \( (-2xy) \), viz., \( -6x^5y + 12x^4y^2 - 8x^3y^3 \), is now subtracted from the expression which stands above it, and the remainder is put down below the line.

Now take three times the square of the part of the root already obtained and put down the result, \( 3x^4 - 12x^3y + 12x^2y^2 \), as part of a divisor.

The third term of the root, viz., \( 4y^2 \), is obtained by dividing \( 12x^4y^2 \), the first term of the remainder, by \( 3x^4 \), the first term of the divisor.

The complete divisor is then formed as shown on the next page, and the product of this divisor by the third term of the root is subtracted from the expression which stands above it.

As no remainder is now left we find the required root \( x^2 - 2xy + 4y^2 \).
\[
3 \times (x^2)^3 = 3x^6
\]

\[
3 \times x^3 \times (-2xy) = -6x^2y
\]

\[
(-2xy)^2 = +4x^2y^2
\]

\[
3x^4 - 6x^2y + 4x^2y^2
\]

\[
3 \times (x^2 - 2xy)^2 = 3x^4 - 12x^2y + 12x^2y^2
\]

\[
3 \times (x^2 - 2xy) \times (4y^2) = +12x^2y^2 - 24xy^3
\]

\[
(4y^2)^2 = +16y^4
\]

\[
3x^4 - 12x^2y + 24x^2y^2 - 24xy^3 + 16y^4
\]

\[
12x^4y^2 - 48x^2y^3 + 96x^2y^4 - 96xy^5 + 64y^6
\]

\[
x^6 - 6x^5y + 24x^4y^2 - 56x^3y^3 + 96x^2y^4 - 96xy^5 + 64y^6 \left( x^2 - 2xy + 4y^2 \right)
\]

\[
-6x^5y + 24x^4y^2 - 56x^3y^3 + 96x^2y^4 - 96xy^5 + 64y^6
\]
Exercise (4).

Find the cube root of:—

1. \(x^3 + 27x^2 + 243x + 729\).
2. \(27x^3 - 216x^2 + 576x - 512\).
3. \(64a^3 - 144a^2b + 108ab^2 - 27b^3\).
4. \(33x^4 - 36x^3 + x^6 - 63x^5 + 8 - 9x^6 + 66x^2\).
5. \(8x^6 + 12x^5 - 30x^4 - 35x^3 + 45x^2 + 27x - 27\).
6. \(1 - 9x^2 + 33x^4 - 63x^6 + 66x^8 - 36x^{10} + 8x^{12}\).
7. \(c^6 - 63c^3x^3 + 8x^6 - 9c^5x + 66c^2x^4 - 36cx^5 + 33c^4x^2\).

6. Cube Roots of Numbers.

Example 1. Given that the greatest number whose cube is contained in 9718142 is 213, find the cube root of 9718142,104.

Since \(9718142 > (213)^3\)

and \(< (214)^3\)

\[\therefore 9718142,104 > (2130)^3\]

and \(< (2140)^3\).

Evidently therefore the required root must lie between 2130 and 2140; let it therefore be represented by 2130 + \(x\), where \(x\) is some integer less than 10.

Hence, we must have

\[
9718142,104 = (2130 + x)^3
\]

\[
= (2130)^3 + 3(2130)^2x + 3(2130)x^2 + x^3
\]

\[
= 9663597000 + 3 \times 4536900x + 3 \times 2130x^2 + x^3,
\]

or, \(54545104 = (3 \times 4536900 + 3 \times 2130x + x^2)\).

Dividing 54545104 by 3 \times 4536900 we get 4 as the quotient, and we also find that \(x = 4\) satisfies the above relation.

Hence we have the required root = 2134. The process may be shewn as follows:—

\[
(2130)^3 = 9663597000
\]

\[
\begin{array}{c|c}
3 \times (2130)^2 & 13610700 \\
3 \times 2130 \times 4 & 25560 \\
4^3 & 16 \\
13636276 & 4 \\
54545104 & 54545104
\end{array}
\]
Or, more briefly thus:—

\[
(213)^3 = 9663597
\]

\[
\begin{array}{c}
3 \times (2130)^2 = 13610700 \\
3 \times 2130 \times 4 = 25560 \\
4^2 = 16 \\
\hline
13636276
\end{array}
\]

\[
\begin{array}{c}
13636276 \\
4 \\
\hline
54545104 \quad 54545104
\end{array}
\]

Note. 1. It should be carefully remembered that the figure 4 of the root is found by dividing 54545104 by 13610700. Sometimes the figure so found may prove too large and then we have to try the next smaller figure.

Note. 2. By a method similar to that of example 2, article 3, we can easily shew that when the greatest integer whose cube is contained in a given number is known, we can determine the greatest integer whose cube is contained in a number formed by annexing three more digits to the given number; i.e., when the greatest integer whose cube is contained in a number like 34567 is known, we can find the greatest integer whose cube is contained in 34567,432.

Hence we have the following means of determining the cube root of any number, say 3278975416.

(1) Divide the number into a number of periods by placing a comma before every third figure from the right, thus:—3,278,975,416.

(2) Find the greatest number whose cube is contained in the first period, namely 3.

(3) Hence find the greatest integer whose cube is contained in 3,278.

(4) Hence find the greatest integer whose cube is contained in 3,278,975.

(5) Hence find the greatest integer whose cube is contained in 3,278,975,416. If the number be a perfect cube this integer will be its cube root and no remainder will be left after the operation.

Example 2. Find the cube root of 68417929.

Putting a comma before every third figure from the right we have 68,417,929.

Then, by Note 2, last example, the successive stages of the required operation will be as follows:—

(1) The greatest number whose cube is contained in the first period, namely 68, is 4.

(2) Hence let us find the greatest number whose cube is contained in 68,417:—
\[ 4^3 = 64 \]
\[
\begin{array}{c|c|c}
3 \times (40)^2 & 4800 & 4417 \\
3 \times 40 \times 0 & = & 0 \\
0^2 & = & 0 \\
\hline
4800 & & \\
0 & & \\
\hline
0 & 0 & \\
\hline
4417 & & 
\end{array}
\]

Thus the greatest integer whose cube is contained in 68,417 is 40.

(3) Hence let us find the cube root of 68,417,929.

\[
\begin{array}{c|c|c}
3 \times (400)^2 & 480000 & 4417929 \\
3 \times 400 \times 9 & = & 10800 \\
9^2 & = & 81 \\
\hline
490881 & & \\
9 & & \\
\hline
4417929 & 4417929 & 
\end{array}
\]

Thus the required root = 409.

Note. 1. Since the remainder in (2), namely 4417, is the same as the remainder left when \((40)^3\) is subtracted from 68,417 in (3), it is clear that the three different stages might very well be combined into one as in example 3, article 3.

Note. 2. The number of figures in the cube root is the same as the number of periods into which the given number is divided.

Example 3. Find the cube root of 568,7432.75 to two places of decimals.

We have \(568,7432.75 = \frac{568,7432750000}{(100)^3}\).

Hence \(\sqrt[3]{568,7432.75} = \frac{\sqrt[3]{568,7432750000}}{100}\).
Now let us find the cube root of $5,687,432,750,000$ or rather the greatest integer whose cube is contained in it.

\[ 1^3 = 1 \]

\[ \begin{array}{c}
3 \times (10)^2 = 300 \\
3 \times 10 \times 7 = 210 \\
7^2 = 49 \\
\end{array} \]

\[ \frac{4687}{559} \]

\[ \frac{7}{3913} \]

\[ \begin{array}{c}
3 \times (170)^2 = 86700 \\
3 \times 170 \times 8 = 4080 \\
8^2 = 64 \\
\end{array} \]

\[ \frac{774432}{90844} \]

\[ \frac{8}{726752} \]

\[ \begin{array}{c}
3 \times (1780)^2 = 9505200 \\
3 \times 1780 \times 5 = 26700 \\
5^2 = 25 \\
\end{array} \]

\[ \frac{47680750}{9531925} \]

\[ \frac{5}{47659625} \]

\[ \begin{array}{c}
3 \times (17850)^2 = 955867500 \\
\end{array} \]

As $21125000$ is not divisible by $955867500$ we must take zero for the next figure of the root and stop here.

Thus the greatest number whose cube is contained in $5,687,432,750,000$ is $17850$; and therefore the required root $= 178.50$.

**Note. 1.** If we were asked to find the cube root to three places of decimals we would have to multiply and divide the given number by $(1000)^2$, and if to four places of decimals, by $(10000)^2$ or $(10^4)^2$, and so on.

**Note. 2.** From the process shown above it is clear that we might with much advantage proceed as follows:
(1) Since the root is to be found to two places of decimals, make the
decimal places six in number by affixing four zeroes to the right of the
proposed number, thus:—5687432.750000.

(2) On the left of the decimal point put a comma before every third
figure, and on the right of the point put a comma after every third figure,
thus:—5,687,432.750,000. The whole number is thus divided into five
periods altogether of which three are on the left, and two on the right, of
the decimal point.

(3) Now extract the cube root of this number as if it were an integer.
and put the decimal point after the first three figures in the root thus found.

Note. 3. It should be observed with care that the number of figures
on the left of the decimal point in the root is the same as the number of
periods on the left of the decimal point in the proposed number. It should
also be observed that if we were asked to find the root to three or four
places of decimals we would have to make the decimal places in the proposed
number respectively nine or twelve in number by affixing zeroes.

Exercise (5).

Find the cube root of:—

1. 15625.  
2. 110592.  
3. 941192.  
4. 8365427.  
5. 28934443.  
6. 95443993.  
7. 194104539.  
8. 223648543.  
9. 843908625.  
10. 673373097125.  
11. 32461759.  
12. 27054036008.  

Find to three places of decimals the cube root of:—

13. 44.6.  
14. 3.00415.  
15. 576.  

7. If the cube root of a number consists of \(2n + 2\) figures, the first \(n + 2\) of them being found by the
ordinary method the remaining \(n\) may be obtained
by division only.

Let \(N\) represent the given number, \(a\) the number formed
by the first \(n + 2\) figures of the root (found by the ordinary
method), and \(x\) the number formed by the remaining \(n\) figures
of the root.

Then we must have

\[ \sqrt[3]{N} = a.10^n + x, \]

\[ N = a^3.10^{3n} + 3a^2.10^{2n}x + 3a.10^n.x^2 + x^3; \]

\[ N - a^3.10^{3n} = x + \frac{x^2}{3a^2.10^{2n}} + \frac{x^3}{a.10^n + 3a^2.10^{2n}}. \]
Now, \( N - a^{3n} \cdot 10^{3n} \) is the number formed by annexing all the remaining periods of \( N \) to the remainder after the group of root-figures represented by \( a \) has been found.

Thus we find that if the above number be divided by \( 3a^2 \cdot 10^{2n} \), the quotient is equal to \( x \) together with \( \frac{x^2}{a \cdot 10^n} + \frac{x^3}{3a^2 \cdot 10^{2n}} \).

Now, since \( x \) consists of \( n \) digits it is less than \( 10^n \) and \( \therefore x^2 < 10^{2n} \); also since \( a \) consists of \( n + 2 \) figures it is greater than \( 10^{n+1} \) and \( \therefore a \cdot 10^n > 10^{2n+1} \); hence

\[
\frac{x^2}{a \cdot 10^n} < \frac{10^{2n}}{10^{2n+1}}, \quad \text{i.e.,} \quad \frac{1}{10},
\]

and also

\[
\frac{x^3}{3a^2 \cdot 10^{2n}} < \frac{10^{3n}}{3 \times 10^{4n+2}}, \quad \text{i.e.,} \quad \frac{1}{3 \times 10^{n+2}}.
\]

Hence, \( \frac{x^2}{a \cdot 10^n} + \frac{x^3}{3a^2 \cdot 10^{2n}} \) is clearly a proper fraction and \( \therefore \) it forms no part of the arithmetical quotient.

Thus when \( N - a^{3n} \cdot 10^{3n} \) is divided by \( 3a^2 \cdot 10^{2n} \), the arithmetical quotient is \( x \), the remaining part of the root; which proves the proposition.

Note. Remarks similar to those made in the note to article 4 apply to the present article.

Miscellaneous examples.

Example 1. If \( x^6 + 3dx^5 + ex^4 + fx^3 + gx^2 + hx + k^2 \) be a perfect cube, find its cube root and determine the co-efficients \( e, f, g, h \) in terms of \( d \) and \( k \).

(Bombay University P. E. Paper, 1889.)

The given expression stands arranged according to descending powers of \( x \).

Hence the first and last terms of the root must be \( x^2 \) and \( k \), the cube roots of the first and last terms of the given expression.

Also as the second term of the root is to be found by dividing the second term of the given expression by three times the square of the first term of the root, \( \therefore \) the second term of the root must be \( = \frac{3dx^5}{3x^2} = dx \).

Thus we have

the 1st term of the root \( = x^2 \)
the 2nd term \( = dx \) and evidently there cannot be any more term of the root between \( dx \) and \( k \).
Hence, the required root = $x^2 + dx + k$.

Therefore we must have

$$x^6 + 3dx^5 + ex^4 + fx^3 + gx^2 + hx + k^3$$

identically = $(x^2 + dx + k)^3$

$$= x^6 + 3x^4(dx + k) + 3x^2(dx + k)^2 + (dx + k)^3$$

$$= x^6 + 3dx^5 + 3(k + d^2)x^4 + d(6k + d^2)x^3$$

$$+ 3k(k + d^2)x^2 + 3dk^2x + k^3.$$

Hence, equating the co-efficients of like powers of $x$ on both sides, we have

$$e = 3(k + d^2), \quad f = d(6k + d^2), \quad g = 3k(k + d^2), \quad h = 3dk^2.$$

**Example 2.** If $a$ be the greatest integer contained in $N^{\frac{1}{3}}$, and the difference be so small that its cube may be neglected, prove that a nearer approximate value of $N^{\frac{1}{3}}$ will be

$$\frac{1}{2} \left\{ a + \left( \frac{4N - a^3}{3a} \right)^{\frac{1}{3}} \right\}$$

(Bombay University P. E. Paper, 1885.)

Let $N^{\frac{1}{3}} = a + x$, where $x$ is, by hypothesis, so small that its cube may be neglected.

Then we have

$$N = a^3 + 3a^2x + 3ax^2 + x^3.$$

Hence, neglecting $x^3$, we have the following equation from which the value of $x$ can be determined approximately:

$$3ax^2 + 3a^2x + a^3 = N.$$

Hence,

$$x^2 + ax = \frac{N - a^3}{3a};$$

or,

$$x^2 + ax + \frac{a^2}{4} = \frac{N - a^3 + a^2/4}{3a};$$

or,

$$\left( x + \frac{a}{2} \right)^2 = \frac{4N - a^3}{4.3a};$$

$$\therefore \quad x + \frac{a}{2} = \frac{1}{2} \left( \frac{4N - a^3}{3a} \right)^{\frac{1}{2}}; \quad \text{[The negative sign is rejected since } x \text{ is, by hypothesis, positive.]}$$

$$\therefore \quad x + a = \frac{a}{2} + \frac{1}{2} \left( \frac{4N - a^3}{3a} \right)^{\frac{1}{2}}.$$
Thus the approximate value of \( a + x \)

\[
\frac{1}{2} \left\{ a + \left( \frac{4N - a^3}{3a} \right)^{\frac{1}{2}} \right\} ;
\]

and this is therefore the required value of \( N^{\frac{1}{2}} \).

---

CHAPTER II.

INDICES.

1. **Definition.** The product of \( m \) factors each equal to \( a \) is represented by \( a^m \). [See Art. 11, page 9.]

Thus the meaning of \( a^m \) is clear when \( m \) is a **positive integer**.

2. **The Index Law and the truths necessarily following from it.**

To prove that \( a^m \times a^n = a^{m+n} \), where \( m \) and \( n \) are any two positive integers.

Since \( a^m = a \times a \times a \times \ldots \times a \) to \( m \) factors

and \( a^n = a \times a \times a \times \ldots \times a \) to \( n \) factors,

\[
\therefore a^m \times a^n = (a \times a \times a \times \ldots \times a) \times (a \times a \times a \times \ldots \times a) = a \times a \times a \times a \times a \times \ldots \text{ to } (m+n) \text{ factors}
\]

\[
= a^{m+n}.
\]

This result is called the **Index Law**.

**Cor. 1.** \( a^m \times a^n \times a^p = a^{m+n+p} \), when \( m \), \( n \) and \( p \) are positive integers.

For \( a^m \times a^n = a^{m+n} \), \( \therefore a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{(m+n)+p} = a^{m+n+p} \).

Hence, \( a^m \times a^n \times a^p \times a^q \ldots = a^{m+n+p+q+\ldots} \).

Thus, the **product of any number of powers of a given quantity is that power of the quantity whose index is equal to the sum of the indices of the factors**.
Cor. 2. \((a^m)^n = a^{mn}\), when \(m\) and \(n\) are any two positive integers.

For \((a^m)^n = a^m \times a^m \times a^m \times \ldots \) to \(n\) factors
\[= a^{m+m+m+\ldots} \] to \(n\) terms \(\text{[by Cor. 1.]}\)

and \(\therefore = a^{mn}\).

Cor. 3. \(a^m \div a^n = a^{m-n}\), when \(m\) and \(n\) are positive integers and \(m\) is greater than \(n\).

For \(a^{m-n} \times a^n = a^{(m-n)+n}\) \(\text{[because} m-n \text{is a positive integer.]}\)
\[= a^m,\]
\(\therefore a^m \div a^n = a^{m-n}\).

3. Assuming the formula \(a^m \times a^n = a^{m+n}\) to be true for all values of \(m\) and \(n\), to find meanings for quantities with fractional or negative indices.

(i) To find the meaning of \(a^\frac{p}{q}\), when \(p\) and \(q\) are any two positive integers.

Since \(a^m \times a^n = a^{m+n}\) for all values of \(m\) and \(n\), putting \(\frac{p}{q}\) for each of them, we have
\[a^\frac{p}{q} \times a^\frac{p}{q} = a^{\frac{p}{q} + \frac{p}{q}} = a^{\frac{2p}{q}}.\]
Similarly, \(a^\frac{p}{q} \times a^\frac{p}{q} \times a^\frac{p}{q} = a^{\frac{p}{q} \times \frac{p}{q}} \times a^{\frac{p}{q}} = a^\frac{3p}{q}\), and so on.

Hence, \(a^\frac{p}{q} \times a^\frac{p}{q} \times a^\frac{p}{q} \ldots \ldots \ldots \) to \(q\) factors
\[= a^{\frac{2p}{q}} = a^\frac{p}{q}.\]

Thus \(a^\frac{p}{q}\) is equal to the \(q^{th}\) root of \(a^p\), and is therefore equivalent to \(\sqrt[q]{a^p}\).

Cor. Hence \(a^\frac{1}{2} = \sqrt{a}\), \(a^\frac{1}{3} = \sqrt[3]{a}\), \(a^\frac{1}{4} = \sqrt[4]{a}\), and so on.

Generally, \(a^\frac{1}{n} = \sqrt[n]{a}\).
Note. From the Index Law it is also easy to see that \( a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times \ldots \) to \( p \) factors = \( a^{\frac{p}{q}} \). Thus \( a^{\frac{p}{q}} \) may as well be regarded as the \( p^{th} \) power of \( a^{\frac{1}{q}} \), i.e., equivalent to \((a^{\frac{1}{q}})^p\). Thus \( a^{\frac{p}{q}} \) may be interpreted either as the \( q^{th} \) root of the \( p^{th} \) power of \( a \), or as the \( p^{th} \) power of the \( q^{th} \) root of \( a \).

(ii) To find the meaning of \( a^0 \).

Since \( a^m \times a^n = a^{m+n} \) is true for all values of \( m \) and \( n \), putting \( m = 0 \) we have

\[
a^0 \times a^n = a^{0+n} = a^n ;
\]

\[
\therefore \ a^0 = a^n \div a^n = 1.
\]

Thus any quantity raised to the power zero is equivalent to 1.

(iii) To find the meaning of \( a^{-n} \), where \( n \) is any positive integer.

Since \( a^m \times a^n = a^{m+n} \) is true for all values of \( m \) and \( n \), putting \( m = -n \), we have

\[
a^{-n} \times a^n = a^{-n+n} = a^0 = 1 ;
\]

\[
\therefore \ a^{-n} = \frac{1}{a^n}, \text{ and } a^n = \frac{1}{a^{-n}}.
\]

Cor. Hence \( a^m \div a^n = a^{m-n} \) for all values of \( m \) and \( n \).

For \( a^m \div a^n = \frac{a^m}{a^n} = a^m \times a^{-n} = a^{m-n} \).

**Example 1.** Find the value of \( 8^\frac{5}{3} \).

\[
8^{\frac{5}{3}} = \left(\frac{2}{8}\right)^5 = 2^5 = 32.
\]

**Example 2.** Find the value of \( 4^{-\frac{5}{2}} \).

\[
4^{-\frac{5}{2}} = \frac{1}{4^{\frac{5}{2}}} = \frac{1}{(\sqrt{4})^5} = \frac{1}{2^5} = \frac{1}{32}.
\]

**Example 3.** Multiply together \( a^\frac{5}{2}, a^\frac{3}{4}, \sqrt[4]{a^{-5}} \) and \( \frac{1}{a^{-3}} \).

The required product = \( a^\frac{5}{2} \times a^\frac{3}{4} \times a^{-\frac{5}{4}} \times a^3 \)

\[
= a^{\frac{5}{2} + \frac{3}{4} - \frac{5}{4} + 3}
\]

\[
= a^{\frac{5}{2} - \frac{1}{2} + 3} = a^{2 + 3} = a^5.
\]
Exercise (6)

Express the following avoiding fractional or negative indices:

1. \( a^{\frac{5}{7}} \).
2. \( x^{-\frac{3}{2}} \).
3. \( \frac{3}{x^{-\frac{4}{3}}} \).
4. \( x^{-\frac{2}{3}} \times 3a^{-\frac{1}{2}} \).
5. \( 8m^{-2} \times m^{-\frac{2}{3}} \).
6. \( x^{-\frac{4}{3}} \div 3a^{-\frac{5}{4}} \).
7. \( x^{-\frac{2}{3}} \div 2x^{-\frac{1}{2}} \).
8. \( \sqrt[5]{x^2} \div \sqrt[5]{x^{-a}} \).
9. \( 2\sqrt{a^{-5}} \times \sqrt{a^3} \).
10. \( \sqrt[4]{x^6} \div 2\sqrt[2]{x^{-5}} \).

Express the following avoiding radical signs and negative indices:

11. \( \left(\frac{3}{x}\right)^{-\frac{3}{7}} \).
12. \( \left(\frac{a}{4}\right)^{-\frac{6}{3}} \).
13. \( \frac{1}{3\sqrt{x^{-2}}} \).
14. \( \frac{1}{(\sqrt[3]{a})^{-2}} \).
15. \( \sqrt[3]{x^4} \div \left(\frac{a}{4}\right)^{-1} \).
16. \( \sqrt[4]{a^{-3}} \div \left(\frac{a}{4}\right)^{-1.2} \).

Find the value of:

17. \( 4^{-\frac{3}{2}} \).
18. \( 8^{\frac{3}{2}} \).
19. \( 9^{\frac{3}{2}} \).
20. \( 16^{\frac{5}{4}} \).
21. \( 81^{-\frac{3}{4}} \).
22. \( \frac{1}{6^{-\frac{2}{3}}} \).
23. \( (125)^{-\frac{2}{3}} \).
24. \( \left(\frac{1}{216}\right)^{-\frac{4}{3}} \).
25. \( \left(\frac{1}{216}\right)^{-\frac{3}{2}} \).
26. Simplify \( \frac{x^{m+n} \times x^{3m-n}}{z^{5m-6n}} \).

4. To prove that \( (a^m)^n = a^{mn} \) is true for all values of \( m \) and \( n \).

(i) Let \( n \) be a positive integer. Then, whatever may be the value of \( m \), we have

\[
(a^m)^n = a^m \times a^m \times a^m \times \ldots \ldots \ldots \text{to } n \text{ factors}
\]

\[
= a^{m+m+m+\ldots} \ldots \ldots \ldots \text{to } n \text{ terms}
\]

\[
= a^{mn}.
\]
(ii) Let $n$ be a positive fraction equal to $\frac{p}{q}$, where $p$ and $q$ are positive integers. Then we have

\[
\left( a^m \right)^n = \left( \frac{a^m}{q} \right)^p = \sqrt[q^p]{a^m} \qquad \text{[Art. 3, (i)]}
\]

\[
= \sqrt[q^p]{a^{mp}} \qquad \text{[by (i)]}
\]

\[
= a^{\frac{mp}{q}} \qquad \text{[Art. 3, (i)]}
\]

\[
= a^{mn}. \]

(iii) Let $n$ be any negative quantity, equal to $-p$, where $p$ is positive. Then we have

\[
(a^m)^n = (a^m)^{-p} = \frac{1}{(a^m)^p} \qquad \text{[Art. 3, (iii)]}
\]

\[
= \frac{1}{a^{mp}} \qquad \text{[by (i) and (ii)]}
\]

\[
= a^{-mp} \qquad \text{[Art. 3, (iii)]}
\]

\[
= a^{m(-p)} = a^{-n}. \]

Thus the proposition is established.

5. To prove that $a^n b^n = (ab)^n$ for all values of $n$.

(i) Let $n$ be a positive integer. Then we have

\[
a^n b^n = (a \cdot a \cdot a \cdots \cdots \text{to } n \text{ factors})
\]

\[
\times (b \cdot b \cdot b \cdots \cdots \text{to } n \text{ factors})
\]

\[
= (ab \cdot ab \cdot ab \cdots \cdots \text{to } n \text{ factors})
\]

\[
= (ab)^n. \]

(ii) Let $n$ be a positive fraction equal to $\frac{p}{q}$, where $p$ and $q$ are positive integers. Then putting $x$ for $a^n b^n$ we have

\[
x = a^{\frac{p}{q}} b^{\frac{p}{q}},
\]

\[
\therefore \quad x^n = \left( a^{\frac{p}{q}} b^{\frac{p}{q}} \right)^n
\]

\[
= \left( a^{\frac{p}{q}} \right)^n \times \left( b^{\frac{p}{q}} \right)^n \qquad \text{[by (i)]}
\]
\[ \begin{align*}
\frac{a^n}{b^{-n}} &= a^n b^{-n} \quad \text{[Art. 3, (iii)]} \\
&= \frac{a^n}{b^{-n}} \quad \text{[Art. 3, (iii)]} \\
&= \frac{1}{(ab)^{-n}} \quad \text{[by (i) and (iii)]} \\
&= (ab)^{-n} \quad \text{[Art. 3, (iii)]} \\
&= (ab)^n.
\end{align*} \]

Thus the proposition is established.

**Cor. 1.** \[ \frac{a^n}{b^{-n}} = a^n b^{-n} = a^n \left( b^{-1} \right)^n = \left( ab^{-1} \right)^n = \left( \frac{a}{b} \right)^n. \]

**Cor. 2.** \[ a^n b^n c^n = (ab)^n c^n = (abc)^n; \]

generally, \[ a^n b^n c^n d^n \ldots = (abcd \ldots)^n. \]

6. Applications of the results proved in the last two articles.

**Example 1.** Simplify \( (a^8 b^5)^{-\frac{3}{4}}. \)

\[ (a^8 b^5)^{-\frac{3}{4}} = (a^8)^{-\frac{3}{4}} \times (b^5)^{-\frac{3}{4}} \]
\[ = a^{8 \times (-\frac{3}{4})} \times b^{5 \times (-\frac{3}{4})} \]
\[ = a^{-\frac{24}{4}} b^{-\frac{15}{4}}. \]

**Example 2.** Simplify \( \sqrt[3]{a^{-2} b} \times \sqrt[3]{ab^{-3}}. \)

\[ \sqrt[3]{a^{-2} b} = (a^{-2} b)^{\frac{1}{3}} = (a^{-2})^{\frac{1}{3}} \times b^{\frac{1}{3}} = a^{-\frac{2}{3}} b^{\frac{1}{3}}; \]
\[ \sqrt[3]{ab^{-3}} = (ab^{-3})^{\frac{1}{3}} = a^{\frac{1}{3}} \times (b^{-3})^{\frac{1}{3}} = a^{\frac{1}{3}} b^{-1}. \]
Hence, the given expression
\[ = a^{-\frac{1}{2}}b^{\frac{1}{3}} \times a^{\frac{1}{3}}b^{-\frac{1}{1}} \]
\[ = a^{-\frac{1}{2} + \frac{1}{3}} \times b^{\frac{1}{3} - \frac{1}{1}} = a^{-\frac{2}{3}}b^{-\frac{1}{3}}. \]

**Example 3.** Simplify \( \sqrt[3]{a^2 b^{-\frac{1}{3}} c^{-\frac{1}{8}}} \div \sqrt[3]{a^4 b^{-1} c^4} \)

\[ \sqrt[3]{a^2 b^{-\frac{1}{3}} c^{-\frac{1}{8}}} = \left( a^2 b^{-\frac{1}{3}} c^{-\frac{1}{8}} \right)^{\frac{1}{3}} \]
\[ = \left( a^\frac{1}{3} \right)^{\frac{1}{3}} \left( b^{-\frac{1}{3}} \right)^{\frac{1}{3}} \left( c^{-\frac{1}{8}} \right)^{\frac{1}{3}} \]
\[ = a^\frac{1}{3} b^{-\frac{1}{3}} c^{-\frac{1}{8}}; \]

and \( \sqrt[3]{a^4 b^{-1} c^4} = \left( a^4 b^{-1} c^4 \right)^{\frac{1}{3}} \)
\[ = \left( a^4 \right)^{\frac{1}{3}} \left( b^{-1} \right)^{\frac{1}{3}} \left( c^4 \right)^{\frac{1}{3}} \]
\[ = a^{\frac{4}{3}} b^{-\frac{1}{3}} c^{\frac{4}{3}}. \]

Hence, the given expression
\[ = a^{\frac{3}{2}} b^{-\frac{1}{3}} c^{-\frac{7}{8}} \div a^{\frac{4}{3}} b^{-\frac{1}{3}} c^{\frac{5}{8}} \]
\[ = a^{\frac{3}{2} - \frac{4}{3}} b^{-\frac{1}{3} - \frac{1}{3}} c^{-\frac{7}{8} - \frac{5}{8}} \]
\[ = a^{\frac{3}{6} - \frac{4}{6}} b^{-\frac{2}{3}} c^{-\frac{12}{8} - \frac{5}{8}} \]
\[ = a^{\frac{1}{6}} b^0 c^{-1} = a^\frac{1}{6} c^{-1}. \]

**Exercise (7).**

Simplify—

1. \( \left( a^{-\frac{2}{3}} \right)^6 \).
2. \( \left( a^{-\frac{2}{3}} b^\frac{5}{3} \right)^3 \).
3. \( \left( a^{-\frac{1}{2}} b^{-3} \right)^{-2} \).
4. \( \left( a^\frac{5}{6} b^\frac{2}{3} \right)^{-\frac{2}{3}} \).
5. \( \left( \sqrt[3]{a^{1/6} b^{1/3}} \right)^6 \).
6. \( \left( \sqrt[3]{x^9 y^{-6}} \right)^{-3} \).
7. \( \sqrt[8]{x^2 \cdot \sqrt[4]{x^{-3}}} \)

8. \( \sqrt[4]{a^{-3}b^4} \times \sqrt[4]{a^2b^{-8}} \)

9. \( \sqrt[4]{x^{-2} \cdot \sqrt[6]{y^5}} \times \sqrt[4]{x^4 \cdot \sqrt[3]{y^3}} \)

10. \( (8x^3 \div 27a^{-3})^{\frac{2}{3}} \)

11. \( (64x^3 \div 27a^{-3})^{\frac{2}{3}} \)

12. \( \sqrt[3]{a^6b^{-2}c^{-4}} \times \sqrt[4]{a^2b^4c^9} \)

13. \( \sqrt[4]{a^{-8}b^4c^{-3}} \div \sqrt[3]{a^2b^4c^{-1}} \)

14. \( \sqrt[ab^{-2}c^3] {\left( \frac{3 \sqrt[3]{a^3b^2c^{-3}} \right)}^{-1}} \)

15. \( \left( \frac{a^{-1}b^2}{a^2b^{-4}} \right)^7 \div \left( \frac{a^2b^{-6}}{a^{-2}b^3} \right)^5 \)

7. Miscellaneous Examples.

Example 1. Divide \( a + b + c + 3a^\frac{1}{3}b^\frac{2}{3} + 3a^\frac{2}{3}b^\frac{1}{3} \) by \( a^\frac{1}{3} + b^\frac{1}{3} + c^\frac{1}{3} \).

Let us proceed by arranging the dividend and the divisor according to descending powers of \( a : - \)

\[
\frac{a^\frac{1}{3} + (b^\frac{1}{3} + c^\frac{1}{3})}{a + 3a^\frac{2}{3}b^\frac{1}{3} + 3a^\frac{1}{3}b^\frac{2}{3} + (b + c)} \left( a^\frac{2}{3} + a^\frac{1}{3}(2b^\frac{1}{3} - c^\frac{1}{3}) \right)
\]

\[
= a^\frac{2}{3}(2b^\frac{1}{3} - c^\frac{1}{3}) + 3a^\frac{1}{3}b^\frac{2}{3} + (b + c)
\]

\[
a^\frac{2}{3}(2b^\frac{1}{3} - c^\frac{1}{3}) + a^\frac{1}{3}(2b^\frac{2}{3} + b^\frac{1}{3}c^\frac{1}{3} - c^\frac{2}{3})
\]

\[
= a^\frac{1}{3}(b^\frac{2}{3} - b^\frac{1}{3}c^\frac{1}{3} + c^\frac{2}{3}) + (b + c)
\]

Thus the required quotient

\[= a^\frac{2}{3} + 2a^\frac{1}{3}b^\frac{1}{3} - a^\frac{1}{3}c^\frac{1}{3} + b^\frac{2}{3} - b^\frac{1}{3}c^\frac{1}{3} + c^\frac{2}{3}.\]

Note. In multiplication as well as in division the arrangement of the expressions concerned according to ascending or descending powers of some common letter should never be overlooked. Such arrangements invariably give neatness to the required operations, if not always indispensable.
Example 2. Divide \( x^n + a^{n-1}x^{n-1} + a^n \) by
\[ x^{n-1} - a^{n-2}x^{n-2} + a^{n-1}. \]

Let \( m = x^{n-2} \) and \( n = a^{n-2} \).

Then \( m^2 = \left(x^{n-2}\right)^2 = x^{2\times(n-2)} = x^{2n-4} \),
and \( m^4 = \left(m^2\right)^2 = \left(x^{2n-4}\right)^2 = x^{2\times2n-4} = x^{3n-8} \).

Similarly, \( n^2 = a^{2n-4} \) and \( n^4 = a^{2n} \).

Hence,
\[
\frac{x^n + a^{n-1}x^{n-1} + a^n}{x^{n-1} - a^{n-2}x^{n-2} + a^{n-1}} = \frac{m^4 + m^2n^2 + n^4}{m^4 - mn + n^2} = \frac{(m^2 + n^2)^2 - m^2n^2}{m^4 - mn + n^2}.
\]
\[
= \frac{m^2 + n^2 + mn(m^2 + n^2 - mn)}{m^2 - mn + n^2} = \frac{m^2 + mn + n^2}{m^2 - mn + n^2} = \frac{x^{2n-4} + x^{n-2}a^{n-2} + a^{n-1}}{x^{2n-4} - x^{n-2}a^{n-2} + a^{n-1}}.
\]

Example 3. Shew that
\[
\frac{1}{1 + x^{m-n} + x^{m-p}} + \frac{1}{1 + x^{n-m} + x^{n-p}} + \frac{1}{1 + x^{p-m} + x^{p-n}} = 1.
\]

The 1st term \( = \frac{x^m}{x^m(1 + x^{m-n} + x^{m-p})} \)
\( = \frac{x^m}{x^m + x^{-n} + x^{-p}} \);

the 2nd term \( = \frac{x^{n}}{x^m(1 + x^{n-m} + x^{n-p})} \)
\( = \frac{x^{n}}{x^m + x^{-m} + x^{-p}} \);
and the 3rd term = \( \frac{x^{-p}}{x^{-p}(1 + x^{p-m} + x^{-n})} \).

\[
= \frac{x^{-p}}{x^{-p} + x^{-m} + x^{-n}}
\]

Hence, the given expression

\[
= \frac{x^{-m}}{x^{-m} + x^{-n} + x^{-p}} + \frac{x^p}{x^{-n} + x^{-m} + x^{-p}} \cdot \frac{x^{-p} + x^{-m} + x^{-n}}{x^{-m} + x^{-n} + x^{-p}} = 1.
\]

**Example 4.** Solve the equation

\[
a^{-z} (a^z + b^{-z}) = \frac{a^2 b^z + 1}{a^z b^z}.
\]

We have \( a^{-z} a^z + a^{-z} b^{-z} = 1 + \frac{1}{a^z b^z} \),

or, \( 1 + (ab)^{-z} = 1 + a^{-2} b^{-2} \)

\[
= 1 + (ab)^{-2}.
\]

Hence \( (ab)^{-z} = (ab)^{-2} \),

\[
\therefore \quad z = 2.
\]

**Example 5.** If \( a^b = b^a \), shew that \( \left( \frac{a}{b} \right)^{\frac{a}{b}} = a^{\frac{a}{b} - 1} \); and if \( a = 2b \), shew that \( b = 2 \).

Since \( a^b = b^a \),

\[
\therefore \quad a = b^{\frac{a}{b}} \quad \text{[extracting the } b\text{th root of both sides]}
\]

Hence, \( \left( \frac{a}{b} \right)^{\frac{a}{b}} = a^{\frac{a}{b}} = \frac{a^{\frac{a}{b}}}{b^{\frac{a}{b}}} = a^{\frac{a}{b} - 1} \).

If \( a = 2b \), from the given relation we have

\[
(2b)^b = (b)^{2b} = (b^2)^b,
\]

\[
\therefore \quad 2b = b^2, \quad \therefore \quad b = 2.
\]
Example 6. If \( x = \left( a + \sqrt{a^2 + b^2} \right)^{\frac{1}{3}} + \left( a - \sqrt{a^2 + b^2} \right)^{\frac{1}{3}}, \)

show that \( x^3 + 3bx - 2a = 0. \)

* Putting \( m \) for \( a + \sqrt{a^2 + b^2}, \)

and \( n \) for \( a - \sqrt{a^2 + b^2}, \) we have

\[
x^3 = (m^{\frac{1}{3}} + n^{\frac{1}{3}})^3
\]

\[
= (m^{\frac{1}{3}})^3 + (n^{\frac{1}{3}})^3 + 3m^{\frac{1}{3}}n^{\frac{1}{3}}(m^{\frac{1}{3}} + n^{\frac{1}{3}})
\]

\[
= m + n + 3(mn)^{\frac{1}{3}}(m^{\frac{1}{3}} + n^{\frac{1}{3}})
\]

\[
= m + n + 3(mn)^{\frac{1}{3}}x.
\]

But \( m + n = 2a, \)

and \( (mn)^{\frac{1}{3}} = \left[ a^2 - (a^2 + b^2) \right]^{\frac{1}{3}} \)

\[
= \left( -b^2 \right)^{\frac{1}{3}} = -b;
\]

\[
\therefore \quad x^3 = 2a - 3bx,
\]

\[
\therefore \quad x^3 + 3bx - 2a = 0.
\]

Exercise (8).

Multiply:

1. \( x^2 + 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} + 2x^{\frac{1}{6}} + 1 \) by \( x^{\frac{1}{6}} - 2x^{\frac{1}{6}} + 1. \)

2. \( a^2 + 3a^{\frac{1}{2}}b^{\frac{1}{3}} + 9b^{\frac{2}{3}} \) by \( a^{\frac{1}{3}} - 3b^{\frac{1}{3}}. \)

3. \( 1 + ab^{-1} + a^2b^{-2} \) by \( 1 - ab^{-1} + a^2b^{-2}. \)

4. \( x + 2y^{\frac{1}{2}} + 3z^{\frac{1}{3}} \) by \( x - 2y^{\frac{1}{2}} + 3z^{\frac{1}{3}}. \)

5. \( x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1} \) by \( x^{-1} - x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}. \)

6. \( a^2 - a^{\frac{1}{3}} + 1 - a^{\frac{1}{3}} + a^{\frac{2}{3}} \) by \( a^{\frac{1}{3}} + 1 + a^{-\frac{1}{3}}. \)

7. \( x^2 + y^3 + z^3 - y^{\frac{1}{3}} z^{\frac{1}{3}} - z^{\frac{1}{3}} x^{\frac{1}{3}} - x^{\frac{1}{3}} y^{\frac{1}{3}} \) by \( x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}}. \)
8. \( a^n + 3b^n - 2c^n \) by \( a^n - 3b^n + 2c^n \).

9. \( \frac{5}{2} + 8ab + 4a^2b^3 + 2a^3b^2 + 32b^5 + 16a^3b^3 \) by \( a^\frac{1}{2} - 2b^\frac{1}{3} \).

10. \( a^\frac{5}{8} + a^x x^{-\frac{3}{8}} + x^{-\frac{5}{8}} + a^\frac{3}{8} x^{-\frac{1}{4}} + a^\frac{1}{8} x^{-\frac{1}{4}} + a^\frac{1}{8} x^{-\frac{1}{8}} \) by

\[
\frac{5}{8} + a^\frac{1}{8} x^{-\frac{1}{4}} - x^{-\frac{5}{8}} - a^\frac{1}{8} x^{-\frac{1}{8}}.
\]

Divide :

11. \( x^\frac{5}{2} - 4x^\frac{3}{2} - 2x^\frac{1}{2} + 6x - x^2 \) by \( x^\frac{3}{2} + 2 - 4x^\frac{1}{2} \).

12. \( 8 + 12x^{-1} + 2x^{-2} + 2x^{-4} \) by \( x^{-2} - 2x^{-1} + 4 \).

13. \( xy^{-1} + 2x^\frac{1}{2}y^{-\frac{1}{2}} + 3 + 2x^{-\frac{1}{2}}y^\frac{1}{2} + x^{-1}y^{-1} \) by \( x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1} \).

14. \( \frac{5}{2} - a^3b + ab^2 - 2a^2b^2 + b^3 \) by \( a^\frac{3}{2} - ab + a^\frac{1}{2}b - b^\frac{3}{2} \).

15. \( 8x^{-n} - 8x^n + 5x^{3n} - 3x^{-3n} \) by \( 5x^n - 3x^{-n} \).

16. \( 8x^3 + y^\frac{3}{2} + z + 6x^\frac{1}{2}y^\frac{1}{2}z^\frac{1}{3} \) by \( 2x^\frac{1}{2} + y^{-\frac{1}{2}} - z^\frac{1}{2} \).

17. Show that \( x^3 + a^3 + x^3 a^3 \) is divisible by \( x^4 + a^4 + x^3 a^3 \).

18. Multiply \( x^{n-1} + a^{n-1} \) by \( x^{n-1} - a^{n-1} \).

19. Divide \( x^n - y^n \) by \( x^{n-1} + y^{n-1} \).

20. Simplify \( \left(\left(a^m\right)^{-\frac{m-1}{m}}\right)^{\frac{1}{m+1}} \).

21. Divide \( 2x^{-\frac{1}{4}} + 3x^\frac{3}{4} - 7x^\frac{1}{4} + x - 2x^\frac{3}{4} \) by \( x^\frac{1}{4} - 2x^{-\frac{1}{4}} \).

22. Find the square of \( x^\frac{3}{4} - x^\frac{1}{4}y^{-\frac{1}{4}} + y^\frac{1}{2} \).

23. Divide \( x^\frac{3}{2} - a^\frac{3}{2} \) by \( x^\frac{3}{2} - a^\frac{3}{2} \).

24. Find the square of \( x^\frac{1}{3} - 2x^\frac{1}{3} + x^\frac{5}{3} \).

25. Divide \( ax^{-1} + a^{-1}x + 2 \) by \( a^\frac{1}{2}x^{-\frac{1}{2}} + a^{-\frac{1}{2}}x^{\frac{1}{2}} - 1 \).

26. Simplify \( \left(\frac{a - b}{a^\frac{1}{2} - b^\frac{1}{2}} - \frac{a^\frac{3}{2} - b^\frac{3}{2}}{a^\frac{1}{2} - b^\frac{1}{2}}\right)^{-1} \).
27. Simplify \[
\frac{x^4 + 3y^3}{x^3 - 3y^3} + \frac{x^2}{x^3 + 3x^3y^3} + 9y^2
\]

28. Simplify \[
\frac{\frac{a^3}{a^3} - \frac{ax^3}{a^3} + \frac{a^2x - x^3}{a^2x^2}}{a^3 - a^2x^2 + 3ax^3 + ax^3 + a^3x^2 - x^5}
\]

29. Simplify \[
\frac{a^2 + b^2 - a^{-2} - b^{-2}}{a^2b^{-2} - a^{-2}b^{-2}} + \frac{(a - a^{-1})(b - b^{-1})}{ab + a^{-1}b^{-1}}
\]

30. Simplify \[
\frac{x - y}{x^2 + x^2y^4} + \frac{x^2 + y^2}{x^2y^4 + x^2y^4}
\]

31. Simplify \[
(a + b + c)(a^{-1} + b^{-1} + c^{-1}) - a^{-1}b^{-1}c^{-1}(b + c)(c + a)(a + b)
\] Solve: —

32. \(2^{x+7} = 4^{x+2}\). 33. \((\sqrt[3]{3})^{x+5} = \left(\frac{5}{3}\right)^{2x+5}\).

34. \((\sqrt[4]{4})^{4x+7} = (\sqrt[4]{64})^{2x+7}\).

35. \((\sqrt[2]{25})^{2x+1} = (\sqrt[2]{125})^{x+6}\).

36. \[
\begin{align*}
2^{3x-1} & = 4^{x-1} \\
3x - y & = 1
\end{align*}
\] 37. \[
\begin{align*}
9^{2x-3} & = (\sqrt[3]{3})^{2y-x} \\
2^{3x} & = 4^y
\end{align*}
\]

38. \[
\begin{align*}
4^{3y-1} & = 16^{x+y} \\
3^{x+3y} & = 9^{2x+3}
\end{align*}
\] 39. \[
\begin{align*}
2^{x+y+z} & = 8^{x+z-y} \\
3^{2x+2z+y} & = 25^{x+z}
\end{align*}
\]

40. \[
\begin{align*}
\left(\sqrt[a]{a}\right)^{x+y} & = \left(\frac{a}{a}\right)^{y+z} \\
\left(\sqrt[b]{b}\right)^{x+1} & = \left(\frac{b}{b}\right)^{y+z} \\
\left(\sqrt[c]{c}\right)^{y} & = \left(\frac{c}{c}\right)^{x+y+1}
\end{align*}
\]
CHAPTER III.

SURDS.

1. Definition. Any root of any arithmetical number which cannot be exactly found is called a surd or an irrational quantity. Thus $\sqrt{2}$, $\sqrt{6}$, $\sqrt[3]{4}$, and $\sqrt[4]{5}$ are all surds.

Note 1. Quantities which are not surds are called rational quantities. Hence every root of an arithmetical number is either rational or irrational. Thus $\frac{1}{8}$, $\sqrt{25}$ and $\sqrt[4]{16}$ are rational quantities, whilst $\sqrt{2}$, $\sqrt{5}$ and $\sqrt{9}$ are all irrational quantities.

Note 2. An algebraical expression also, such as $\sqrt{x}$, is called a surd although the value of $x$ may be such that $\sqrt{x}$ is not in reality a surd. For instance, if $x = 4$, $\sqrt{x} = \sqrt{4} = 2$, and is therefore not really a surd.

2. To express in the form of a surd the product of a rational quantity and a surd.

Example 1. $\sqrt{5} \times 3 = \left(5^2\right)^{\frac{1}{2}} \times 3^{\frac{1}{2}}$

$= \left(5^2 \times 3\right)^{\frac{1}{2}}$ [Art. 5, Chap. II.]

$= \sqrt{5^2 \times 3} = \sqrt{75}$.

Example 2. $2 \sqrt{3} = \left(2^3\right)^{\frac{1}{2}} \times 9^{\frac{1}{3}}$

$= \left(2^3 \times 9\right)^{\frac{1}{3}}$ [Art. 5, Chap. II.]

$= \sqrt[3]{2^3 \times 9} = \sqrt[3]{54}$.

Exercise (9).

Express as a complete surd:

1. $3 \sqrt{5}$.
2. $2 \sqrt[3]{3}$.
3. $2 \sqrt[4]{6}$.
4. $4 \sqrt[4]{5}$.
5. $a \sqrt{b}$.
6. $x^3 \sqrt{y}$.
7. $a^4 \sqrt{b^2}$. 
3. A surd may sometimes be expressed as the product of a rational quantity and a surd.

Example 1. \( \sqrt{32} = \sqrt{16 \times 2} \)
\[
= (4^2 \times 2)^{\frac{1}{2}} \\
= (4^1)^{\frac{1}{2}} \times 2^{\frac{1}{2}} \\
= 4 \times 2^{\frac{1}{2}} = 4 \sqrt{2}
\]

Example 2. \( \sqrt[3]{40} = \sqrt[3]{8 \times 5} \)
\[
= (2^3 \times 5)^{\frac{1}{3}} \\
= (2^1)^{\frac{1}{3}} \times 5^{\frac{1}{3}} \\
= 2 \times 5^{\frac{1}{3}} \\
= 2^{\frac{2}{3}} \sqrt[3]{5}
\]

Exercise (10.)

Simplify:

1. \( \sqrt{18} \)  
2. \( \sqrt{80} \)  
3. \( \sqrt[3]{250} \)  
4. \( \sqrt[5]{128} \)  
5. \( \sqrt[4]{405} \)  
6. \( \sqrt[3]{1372} \)  
7. \( \sqrt[4]{1875} \)  
8. \( \sqrt[3]{a^6 b} \)  
9. \( \sqrt[3]{x^4 a} \)  
10. \( \sqrt[3]{-2560} \)  
11. \( \sqrt[3]{-192a^3 b^4} \)  
12. \( \sqrt[3]{500a^7 x^4} \)

4. Similar surds. Two or more surds are said to be similar or like when they can be so reduced as to have the same irrational factor. Thus \( \sqrt[4]{45} \) and \( \sqrt[3]{80} \) are similar surds for they are respectively equivalent to \( 3 \sqrt[4]{5} \) and \( 4 \sqrt[3]{5} \). The sum of any number of similar surds may be found as follows:

Example 1. \( \sqrt{147} + \sqrt{27} \)
\[
= \sqrt{49 \times 3} + \sqrt{9 \times 3} \\
= 7 \sqrt{3} + 3 \sqrt{3} = 10 \sqrt{3}
\]

Example 2. \( \sqrt[3]{625} - \sqrt[3]{135} + \sqrt[3]{40} \)
\[
= \sqrt[3]{125 \times 5} - \sqrt[3]{27 \times 5} + \sqrt[3]{8 \times 5}
\]
\[ = \sqrt[3]{5^3} \times 5 - \sqrt[3]{3^3} \times 5 + \sqrt[3]{2^3} \times 5, \]
\[ = 5 \sqrt[3]{5} - 3 \sqrt[3]{5} + 2 \sqrt[3]{5} \]
\[ = 4 \sqrt[3]{5}. \]

**Exercise (II).**

Simplify:—

1. \( \sqrt{12} + \sqrt{75} \)
2. \( \sqrt{18} + \sqrt{32} \)
3. \( \sqrt{26} + \sqrt{180} \)
4. \( \sqrt{98} - \sqrt{50} \)
5. \( \sqrt[3]{128} - \sqrt[3]{54} \)
6. \( \sqrt[4]{80} + \sqrt[4]{405} \)
7. \( \sqrt[6]{68} - \sqrt[6]{243} \)
8. \( 2 \sqrt{27} - \sqrt{75} + \sqrt{12} \)
9. \( 2 \sqrt{405} - 3 \sqrt{125} + \sqrt{45} \)
10. \( 4 \sqrt{192} - 4 \sqrt{375} + 2 \sqrt{34} \)
11. \( 3 \sqrt{40} + 2 \sqrt{625} - 4 \sqrt{320} \)
12. \( 5 \sqrt[3]{54} - 2 \sqrt[3]{16} + 4 \sqrt[3]{686} \)
13. \( \sqrt[4]{49x^3} + \sqrt[5]{50x^5} + \sqrt[5]{5xy^2} \)
14. \( x^2 / x^3a + y^3 / \sqrt[3]{-8y^3a} - z^3 / 27z^3a \)
15. \( 2 \sqrt[3]{32a^4x} + 3 \sqrt[4]{512a^4x} - 4a \sqrt[4]{162x} \)

5. Surds of the same order. Surds are said to be of the same order or *equiradical* when they have all got the same root symbol. Thus \( \sqrt[5]{5}, \sqrt[5]{a^3} \) and \((a + x)^5\) are all surds of the same (the second) order.

A surd of the second order is often called a **quadratic surd**; whilst one of the third order, as \( \sqrt[3]{1/4} \) or \( \sqrt[3]{a^2} \), is called a **cubic surd**.

Surds of different orders may be reduced to equivalent surds of the same order.

**Example 1.** Reduce \( \sqrt[5]{5} \) and \( \sqrt[3]{4} \) to surds of the same order.

The given surds are respectively of the 2nd and 3rd orders; and the L. C. M. of 2 and 3 is 6. Hence we can at once reduce them to surds of the 6th order, thus:—

\[ \sqrt[5]{5} = 5^{\frac{1}{5}} = 5^\frac{3}{15} = \sqrt[15]{5^3} \]
\[ = \sqrt[15]{125} \]
\[ \sqrt[3]{4} = 4^{\frac{1}{3}} = 4^\frac{2}{6} = \sqrt[6]{4^2} \]
\[ = \sqrt[6]{16}. \]

Thus the required surds are \( \sqrt[6]{125} \) and \( \sqrt[6]{16} \).
Example 2. Reduce $\sqrt[6]{3}$ and $\sqrt[3]{2}$ to surds of the same order.

The L. C. M. of 6 and 8 is 24.

Thus we have:

$\sqrt[6]{3} = 3^{\frac{1}{6}} = 3^{\frac{2}{12}} = 2^{4}\sqrt{3}$

and $\sqrt[3]{2} = 2^{\frac{1}{3}} = 2^{\frac{2}{6}} = 2^{4}\sqrt{3}$

Thus the required surds are $2^{4}\sqrt{8}$ and $2^{4}\sqrt{3}$.

Example 3. Which is the greater $\sqrt[3]{9}$ or $\sqrt[4]{20}$?

We have $\sqrt[3]{9} = 9^{\frac{1}{3}} = 9^{\frac{4}{12}} = 12\sqrt[3]{81}$

and $\sqrt[4]{20} = 20^{\frac{1}{4}} = 20^{\frac{3}{12}} = 12\sqrt[3]{8000}$.

Thus the given surds are respectively equivalent to $12\sqrt{8561}$ and $12\sqrt{8000}$, and as the latter is greater than the former, therefore $\sqrt[4]{20} > \sqrt[3]{9}$.

Exercise (12).

Reduce to surds of the same order:—

1. $\sqrt{3}$ and $\sqrt{2}$.
2. $\sqrt[4]{3}$ and $\sqrt[5]{5}$.
3. $\sqrt[2]{2}$ and $\sqrt[3]{3}$.
4. $\sqrt[3]{3}$ and $\sqrt[4]{5}$.
5. $\sqrt[4]{4}$ and $\sqrt[6]{6}$.

Which is the greater:—

6. $\sqrt{2}$ or $\sqrt[3]{3}$?
7. $\sqrt[3]{3}$ or $\sqrt[4]{4}$?
8. $\sqrt[4]{6}$ or $\sqrt[10]{10}$?

Arrange according to descending order of magnitude:—

9. $\sqrt[6]{6}$, $\sqrt{2}$ and $\sqrt[4]{4}$.
10. $\sqrt[3]{3}$, $\sqrt[10]{10}$ and $\sqrt[2]{25}$.

6. Multiplication and division of surds.

Example 1. $\sqrt[3]{6} \times \sqrt[3]{10} = 6^{\frac{1}{3}} \times 10^{\frac{1}{3}}$

$= (6 \times 10)^{\frac{1}{3}}$

$= \sqrt[3]{60}$

Note. In this example the given surds are of the same order.
Example 2. \( \sqrt[3]{5} \times \sqrt[8]{8} = 5^{\frac{3}{8}} \times 8^{\frac{1}{8}} \)

\[ = \left( 5^{3} \right)^{\frac{1}{8}} \times \left( 8^{2} \right)^{\frac{1}{8}} \quad [\text{Art. 4, Chap. II.}] \]

\[ = \left( 5^{3} \times 8^{2} \right)^{\frac{1}{8}} \quad [\text{Art. 5, Chap. II.}] \]

\[ = \sqrt[12]{125} \times 64 \]

\[ = \sqrt[12]{8000} \cdot \]

Note. In this example the given surds are of different orders.

Example 3. \( \sqrt[2]{2} \times \sqrt[5]{2} = 2^{\frac{1}{2}} \times 2^{\frac{1}{5}} \)

\[ = 2^{\frac{1}{2} + \frac{1}{5}} \]

\[ = 2^{\frac{3}{10}} = \sqrt[10]{2} \cdot \sqrt[5]{2} \]

\[ = \sqrt[10]{256} \cdot \]

Note. In this example the given surds have got the same quantity under the radical sign. They may as well be regarded as surds of different orders and treated like those in the last example.

Example 4. \( 4 \sqrt[12]{18} \times \sqrt[75]{2} \)

\[ = 4 \cdot 3 \sqrt{2} \times 5 \sqrt{3} \]

\[ = 60 \sqrt{2} \cdot \sqrt{3} = 60 \sqrt{6} \cdot \]

Note. In this example the given surds have been reduced to simpler forms before multiplication.

Example 5. \( \sqrt[3]{4} \div \sqrt[6]{6} = 4^{\frac{1}{3}} \div 6^{\frac{1}{6}} \)

\[ = 4^{\frac{2}{6}} \div 6^{\frac{1}{6}} \]

\[ = \frac{\left( 4^{2} \right)^{\frac{1}{6}}}{\left( 6^{3} \right)^{\frac{1}{6}}} \quad [\text{Art. 4, Chap. II.}] \]

\[ = \frac{\left( 4^{3} \right)^{\frac{1}{12}}}{\left( 6^{3} \right)^{\frac{1}{12}}} \quad [\text{Cor. 1, Art. 5, Chap. II.}] \]

\[ = \sqrt[12]{\frac{2}{27}} \cdot \]
Example 6. Express \( \sqrt{5} \div 3 \sqrt{3} \) as a fraction with a rational denominator.

We have:

\[
\sqrt{5} \div 3 \sqrt{3} = \frac{\sqrt{5}}{3 \sqrt{3}} = \frac{\sqrt{5} \times \sqrt{3}}{3 \sqrt{3} \times \sqrt{3}}
\]

\[
= \frac{\sqrt{15}}{3 \times 3} = \frac{\sqrt{15}}{9}.
\]

Note. For arithmetical calculations it is always most convenient to reduce the quotient of one surd by another to the form of a fraction with a rational denominator. Hence even when the numerical value of a surd fraction is not required it is usual to express it in the above form.

Exercise (13).

Simplify:

1. \( \sqrt{5} \times \sqrt{10} \).
2. \( \sqrt{8} \times \sqrt{6} \).
3. \( \sqrt{27} \times \sqrt{3} \).
4. \( \sqrt{15} \times \sqrt{6} \).
5. \( \sqrt{20} \times \sqrt{45} \).
6. \( \frac{\sqrt{5}}{3} \times \sqrt{25} \).
7. \( \sqrt{6} \times x \times \sqrt{27a^2x^3} \).
8. \( \sqrt{2} \times \sqrt{5} \).
9. \( \sqrt{2} \times \sqrt{6} \).
10. \( \sqrt{4} \times \sqrt{8} \).
11. \( \sqrt{9} \times \sqrt{27} \).
12. \( \sqrt{2} \times \sqrt{3} \).
13. \( \sqrt{3} \times \sqrt{3} \).
14. \( \sqrt{2} \times \sqrt{2} \).
15. \( \sqrt{4} \times \sqrt{4} \).
16. \( 5 \sqrt{8} \times 2 \sqrt{6} \).
17. \( 8 \sqrt{12} \times 3 \sqrt{24} \).
18. \( 4 \sqrt{72} \times 5 \sqrt{576} \).
19. \( 7 \sqrt{8a^3x^2} \times 5 \sqrt{27b^3x^2} \).
20. \( 8 \sqrt{10} \div 4 \sqrt{15} \).
21. \( 3 \sqrt{12} \div 6 \sqrt{27} \).
22. \( \sqrt{36} \div \sqrt{18} \).
23. \( \sqrt{8} \div \sqrt{6} \).

Given \( \sqrt{2} = 1.414 \), \( \sqrt{3} = 1.732 \), \( \sqrt{5} = 2.236 \), find to 3 places of decimals the numerical value of:

24. \( \sqrt{2} \div \sqrt{6} \).
25. \( \sqrt{72} \div \sqrt{40} \).
26. \( \sqrt{275} \div \sqrt{22} \).
27. \( 10 \sqrt{108} \div \sqrt{15} \).

7. Compound Surds. An expression consisting of two or more simple surds connected by the sign + or - is called a compound surd. Thus \( 5 \sqrt{2} + 4 \sqrt{3} \) are simple surds, but \( 5 \sqrt{2} + 4 \sqrt{3} \) and \( 5 \sqrt{2} - 4 \sqrt{3} \) are compound surds.

Two or more compound surds are multiplied together in the same way as two or more compound algebraical expressions.
Example 1. Multiply \( \sqrt{x} + 2 \sqrt{3} \) by \( \sqrt{x} - \sqrt{3} \).

\[
(3 \sqrt{x} + 2 \sqrt{3})(\sqrt{x} - \sqrt{3}) = 3 \sqrt{x} \cdot \sqrt{x} + 2 \sqrt{3} \cdot \sqrt{x} - 3 \sqrt{x} \cdot \sqrt{3} - 2 \sqrt{3} \cdot \sqrt{3} \\
= 3x + 2 \sqrt{3x} - 3 \sqrt{3x} - 6 \\
= 3x - \sqrt{3x} - 6.
\]

Example 2. Multiply \( 7 \sqrt{2} + \sqrt{3} \) by \( 7 \sqrt{2} - \sqrt{3} \).

\[
(7 \sqrt{2} + \sqrt{3})(7 \sqrt{2} - \sqrt{3}) = (7 \sqrt{2})^2 - (\sqrt{3})^2 \\
= 49 \cdot 2 - 3 \\
= 98 - 3 = 95.
\]

Example 3. Find the square of \( \sqrt{3+a} + x + \sqrt{3a-x} \).

\[
(\sqrt{3+a} + x + \sqrt{3a-x})^2 = (\sqrt{3+a} + x)^2 + (\sqrt{3a-x})^2 + 2 \sqrt{(3+a+x)(3a-x)} \\
= (3+a+x)(3a-x) + 2 \sqrt{3a^2 - x^2} \\
= 6a + 2 \sqrt{9a^2 - x^2}.
\]

Exercise (14).

Multiply :-

1. \( \sqrt{a} + \sqrt{b} \) by \( \sqrt{ab} \).
2. \( \sqrt{a} + \sqrt{b} \) by \( \sqrt{a} - \sqrt{b} \).
3. \( 3 \sqrt{a} - 5 \) by \( 2 \sqrt{a} \).
4. \( 4 \sqrt{x} + 3 \sqrt{y} \) by \( 4 \sqrt{x} - 3 \sqrt{y} \).
5. \( 2 \sqrt{x} + 5 + 4 \) by \( 3 \sqrt{x} - 5 - 6 \).
6. \( 3 \sqrt{5} - 4 \sqrt{2} \) by \( 2 \sqrt{5} + 3 \sqrt{2} \).
7. \( \sqrt{2} + 2 \sqrt{3} + \sqrt{7} \) by \( \sqrt{2} + 2 \sqrt{3} - \sqrt{7} \).
8. \( 3 - \sqrt{5} + \sqrt{8} \) by \( 3 - \sqrt{5} - \sqrt{8} \).
9. \( \sqrt{11} + \sqrt{6} - \sqrt{3} \) by \( \sqrt{11} - \sqrt{6} + \sqrt{3} \).
10. \( \sqrt{a} + \sqrt{b} + \sqrt{c} \) by \( \sqrt{2} + \sqrt{3} \).

Find the square of :-

11. \( \sqrt{x+a} - \sqrt{x-a} \).
12. \( 2 \sqrt{8} + 5 \sqrt{6} \).
13. \( 2 \sqrt{5} + 3 \sqrt{7} \).
14. \( \sqrt{a^2 + 2b^2} - \sqrt{a^2 - 2b^2} \).
15. \( 2 \sqrt{x^2 + y^2} + 5 \sqrt{x^2 - y^2} \).
8. Rationalisation. If two surds be such that their product is rational, each of them is said to be rationalised when multiplied by the other. Thus $2\sqrt{5} + \sqrt{2}$ are rationalised when respectively multiplied by $\sqrt{5}$ and $\sqrt{3} - \sqrt{2}$; for,

$$2\sqrt{5} \times \sqrt{5} = 10,$$

and $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 3 - 2 = 1$.

Two binomial quadratic surds which differ only in the sign which connects their terms are said to be **conjugate** or **complementary** to each other. Thus $\sqrt{3} + \sqrt{2}$ and $2\sqrt{5} - \sqrt{7}$ are respectively conjugate or complementary to $\sqrt{3} - \sqrt{2}$ and $2\sqrt{5} + \sqrt{7}$.

Evidently therefore every binomial quadratic surd is rationalised when multiplied by the complementary surd.

Hence a fraction with a binomial quadratic surd for its denominator can be easily reduced to an equivalent fraction with a rational denominator.

**Example 1.** Given $\sqrt{2} = 1.414$, find to three places of decimals the value of $\frac{1 + \sqrt{2}}{3 - 2\sqrt{2}}$.

$$\frac{1 + \sqrt{2}}{3 - 2\sqrt{2}} = \frac{(1 + \sqrt{2})(3 + 2\sqrt{2})}{(3 - 2\sqrt{2})(3 + 2\sqrt{2})} = \frac{3 + 3\sqrt{2} + 2\sqrt{2} + 4}{9 - 8}$$

$$= 7 + 5\sqrt{2}$$

$$= 7 + 5 \times 1.414$$

$$= 7 + 7.070 = 14.070.$$  

**Example 2.** Rationalise the denominator of

$$\frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{\sqrt{1 + x^2} + \sqrt{1 - x^2}}.$$

The given expression

$$= \frac{(\sqrt{1 + x^2} - \sqrt{1 - x^2})^2}{(\sqrt{1 + x^2} + \sqrt{1 - x^2})(\sqrt{1 + x^2} - \sqrt{1 - x^2})}$$

$$= \frac{(1 + x^2) + (1 - x^2) - 2\sqrt{1 - x^4}}{(1 + x^2) - (1 - x^2)}$$

$$= \frac{2 - 2\sqrt{1 - x^4}}{2x^2} = \frac{1 - \sqrt{1 - x^4}}{x^2}$$
Example 3. Simplify \( \frac{3 + \sqrt{6}}{5\sqrt{3} - 2\sqrt{12} - \sqrt{32} + \sqrt{50}} \).

The denominator = \( 5\sqrt{3} - 2 \times 2\sqrt{3} - 4\sqrt{2} + 5\sqrt{2} \)

= \( \sqrt{3} + \sqrt{2} \).

Hence, the given fraction

= \( \frac{3 + \sqrt{6}}{\sqrt{3} + \sqrt{2}} \)

= \( \frac{(3 + \sqrt{6})(\sqrt{3} - \sqrt{2})}{(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})} \)

= \( \frac{3\sqrt{3} - 3\sqrt{2} + 3\sqrt{2} - 2\sqrt{3}}{3 - 2} \)

= \( \sqrt{3} \).

Example 4. Simplify \( \frac{3\sqrt{2}}{\sqrt{3} + \sqrt{6}} - \frac{4\sqrt{3}}{\sqrt{6} + \sqrt{2}} + \frac{\sqrt{6}}{\sqrt{2} + \sqrt{3}} \).

The 1st term

= \( \frac{3\sqrt{2}}{\sqrt{3}(1 + \sqrt{2})} = \frac{\sqrt{6}}{\sqrt{2} + 1} = \frac{\sqrt{6}(\sqrt{2} - 1)}{(\sqrt{2} + 1)(\sqrt{2} - 1)} \)

= \( 2\sqrt{3} - \sqrt{6} \)

The 2nd term

= \( \frac{4\sqrt{3}}{\sqrt{2}(\sqrt{3} + 1)} = \frac{2\sqrt{6}}{\sqrt{3} + 1} = \frac{2\sqrt{6}(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)} \)

= \( 2(3\sqrt{2} - \sqrt{6}) \)

= \( 3\sqrt{2} - \sqrt{6} \).

The 3rd term

= \( \frac{\sqrt{6}}{\sqrt{2} + \sqrt{3}} = \frac{\sqrt{6}(\sqrt{3} - \sqrt{2})}{(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})} = 3\sqrt{2} - 2\sqrt{3} \).

Hence the given expression

= \( (2\sqrt{3} - \sqrt{6}) - (3\sqrt{2} - \sqrt{6}) + (3\sqrt{2} - 2\sqrt{3}) \)

= 0

Example 5. Bring \( \frac{7}{2^{1/3} + 2^{1/4} + 1} \) to a form with a rational denominator.

(Bombay University P. E. Paper, 1890.)
The given expression
\[
\frac{\sqrt[4]{7}}{(2^{\frac{1}{2}} + 1) + 2^{\frac{1}{2}}} = \frac{7\{(2^{\frac{1}{2}} + 1) - 2^{\frac{1}{2}}\}}{\{(2^{\frac{1}{2}} + 1) + 2^{\frac{1}{2}}\}\{(2^{\frac{1}{2}} + 1) - 2^{\frac{1}{2}}\}}
\]

of which the denominator
\[
= (2^{\frac{1}{2}} + 1)^2 - 2^{\frac{1}{2}}
= (2 + 2^{\frac{1}{2}} + 1) - 2^{\frac{1}{2}} = 3 + 2^{\frac{1}{2}}.
\]

Hence, the given expression
\[
= \frac{7\{(2^{\frac{1}{2}} + 1) - 2^{\frac{1}{2}}\}}{3 + 2^{\frac{1}{2}}}
= \frac{7(1 - 2^{\frac{1}{2}} + 2^{\frac{1}{2}})(3 - 2^{\frac{1}{2}})}{(3 + 2^{\frac{1}{2}})(3 - 2^{\frac{1}{2}})}
= \frac{7(3 - 3\cdot2^{\frac{1}{2}} + 3\cdot2^{\frac{1}{2}} - 2^{\frac{3}{2}} - 2)}{9 - 2}
= 1 - 3\cdot2^{\frac{1}{2}} + 2\cdot2^{\frac{1}{2}} + 2^{\frac{3}{2}}.
\]

Note. It may be observed in this connection that any fraction of the form \(\frac{x}{\sqrt{a} + \sqrt{b} + \sqrt{c}}\) can be reduced to an equivalent fraction with a rational denominator as follows:—Multiply the numerator and denominator by \(\sqrt{a} + \sqrt{b} - \sqrt{c}\); the denominator thus becomes \(\sqrt{a} + \sqrt{b} - \sqrt{c}\) or \(a + b - c + 2\sqrt{ab}\). Now multiply both numerator and denominator by \((a + b - c) - 2\sqrt{ab}\), on which the denominator becomes \((a + b - c)^2 - 4ab\), which is rational.

**Exercise (15).**

Reduce to an equivalent fraction with a rational denominator:

1. \(\frac{5\sqrt{3} + \sqrt{7}}{4\sqrt{3} + 2\sqrt{7}}\)
2. \(\frac{3\sqrt{2} + 2\sqrt{3}}{3\sqrt{2} - 2\sqrt{3}}\)
3. \(\frac{4 + 3\sqrt{2}}{3 - 2\sqrt{2}}\)
4. \(\frac{3\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}}\)
5. \(\frac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}}\)
6. \(\frac{\sqrt{x^2 + 1} - \sqrt{x^2 - 1}}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}\)
7. \(\frac{1}{1 + \sqrt{2} + \sqrt{3}}\)
8. \(\frac{12}{3 + \sqrt{5} - 2\sqrt{3}}\)
9. \(\frac{\sqrt[3]{3} + \sqrt[5]{5}}{\sqrt[2]{2} + \sqrt[3]{3} + \sqrt[5]{5}}\)  

10. \(\frac{1}{\sqrt{10} + \sqrt{14} + \sqrt{15} + \sqrt{21}}\)

Given \(\sqrt{2} = 1.414\), \(\sqrt{3} = 1.732\), \(\sqrt{5} = 2.236\), find to three places of decimals the value of:

11. \(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\)  
12. \(\frac{\sqrt{3}}{2 - \sqrt{3}}\)  
13. \(\frac{8 - 5\sqrt{2}}{3 - 2\sqrt{2}}\)

14. \(\frac{3}{\sqrt{3} - \sqrt{2}}\)  
15. \(\frac{3 + \sqrt{5}}{3 - \sqrt{5}}\)  
16. \(\frac{\sqrt{5} + \sqrt{3}}{4 + \sqrt{15}}\)

Simplify:

17. \(\frac{1}{x + \sqrt{x^2 - 1}} + \frac{1}{x - \sqrt{x^2 - 1}}\)

18. \(\sqrt{10} + \sqrt{20} + \sqrt{40} - \sqrt{5} - \sqrt{80}\)

19. \(\sqrt{2}(\sqrt{3} + 1)(2 - \sqrt{3})\)  
(\(\sqrt{3} + 1)(3\sqrt{3} - 5)(2 + \sqrt{2})\)

20. \((3 + 2\sqrt{2})^3 + (3 - 2\sqrt{2})^3\)

21. \(\frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} - \frac{x - \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}}\)

22. \(\sqrt{x^2 + 1 + \sqrt{x^2 - 1}} + \sqrt{x^2 + 1 - \sqrt{x^2 - 1}}\)
\(\sqrt{x^2 + 1 - \sqrt{x^2 - 1}} + \sqrt{x^2 + 1 + \sqrt{x^2 - 1}}\)

9. To find a factor which will rationalise any given binomial surd.

Every binomial surd must be of the form \(\sqrt{a} + \sqrt{b}\) or \(\sqrt{a} - \sqrt{b}\). For instance, \(3^\frac{3}{5} + 2^\frac{4}{7} = 5\sqrt[5]{3^3} + 7\sqrt[7]{2^4} = 5\sqrt[5]{27} + 7\sqrt[7]{16}\); and \(4^\frac{3}{5} - 3^\frac{4}{1} = 4\sqrt[5]{4^3} - 11\sqrt[5]{3^4} = 4\sqrt[5]{64} - 11\sqrt[5]{81}\).

I. Let it be of the form \(\sqrt{a} + \sqrt{b}\).

Let \(\sqrt{a} = x\), and \(\sqrt{b} = y\); and let \(n\) be the L.C.M. of \(p\) and \(q\).

Then \(x^n\) and \(y^n\) are clearly rational quantities.

Hence, if a factor \(P\) be found such that \((x + y)P = x^n + y^n\) or \(x^n - y^n\), \(P\) will be the very factor we seek.
III.]

SURDS.

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Now, we know that if \( n \) be an even integer \( x + y \) divides \( x^n - y^n \) only, and we have \( x^n - y^n = (x + y)(x^{n-1} - x^{n-2}y + \ldots + xy^{n-2} - y^{n-1}) \); and if \( n \) be an odd integer \( x + y \) divides \( x^n + y^n \) only, and we have

\[ x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \ldots - xy^{n-2} + y^{n-1}). \]

Hence, if \( n \) be even, the rationalising factor and the rational product are respectively \( x^{n-1} - x^{n-2}y + \ldots + xy^{n-2} - y^{n-1}, \)
and \( x^n - y^n \); whilst if \( n \) be odd, these quantities are respectively \( x^{n-1} - x^{n-2}y + \ldots - xy^{n-2} + y^{n-1}, \) and \( x^n + y^n \).

II. If the binomial surd be of the form \( \sqrt[3]{a} - \sqrt[3]{b}, \) taking \( x, \ y \) and \( n \) to mean the same as before, we see that a factor \( Q \) will rationalise \( x - y \) if \( Q \) be such that \( (x - y).(Q) = x^n - y^n \) or \( x^n + y^n \).

Now, we know that \( x - y \) divides \( x^n - y^n \) only for all values of \( n, \) and that \( x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1}) \).

Hence, in this case, whether \( n \) be odd or even, the rationalising factor is \( x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1} \), and the rational product is \( x^n - y^n \).

Example 1. Find the factor which will rationalise \( \sqrt[3]{3} + \sqrt[2]{2} \).

The given surd = \( 3^{\frac{1}{3}} + 2^{\frac{1}{2}} \).

Putting \( x = 3^{\frac{1}{3}}, y = 2^{\frac{1}{2}}, \) we see that \( x^6 \) and \( y^6 \) are both rational, and we also know that \( x^6 - y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5) \).

Hence, the required factor

\[ = x^6 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5 \]

\[ = \left(3^{\frac{1}{3}}\right)^6 - \left(3^{\frac{1}{3}}\right)^4\cdot2^{\frac{1}{2}} + \left(3^{\frac{1}{3}}\right)^3\cdot\left(2^{\frac{1}{2}}\right)^2 - \left(3^{\frac{1}{3}}\right)^2\cdot\left(2^{\frac{1}{2}}\right)^3 + 3^{\frac{1}{3}}\cdot\left(2^{\frac{1}{2}}\right)^4 \]

\[ - \left(2^{\frac{1}{2}}\right)^5 \]

\[ = 3^\frac{4}{3} - 3^\frac{2}{3} \cdot 2^{\frac{1}{2}} + 3.2 - 3^\frac{1}{3} \cdot 2^2 + 3^\frac{1}{3} \cdot 2^2 - 2^5 \]

and the rational product

\[ = \left(3^{\frac{1}{3}}\right)^6 - \left(2^{\frac{1}{2}}\right)^6 = 3^a - 2^a = 1. \]

Example 2. Reduce \( \frac{3\sqrt[3]{3} - 3\sqrt[5]{5}}{\sqrt[3]{3} - \sqrt[5]{5}} \) to an equivalent fraction with a rational denominator.
The denominator of the given expression

\[ = 3^2 - 5^\frac{1}{3}. \]

Putting \( x = 3^{\frac{1}{3}}, y = 5^{\frac{1}{3}} \), we see that \( x^6 \) and \( y^6 \) are both rational, and we also know that

\[ x^6 - y^6 = (x - y)(x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5). \]

Hence, the rationalising factor of the denominator

\[ = x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5 \]
\[ = \left(3^{\frac{1}{3}}\right)^5 + \left(3^{\frac{1}{3}}\right)^4 \cdot 5^{\frac{1}{3}} + \left(3^{\frac{1}{3}}\right)^3 \cdot 5^{\frac{2}{3}} + \left(3^{\frac{1}{3}}\right)^2 \cdot 5^{\frac{1}{3}} + 3^{\frac{1}{3}} \cdot 5^{\frac{2}{3}} + 5^{\frac{5}{3}}; \]

and the rationalised denominator

\[ = \left(3^{\frac{1}{3}}\right)^6 - \left(5^{\frac{1}{3}}\right)^6 = 3^2 - 5^2 = 2. \]

Hence, the given expression

\[ = \frac{\left(3^{\frac{1}{3}}-5^{\frac{1}{3}}\right)\left(3^{\frac{5}{3}} + 3^2 \cdot 5^{\frac{2}{3}} + 3^\frac{1}{3} \cdot 5^{\frac{1}{3}} + 3^{\frac{1}{3}} \cdot 5^{\frac{2}{3}} + 5^{\frac{5}{3}}\right)}{2}. \]

**Exercise (16).**

Find the factor which will rationalise:—

1. \( \sqrt{3} - \sqrt[4]{4} \).

2. \( \sqrt{6} + 4\sqrt[3]{3} \).

3. \( 3 + \sqrt[4]{4} \).

4. \( 2^{\frac{5}{4}} - 3^{\frac{3}{5}} \).

5. \( 3^{\frac{4}{3}} + 4^{\frac{5}{6}} \).

6. \( 4 - 7\sqrt[5]{5} \).

Reduce to an equivalent fraction with a rational denominator:—

7. \( \frac{2 - \sqrt[4]{4}}{2 + \sqrt[4]{4}} \).

8. \( \frac{\sqrt{3} + \sqrt[2]{2}}{\sqrt{3} - \sqrt[2]{2}} \).

9. \( \frac{\sqrt[8]{8} - 4\sqrt[3]{2}}{\sqrt[8]{8} + 4\sqrt[3]{2}} \).

10. \( \frac{3 - \sqrt[9]{9}}{3 + \sqrt[9]{9}} \).

10. The square root of a rational quantity cannot be partly rational and partly a quadratic surd. If possible let \( \sqrt{n} = a + \sqrt{m} \).
Then, squaring both sides, we have
\[ n = a^2 + m + 2a \sqrt{m}, \]
whence,
\[ \sqrt{m} = \frac{n - a^2 - m}{2a}. \]

Thus a surd is equal to a rational quantity, which is impossible.

11. If \( a + \sqrt{b} = x + \sqrt{y}, \) where \( a \) and \( x \) are rational, and \( \sqrt{b} \) and \( \sqrt{y} \) are irrational, then will \( a = x, \) and \( b = y. \)

For if \( a \) be not equal to \( x, \) let \( a = x + m. \)
Then we have \( x + m + \sqrt{b} = x + \sqrt{y}; \)
\[ \therefore \ m + \sqrt{b} = \sqrt{y}. \]

Thus \( \sqrt{y} \) is partly rational and partly a quadratic surd, which is impossible by the last article.

Therefore \( a = x, \) and consequently \( \sqrt{b} = \sqrt{y}, \) or \( b = y. \)

Note. It should be distinctly borne in mind that the results proved above are true only when \( \sqrt{b} \) and \( \sqrt{y} \) are really irrational. For instance, from the relation \( 5 + \sqrt{9} = 3 + \sqrt{25}, \) we cannot conclude that \( 5 = 3 \) and \( 9 = 25. \)

12. To find the square root of \( a + \sqrt{b}, \) where \( \sqrt{b} \) is a surd.

Let \( \sqrt{a + \sqrt{b}} = \sqrt{x + \sqrt{y}}. \)
Then, squaring both sides, we have
\[ a + \sqrt{b} = x + y + 2 \sqrt{xy}. \]
Hence, by the last article,
\[ a = x + y \]
and \( \sqrt{b} = 2 \sqrt{xy} \) \( \ldots \ldots \) (1)
Hence, \( a^2 - \delta = (x + y)^2 - 4xy \)
\[ = (x - y)^2, \]
\[ \therefore \sqrt{a^2 - b} = x - y. \]
Thus we have \( x + y = a \)
and \( x - y = \sqrt{a^2 - b}. \)
Hence, by addition and subtraction,
\[ 2x = a + \sqrt{a^2 - b}, \text{ and } 2y = a - \sqrt{a^2 - b}; \]
\[ \therefore \ x = \frac{1}{2}(a + \sqrt{a^2 - b}), \text{ and } y = \frac{1}{2}(a - \sqrt{a^2 - b}). \]
Thus \[ \sqrt{a + \sqrt{b}} = \sqrt{\frac{1}{2}(a + \sqrt{a^2 - b}) + \sqrt{\frac{1}{2}(a - \sqrt{a^2 - b})}}. \]

**Note.** From the values of \( x \) and \( y \) found above it is clear that unless \( \sqrt{a^2 - b} \) is rational the square root obtained is by far more complicated than the original expression. Thus the process given above is of no great practical value except when \( a^2 - b \) is a perfect square.

**Cor.** From (1) we have \( a - \sqrt{b} = x + y - 2\sqrt{xy} = (\sqrt{x} - \sqrt{y})^2 \);
\[ \therefore \sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}. \] Thus if \( \sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y} \)
then will \( \sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}. \)

**Example 1.** Find the square root of \( 7 + 2\sqrt{10} \).

Let \( \sqrt{7 + 2\sqrt{10}} = \sqrt{x} + \sqrt{y}. \)

Then, squaring both sides,
\[ 7 + 2\sqrt{10} = x + y + 2\sqrt{xy}. \]

Hence,
\[ \begin{align*}
  x + y &= 7 \\
  xy &= 10
\end{align*} \]

The relations are evidently satisfied by the numbers 5 and 2.
Hence, the required root = \( \sqrt{5} + \sqrt{2} \).

**Example 2.** Find the square root of \( 16 - 5\sqrt{7} \).

Let \( \sqrt{16 - 5\sqrt{7}} = \sqrt{x} - \sqrt{y}. \)

Then \( 16 - 5\sqrt{7} = x + y - 2\sqrt{xy}. \)

Therefore
\[ \begin{align*}
  x + y &= 16 \\
  2\sqrt{xy} &= 5\sqrt{7}
\end{align*} \]

Hence,
\[ (x - y)^2 = (x + y)^2 - 4xy \]
\[ = 16^2 - (5\sqrt{7})^2 \]
\[ = 256 - 175 = 81; \]
\[ \therefore \ x - y = 9. \]
Thus we have \( \begin{align*}
  x + y &= 16 \\
  x - y &= 9
\end{align*} \).
whence, \( x = \frac{25}{2}, \) and \( y = \frac{7}{2}. \)

Thus the required root \( = \sqrt{\frac{25}{2}} - \sqrt{\frac{7}{2}}. \)

**Example 3.** Find the square root of \( \sqrt{27} + \sqrt{15}. \)

\( \sqrt{27} + \sqrt{15} = 3 \sqrt{3} + \sqrt{3} \sqrt{5} = \sqrt{3(3 + \sqrt{5})}. \)

Hence, \( \sqrt{\sqrt{27} + \sqrt{15}} = \sqrt{3} \sqrt{3 + \sqrt{5}}. \)

Now, proceeding as in the last example, we find that

\( \sqrt{3 + \sqrt{5}} = \sqrt{\frac{1}{2} + \sqrt{\frac{5}{2}}}. \)

Therefore \( \sqrt{\sqrt{27} + \sqrt{15}} = \sqrt{3} \left( \sqrt{\frac{1}{2} + \sqrt{\frac{5}{2}}} \right). \)

**Exercise (17).**

Find the square root of:—

1. \( 8 + 2 \sqrt{15}. \)
2. \( 14 - 6 \sqrt{5}. \)
3. \( 17 + 12 \sqrt{2}. \)
4. \( 37 - 20 \sqrt{3}. \)
5. \( 31 + 4 \sqrt{33}. \)
6. \( 73 - 12 \sqrt{35}. \)
7. \( 47 + 4 \sqrt{33}. \)
8. \( 6 - \sqrt{35}. \)
9. \( \sqrt{18} - \sqrt{16}. \)
10. \( \sqrt{32} - \sqrt{24}. \)
11. \( \sqrt{27} + \sqrt{24}. \)
12. \( 5 \sqrt{5} + \sqrt{120}. \)
13. \( a^2 + 2x \sqrt{a^2 - x^2}. \)
14. \( 2a + 2 \sqrt{a^2 - b^2}. \)
15. \( a + x + \sqrt{2ax + x^2}. \)
16. \( 2x - 1 + 2 \sqrt{x^2 - x - 6}. \)
17. \( x + y + z + 2 \sqrt{iz + yz}. \)

13. If \( \sqrt[3]{a + \sqrt{b}} = x + \sqrt{y}, \) then will \( \sqrt[3]{a - \sqrt{b}} = x - \sqrt{y}. \)

Since \( \sqrt[3]{a + \sqrt{b}} = x + \sqrt{y}, \)

cubing both sides, we have

\( a + \sqrt{b} = x^3 + 3x^2 \sqrt{y} + 3xy + y \sqrt{y}. \)

Hence, equating rational and irrational parts, we have

\( a = x^3 + 3xy \)

and \( \sqrt{b} = 3x^2 \sqrt{y} + y \sqrt{y}. \)
Hence, \(a - \sqrt[3]{b} = x^2 - 3x^2 \sqrt[3]{y + 3xy - y \sqrt[3]{y}}\).
\[
\therefore \quad \sqrt[3]{a - \sqrt[3]{b}} = \sqrt[3]{x - y}.
\]

**Cor.** Conversely if \(\sqrt[3]{a - \sqrt[3]{b}} = x - \sqrt[3]{y}\), then \(\sqrt[3]{a + \sqrt[3]{b}} = x + \sqrt[3]{y}\).

**Example 1.** Find the cube root of \(38 + 17 \sqrt[3]{5}\).

Let \(\sqrt[3]{38 + 17 \sqrt[3]{5}} = x + \sqrt[3]{y}\),

then \(\sqrt[3]{38 - 17 \sqrt[3]{5}} = x - \sqrt[3]{y}\).

Therefore, by multiplication,

\[
x^2 - y = \sqrt[3]{1444} - \sqrt[3]{1445} = \sqrt[3]{-1} = -1.
\]

Again, since \(\sqrt[3]{38 + 17 \sqrt[3]{5}} = x + \sqrt[3]{y}\),
\[
\therefore \quad 38 + 17 \sqrt[3]{5} = x^3 + 3x^2 \sqrt[3]{y} + 3xy + y \sqrt[3]{y},
\]

whence \(38 = x^3 + 3xy\).

Thus we have \(x^3 + 3xy = 38\)

and \(x^2 - y = -1\)\)

Hence, \(x^3 + 3x(x^2 + 1) = 38\)

or, \(4x^3 + 3x = 38\).

By trial, we find \(x = 2\), and hence \(y = x^2 + 1 = 5\).

Thus the required root = \(2 + \sqrt[3]{5}\).

**Notes.** The method shown above is practically of no use unless the value of \(x^2 - y\) as found above be rational.

**Example 2.** Find the cube root of \(21 \sqrt[3]{6} - 23 \sqrt[3]{5}\).

\[
21 \sqrt[3]{6} - 23 \sqrt[3]{5} = 6 \sqrt[3]{\frac{21}{6} - \frac{23 \sqrt[3]{5}}{6 \sqrt[3]{6}}}
\]

\[
= (\sqrt[3]{6})^3 \left(\frac{7}{2} - \frac{23}{6} \sqrt[3]{\frac{5}{6}}\right);
\]

\[
\therefore \quad \sqrt[3]{21 \sqrt[3]{6} - 23 \sqrt[3]{5}} = \sqrt[3]{6} \sqrt[3]{\frac{7}{2} - \frac{23 \sqrt[3]{5}}{6}}.
\]

Let \(\sqrt[3]{\frac{7}{2} - \frac{23}{6} \sqrt[3]{\frac{5}{6}}} = x - \sqrt[3]{y}\),

then \(\sqrt[3]{\frac{7}{2} + \frac{23}{6} \sqrt[3]{\frac{5}{6}}} = x + \sqrt[3]{y}\).
Hence \( x^2 - y = \sqrt{\frac{49}{4} - \frac{2645}{216}} = \sqrt{\frac{1}{216}} = \frac{1}{6} \ldots (1) \),

also, \( \frac{7}{2} - \frac{23}{6} \sqrt{\frac{5}{6}} = x^3 - 3x^2 \sqrt{y} + 3xy - y \sqrt{y} \);

\[ \therefore x^3 + 3xy = \frac{7}{2} \ldots \ldots \ldots \ldots (2) \]

Hence, from (1) and (2), we have

\[ x^3 + 3x \left( x^2 - \frac{1}{6} \right) = \frac{7}{2} ; \]

or,

\[ 8x^3 - x = 7. \]

By trial, we find \( x = 1 \), and \( \therefore y = \frac{5}{6} \).

Thus \( \sqrt[3]{\frac{7}{2} - \frac{23}{6} \sqrt{\frac{5}{6}}} = 1 - \sqrt{\frac{5}{6}} ; \)

\[ \therefore \text{the required root} = \sqrt{6} \left( 1 - \sqrt{\frac{5}{6}} \right) = \sqrt{6} - \sqrt{5}. \]

**Exercise (18).**

Find the cube root of:

1. \( 19 + 9 \sqrt[6]{6}. \)
2. \( 26 - 15 \sqrt[3]{3}. \)
3. \( 11 \sqrt[5]{5} + 17 \sqrt[2]{2}. \)
4. \( 99 \sqrt[2]{2} - 59 \sqrt[5]{3}. \)
5. \( 264 \sqrt[3]{3} + 150 \sqrt[6]{6}. \)

14. **Miscellaneous Examples.**

**Example 1.** Find the square root of \( 6 + \sqrt{12} - \sqrt{24} - \sqrt{8} \).

Assume \( \sqrt{6 + \sqrt{12} - \sqrt{24} - \sqrt{8}} = \sqrt{x} + \sqrt{y} - \sqrt{z} ; \)

Then we must have

\[ 6 + \sqrt{12} - \sqrt{24} - \sqrt{8} = x + y + z + 2 \sqrt{xy} - 2 \sqrt{yz} - 2 \sqrt{zx}. \]

If now, \( x, y, z \) be such that

\[ \begin{align*}
2 \sqrt{xy} &= \sqrt{\frac{12}{2}} \\
2 \sqrt{yz} &= \sqrt{\frac{24}{4}} \\
2 \sqrt{zx} &= \sqrt{\frac{8}{8}}
\end{align*} \]

and also \( x + y + z = 6 \), then the required root will be found.
From the first three equations we have
\[ \sqrt{xy} = \sqrt{3} \quad \ldots (1) \quad \therefore \text{by multiplication,} \]
\[ \sqrt{yz} = \sqrt{2} \cdot \sqrt{3} \quad \ldots (2) \quad \therefore \quad xyz = 2.3 = 6 ; \]
\[ \sqrt{zx} = \sqrt{2} \quad \ldots (3) \quad \therefore \quad \sqrt{xyz} = \sqrt{6} \quad \ldots (4) \]
Dividing (4) by (2), (3) and (1) respectively, we have
\[ \sqrt{x} = \sqrt{1}, \quad \sqrt{y} = \sqrt{3}, \quad \sqrt{z} = \sqrt{2} ; \]
and these values of \( x, y, z \) also satisfy the equation \( x + y + z = 6 \).

Hence the required root \( = 1 + \sqrt{3} - \sqrt{2} \).

Example 2. Simplify \( \frac{2 + \sqrt{3}}{\sqrt{2} + \sqrt{2} + \sqrt{3}} + \frac{2 - \sqrt{3}}{\sqrt{2} - \sqrt{2} - \sqrt{3}} \)
(Bombay University P. E. Paper, 1888).

The 1st term of the given expression
\[ = \frac{\sqrt{2}(2 + \sqrt{3})}{2 + \sqrt{4} + 2\sqrt{3}} \]
\[ = \frac{\sqrt{2}(2 + \sqrt{3})}{2 + (1 + \sqrt{3})} \]
\[ = \frac{\sqrt{2}(2 + \sqrt{3})}{\sqrt{3}(\sqrt{3} + 1)} \]
\[ = \frac{4 + 2\sqrt{3}}{\sqrt{2} \cdot \sqrt{3}(\sqrt{3} + 1)} \]
\[ = \frac{(\sqrt{3} + 1)^2}{\sqrt{6}(\sqrt{3} + 1)} \]
\[ = \frac{\sqrt{3} + 1}{\sqrt{6}} ; \]

and the 2nd term
\[ = \frac{\sqrt{2}(2 - \sqrt{3})}{2 - \sqrt{4} - 2\sqrt{3}} \]
\[ = \frac{\sqrt{2}(2 - \sqrt{3})}{2 - (\sqrt{3} - 1)} \]
\[ = \frac{\sqrt{2}(2 - \sqrt{3})}{\sqrt{3}(\sqrt{3} - 1)} \]
\[ = \frac{4 - 2\sqrt{3}}{\sqrt{2} \cdot \sqrt{3}(\sqrt{3} - 1)} \]