11. If one root of the equation \( ax^2 + bx + c = 0 \), be the square of the other, prove that \( b^2 + a^2c + ac^2 = 3abc \).

12. If \( ax^2 + bx + c = a'x^2 + b'x + c' \), when \( x = 183, 281 \) and \( 397 \) respectively, prove that \( a = a' \), \( b = b' \) and \( c = c' \).

\[ (ax^2 + bx + c) - (a'x^2 + b'x + c') = 0, \]
\[ i.e., \quad (a-a')x^2 + (b-b')x + (c-c') = 0 \] for three distinct values of \( x \).
\[ \therefore \] by Art. 244, \( a-a' = 0 \), \( b-b' = 0 \) and \( c-c' = 0 \).}

13. Find \( a, b, c \), if \( (a - 12)x^2 + (b - 31)x - 181 - c \) for any value of \( x \).

14. Find \( k \), if the roots of \( 5x^2 + 7kx + 3 = 0 \) be the reciprocals of the roots of \( 3x^2 + (8 - k)x + 5 = 0 \).

15. Find \( a \) and \( k \), if the roots of \( 3x^2 + 2kx + k + 2 = 0 \) be the reciprocals of the roots of \( 2ax^2 + (k + a)x + 3 = 0 \).

---

CHAPTER XXXV
EQUATIONAL PROBLEMS

248. What are eggs a dozen when two more in a shilling's worth lowers the price one penny per dozen?

Let \( x = \) the number of eggs we get for a shilling.

Then the price of each egg = \( \frac{12}{x} \) pence,

and \( \therefore \) the price of a dozen = \( \frac{144}{x} \) pence. \( \ldots (1) \)

If two more were obtained for a shilling, i.e., if \( (x + 2) \) eggs were worth a shilling, the price of a dozen would, for a similar reason, be \( \frac{144}{x + 2} \) pence.

But by the condition of the problem, the latter price is one penny less than the former price, hence,

\[ \frac{144}{x + 2} = \frac{144}{x} - 1; \]
\[ \therefore \quad x^2 + 2x = 288, \]
\[ \therefore \quad x^2 + 2x + 1 = 289; \]
\[ \therefore \quad x + 1 = 17. \]
\[ \therefore \quad x = 16. \]

Hence, from (1), the price per dozen = 9d.
249. Find two numbers, whose difference multiplied by the difference of their squares = 160; and whose sum, multiplied by the sum of their squares gives the number 560.

Let \(x + y\) and \(x - y\) be the numbers.

Then, by the 1st condition of the problem,
\[
2y(4xy) = 160,
\]
or,
\[
xy^2 = 20.
\]

By the 2nd condition of the problem,
\[
2x^2 + 2y^2 = 560,
\]
or,
\[
x(x^2 + y^2) = 145.
\]

From (1) and (2), by subtraction,
\[
x^2 = 125 = 5^2;
\]
\[
\therefore \quad x = 5.
\]

Hence, from (1),
\[
xy^2 = 5y^2 = 20,
\]
\[
i.e., \quad y^2 = 4;
\]
\[
\therefore \quad y = 2.
\]
\[
\therefore \quad x = 5, \text{ and } y = 2.
\]

Hence, the required numbers are 7 and 3.

250. A sets off from London to York and B at the same time from York to London, and they travel uniformly; A reaches York 6 hours and B reaches London 36 hours, after they have met on the road. Find in what time each has performed the journey.

\[
\begin{array}{ccc}
L & m & M \\
& n & Y \\
\end{array}
\]

Let \(L\) and \(Y\) represent London and York respectively, and \(M\) the place where the travellers meet. Let \(m, n\) be the measures of \(LM, MY\) respectively in kilometres.

Now, since A travels \(n\) kilometres (i.e., from \(M\) to \(Y\)) in 16 hours he travels 1 kilometre in \(\frac{16}{n}\) hours and \(\therefore m\) kilometres in \(\frac{16}{n} \cdot m\) hours;

\[
\text{hence, the time in which A travelled from } L \text{ to } M = \frac{16}{n} \cdot m \text{ hours.}
\]

Similarly, the time in which B travelled from \(Y\) to \(M = \frac{36}{n} \cdot m \text{ hours.}

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Now, since they started at the same instant, the time in which $A$ travelled from $L$ to $M$ is evidently equal to the time in which $B$ travelled from $Y$ to $M$.

$$\frac{16}{n} \cdot m = \frac{36}{m} \cdot n,$$

whence

$$\frac{m}{n} = \frac{3}{2}.$$

Hence, the time in which $A$ performed the journey

$$= \left(\frac{16}{n} \cdot m + 16\right) \text{ hours} = 40 \text{ hours};$$

and the time in which $B$ performed the journey

$$= \left(\frac{36}{m} \cdot n + 36\right) \text{ hours} = 60 \text{ hours}.$$

251. A fraudulent tradesman contrives to employ his false balance both in buying and selling a certain article, thereby gaining 11 per cent. more on his outlay than he would gain, were the balance true. If, however, the scale-pan, in which the article is weighed when bought and sold respectively, were interchanged, he would neither gain nor lose by the transaction. Determine the legitimate gain per cent. on the article.

[In a false balance if any weight be placed on one of the scale-pan, the weight to be put on the other pan in order to make the beam horizontal will be different. For instance, if in buying rice a five-kilogram counterpoise be put on the pan, the quantity of rice put on the other will be either more or less than 5 kilograms. Suppose when the five-kilogram counterpoise is put on the scale-pan $A$, we are required to put on the pan $B$, a quantity of rice whose real weight is greater than 5 kilograms; but whatever may be its real weight, as its weight now is supposed to be equal to the weight of the counterpoise, we take it to be 5 kilograms. Thus, we take for 5 kilograms what is really more than 5 kilograms. Hence, if the merchant contrives to put the counterpoise on $A$ and the article bought on $B$, he will evidently take away more of the article than he is supposed to do; let the supposed weight of the article, so bought, be $w$ kilograms; if then $W$ kilograms be the real weight of the article, $w$ is less than $W$. Again, in selling the article if he puts the counterpoise on $B$ and the article on $A$ and if $W$ be the weight of the counterpoise, then $W'$ is greater than $W$. By this contrivance then the merchant buys $W$ kgs. of the article at the price of $w$ kgs. and sells away these $W$ kgs. again at the price of $W'$ kgs. Hence, in such a transaction the merchant’s gain is two-fold, he buys more of the article than he pays for and the whole quantity thus bought he sells away at the price of a still greater quantity.]

Let $w$ and $W'$ be the apparent weights of the article when bought and sold respectively.
Then, evidently \( w \) is less, and \( W' \) greater, than the true weight.

Let \( p \) = prime cost of unit of weight,

\( x \) = the legitimate gain per cent.

Then, the selling price of a unit of weight

\[ -p + x \text{ hundredths of } p = p \left( 1 + \frac{x}{100} \right) \]

Hence, the price paid by the merchant in buying the article, i.e., his outlay = \( w \cdot p \), and the price realised by selling it = \( W \cdot p \left( 1 + \frac{x}{100} \right) \):

\[ W \cdot p \left( 1 + \frac{x}{100} \right) = w \cdot p + (x + 11) \text{ hundredths of } w \cdot p. \]

\[ = w \cdot p \left( 1 + \frac{x + 11}{100} \right). \]...

(1)

If the scale-pans were interchanged, the cost of buying the article would be \( W' \cdot p \) and the price realised by sale, \( w \cdot p \left( 1 + \frac{x}{100} \right) \); hence by the 2nd condition of the problem,

\[ w \cdot p \left( 1 + \frac{x}{100} \right) = W' \cdot p. \]...

(2)

From (1) and (2),

\[ 1 + \frac{x + 11}{100} = 1 + \frac{x}{100}, \]

or,

\[ \frac{x}{100} + x + 11 = 1 + \frac{x}{100} \]

or,

\[ \left( \frac{x}{100} + \frac{1}{4} \right) = \left( \frac{6}{10} \right)^2; \]

\[ \frac{x}{100} = \frac{6}{10} - \frac{1}{2} = \frac{1}{10}; \]

\[ x = 10, \]

i.e., the legitimate gain is 10 per cent.

252. A body of men were formed into a hollow square, three deep, when it was observed that with the addition of 25 to their number, a solid square might be formed, of which the number of men in each side would be greater by 22 than the square root of the number of men in each side of the hollow square. Required the number of men in the hollow square.
A number of men are said to be arranged in a solid square when they are arranged in parallel rows and the number of rows is equal to the number of men in each row. The following diagram, in which $A_1$, $B_1$, $C_1$, &c., represent men, will clearly illustrate the matter.

The above diagram represents an arrangement in which there are 8 rows, each containing 8 men. This is a solid square. If the square $C_1F_1E_1D_1$ be removed from inside, the remainder will be a hollow square two deep having 8 men in each side; if, however, the square $D_4E_4F_4E_4$ be removed, the remainder will be a hollow square three deep.

Hence, the number of men in a hollow square two deep having $x$ men in each side $= x^2 - (x-6)^2$; in one three deep $= x^2 - (x-6)^2$; and so on; thus, the number of men in a hollow square $n$ deep having $x$ men in each side $= x^2 - (x-2n)^2$.

Let $x = $ the number of men in a side of the hollow square; then the whole number of men $= x^2 - (x-6)^2$ \[ \cdots \cdots \cdots \] \[ (1) \]

Hence, by the 2nd condition of the problem,

$\begin{align*}
\quad & x^2 - (x-6)^2 + 25 = (x^\frac{1}{3} + 22)^2, \\
or, & 12x - 11 = x + 44x^\frac{1}{3} + 484, \\
\therefore & 11x - 44x^\frac{1}{3} = 495, \\
or, & x - 4x^\frac{1}{3} = 45, \\
\therefore & x - 4x^\frac{1}{3} + 4 = 49.
\end{align*}$
\[ x^3 - 2 = 7 ; \text{ whence } x = 3. \]

Hence, from (1), the whole number of men

\[ -81^2 - 75^2 = 156 \times 6 = 936. \]

253. \( K \) engages to play a game of chess with \( B \) on the following condition that \( B \) should name a certain number and put into \( K \)'s possession twenty-four rupees together with as many rupees as equal to the square of this number and that at the conclusion of the game \( K \) should return to \( B \) only a number of rupees equal to eight times the number named. What number could \( B \) name with the greatest advantage possible to himself?

Let \( x \) be the number which \( B \) should name; then he has to deposit with \( K \), \((24 + x^2)\) rupees and get back at the end of the game only \( 8x \) rupees;

hence, \( B \) has altogether to lose \((x^2 + 24 - 8x)\) rupees.

\[ \therefore x \text{ must be such that this loss may be as small as possible.} \]

Now, since \( x^2 - 8x + 24 = (x - 4)^2 + 8 \), which is always greater than 8 except when \( x = 4 \), the loss will for all values of \( x \) be greater than Rs. 8 except when \( x \) has this value.

Hence, in order that the loss may be a minimum \( B \) should name the number 4.

254. With the object of examining a student of the 1st year as regards his progress in Algebra, I undertake to engage in a certain contract with him, which is as follows: he is to give me a certain number of books, each worth as many rupees as the number of books, and to get from me in return six times as many rupees as any of those books is worth and also 21 rupees more. How many books should he bring me, with the greatest possible advantage to himself?

Let \( x \) be the number of books that the student brings me; then, since the price of each book is \( x \) rupees, evidently I get \( x^2 \) rupees from him; and in return I give him \((6x + 21)\) rupees.

Hence, his gain (or loss as the case may be) = \((21 + 6x - x^2)\) rupees.

Now, \( 21 + 6x - x^2 = 21 - (x^2 - 6x) = 30 - (x^2 - 6x + 9) = 30 - (x - 3)^2 \).

Evidently, therefore, the student is a loser if \( x - 3 \) be greater than 5, i.e., if \( x \) be greater than 8; and he is a gainer if \( x \) be 8 or less than 8.

But not only should the student be a gainer but his gain must be the greatest possible, which evidently is the case when \((x - 3)^2\) is the least possible, i.e., when \( x = 3 \).

Hence, the student should bring me only three books.
255. Rama, Lakshmana and Bharata went to visit a Rishi and brought their wives with them. The Rishi knew the wives' names to be Urmila, Mandavi and Sita, but forgot which was the wife of each hero. They told the Rishi that they had given presents to Pandits, and that each of the six had rewarded as many Pandits, as he or she had given gold mudras to each Pandit. Rama had rewarded 23 more Pandits than Urmila, and Lakshmana had rewarded 11 Pandits more than Mandavi, likewise each hero had given away 63 gold mudras more than his wife. The Rishi having thought on what they said, dismissed them with his blessing, naming correctly the wife of each hero. From the conditions given, do you also find out the names of the wives?

Let \( x \) = the number of Pandits rewarded by any hero,
and \( y \) = the number of Pandits rewarded by his wife;
then the number of gold mudras given away by the hero = \( x^2 \);
and the number of gold mudras given away by his wife = \( y^2 \).

Hence, by the last condition of the problem, we have

\[
x^2 - y^2 = 63, \quad \text{or}, \quad (x+y)(x-y) = 63.
\]

But \( 63 = 63 \times 1 \), or, \( 21 \times 3 \), or, \( 9 \times 7 \);

hence, since \( x+y \) and \( x-y \) are positive integers, and \( x+y \) is necessarily greater than \( x-y \), we get the following three pairs of values for \( x+y \), and \( x-y \) and \( no \ other \).

\[
\begin{align*}
(1) \quad x+y &= 63, \\
& \quad x-y = 1 \\
(2) \quad x+y &= 21, \\
& \quad x-y = 3 \\
(3) \quad x+y &= 9, \\
& \quad x-y = 7
\end{align*}
\]

Hence, we have the following three pairs of values for \( x \) and \( y \):

\[
\begin{align*}
(1) \quad x &= 32, \\
& \quad y = 31 \\
(2) \quad x &= 12, \\
& \quad y = 9 \\
(3) \quad x &= 8, \\
& \quad y = 1
\end{align*}
\]

... \( \text{(A)} \)

i.e., the 'wife of the hero who rewarded 32 Pandits, rewarded 31 Pandits;
the wife of the hero who rewarded 12 Pandits, rewarded 9 Pandits;
... \( \text{... \( \text{(a)} \}} \)
and the wife of the hero who rewarded 8 Pandits, rewarded only one Pandit.
... \( \text{... \( \beta \)} \}

Now, let us find out the names of the wives from the other conditions of the problem.

The number of Pandits rewarded by Rama may be 32, 12 or 8; but since he is known to have rewarded 23 more Pandits than somebody else, the number of Pandits rewarded by him must be 32.

The number of Pandits rewarded by Lakshmana may then be either 12 or 8, but as he is known to have rewarded 11 more Pandits than somebody else, the number of Pandits rewarded by him must be 12, ... \( \text{(a)} \)

Hence, the number of Pandits rewarded by Bharata must be 8. ... \( \text{(b)} \)
Again, since the number of Pandits rewarded by Urmila is 23 less than the number rewarded by Rama, it must be 9; hence, by (a) and (a), Urmila is the wife of Lakshmana;

also, since the number of Pandits rewarded by Mandavi is 11 less than the number rewarded by Lakshmana, it must be 1; and, therefore, by (β) and (β), Mandavi is the wife of Bharata; evidently therefore, Sita is the wife of Rama.

Thus, we have

\[
\begin{align*}
\text{Rama} & \}, & \text{Lakshmana} & \}, & \text{Bharata} & \}, \\
\text{Sita} & \}, & \text{Urmila} & \}, & \text{Mandavi} & \}
\end{align*}
\]

**EXERCISE 135**

1. A person bought a certain number of oxen for £80; if he had bought 4 more for the same sum, each ox would have cost £1 less; find the number of oxen and the price of each.

2. A gentleman sends a lad into the market to buy a shilling's worth of oranges. The lad having eaten a couple, the gentleman pays at the rate of a penny more than the market price. How many did the gentleman get for his shilling?

3. The plate of a looking glass is 18 centimetres by 12, and is to be framed with a frame of equal width, whose area is to be equal to that of the glass. Required the width of the frame.

4. A and B lay out some money on speculation. A disposes of his bargain for £11, and gains as much per cent. as B lays out; B's gain is £36, and it appears that A gains four times as much per cent. as B. Required the capital of each.

5. A boat's crew row 3½ kilometres down a river and back again in 1 hour and 40 minutes. Supposing the river to have a current of 2 kilometres per hour, find the rate at which the crew would row in still water.

6. What two numbers are those whose sum multiplied by the greater is 204; and whose difference multiplied by the less is 35?

7. What two numbers are those whose sum added to the sum of their squares is 42 and whose product is 15?

8. A and B distribute £60 each among a certain number of persons. A relieves 40 persons more than B does, and B gives to each 5s. more than A. How many persons did A and B respectively relieve?

9. The product of two numbers added to their sum is 28; and five times their sum taken from the sum of their squares leaves 8; required the numbers.

10. A horse dealer buys a horse, and pays a certain sum for it; he afterwards sells it again for Rs. 171, and gains exactly as much per cent. as the horse had cost him. How much did he pay for the horse?

11. The small wheel of a bicycle makes 135 revolutions more than the large wheel in a distance of 250 metres; if the circumference of each
were one-third metre more, the small wheel would make 27 revolutions
more than the large wheel in a distance of 70 metres. Find the circum-
ference of each wheel.

12. By lowering the price of apples and selling them one penny
a dozen cheaper, an apple-woman finds that she can sell 60 more than she
used to do for 5s. At what price per dozen did she sell them at first?

13. There is a number between 10 and 100; when multiplied by
the digit on the left the product is 280, if the sum of the digits be mul-
plied by the same digit the product is 55; required the number.

14. A and B are two stations 300 kilometres apart. Two trains start
simultaneously from A and B, each to the opposite station. The train
from A reaches B nine hours, the train from B reaches A four hours,
after they meet. Find the rate at which each train travels.

15. By selling a horse for £24, I lose as much per cent. as it costs me.
What was the prime cost of it?

16. Find three numbers, such that if the first be multiplied by the
sum of the second and the third, the second by the sum of the first and
the third and the third by the sum of the first and the second, the
products shall be 408, 480 and 501 respectively.

17. There are two square buildings that are paved with stones, a
metre square each. The side of one building exceeds that of the other by
12 metres, and both their pavements taken together contain 2120 stones.
What are the lengths of them separately?

18. There are three numbers, the difference of whose differences
is 5; their sum is 44, and continued product 1950; find the numbers.

19. A train A starts to go from P to Q, two stations 240 kilometres
apart, and travels uniformly. An hour later, another train B starts
from P, and after travelling for 2 hours, comes to a point that A had
passed 45 minutes previously. The pace of B is now increased by
5 kilometres an hour, and it overtakes A just on entering Q. Find the
rates at which they started.

20. A square court-yard has a rectangular gravel walk round it
inside. The side of the court wants 2 metres of being 6 times the
breadth of the gravel walk; and the number of square metres in the
walk exceeds the number of metres in the periphery of the court by 92.
Required the area of the court.

21. Divide the number 26 into three such parts that their squares
may have equal differences, and that the sum of those squares may
be 300.

22. The number of soldiers present at a review is such that they
could all be formed into a solid square and also could be formed into
four hollow squares each 4 deep and each containing 24 more men
in the front rank than when formed into a solid square; find the whole
number.
23. A and B run a race round a two-kilometre course. In the first hit B reaches the winning post 2 minutes before A. In the second hit A increases his speed 2 kilometres an hour, and B diminishes his by the same quantity; and A then reaches the winning post 2 minutes before B. Find at what rate each ran in the first hit.

24. From a vessel of wine containing a gallons, b gallons are drawn off and the vessel is filled up with water. Find the quantity of wine remaining in the vessel when this has been repeated 4 times.

25. A wall was built round a rectangular court to a certain height. Now the length of one side of the court was two metres less, whilst three times the length of the other was 25 metres greater than 8 times the height of the wall; and the number of square metres in the court was greater than the number in the wall by 178. Required the dimensions of the court, and the height of the wall.

26. A person bought a number of £20 railway shares when they were at a certain rate per cent. discount for £1,500; and afterwards when they were at the same rate per cent. premium sold them all but 60 for £1,000. How many did he buy and what did he give for each of them?

27. The sum of 4 numbers is 54; the sum of the product of the 1st and 2nd, and 3rd and 4th is 250; of the 1st and 3rd, and 2nd and 4th is 234; and of the 1st and 4th, and 2nd and 3rd is 225. Find them.

28. To complete a certain work A requires \( \frac{1}{m} \) times as long a time as B and C together; B requires \( \frac{1}{n} \) times as long as A and C together, and C requires \( \frac{1}{p} \) times as long as A and B together. Compare the times in which each would do it and prove that

\[
\frac{1}{m+1} + \frac{1}{n+1} + \frac{1}{p+1} = 1.
\]

29. In a certain village there lived in the year 1872 a number of families each consisting of as many members as there were families. Ten years afterwards it was found that during this interval there were 670 births in the village and that on the average 50 lives were lost per family. Prove that the number of persons, living in the village at the time of this calculation, could not be less than 45, and if this number be actually 45, find out the number of souls that lived in the village in the year 1872.

30. Suppose you agree to give me out of your landed property a square plot of ground and receive in exchange a circular plot of land whose area is 76 square metres and also a rectangular plot, one of whose sides is 36 metres and the other is equal to a side of the piece of land you give me. What must be the area of the plot you give me, so that you can profit most by the exchange.
CHAPTER XXXVI
GRAPHS OF QUADRATIC EQUATIONS AND EXPRESSIONS
AND THEIR APPLICATIONS

256. The graphs of $XY=0$, $X$ and $Y$ being expressions of the first degree in $x$ and $y$.

Example 1. Draw the graph of the equation $x^2 = 25$.

The equation $x^2 = 25$ may be written as

\[
\begin{align*}
x^2 - 25 &= 0 \\
(x - 5)(x + 5) &= 0
\end{align*}
\]

Evidently, the given equation is satisfied (i) by all those points which satisfy the equation $x - 5 = 0$; (ii) by all those points which satisfy the equation $x + 5 = 0$.

Hence, the required graph consists of two straight lines, one being the graph of the equation $x - 5 = 0$ and the other being the graph of the equation $x + 5 = 0$, as shown in the diagram with twice the length of a side of a small square as unit of length.

Example 2. Draw the graph of the equation $x^2 - 3x - 28 = 0$.

Factorising the left-hand side of the equation, we have

\[(x - 7)(x + 4) = 0.\]

Hence, proceeding as in example 1, we notice that the required graph consists of two straight lines, one being the graph of the equation $x - 7 = 0$ and the other being the graph of $x + 4 = 0$, as shown in the diagram with twice the length of a side of small a square as unit of length.
Example 3. Draw the graph of the equation $y^2 = 4x^2$.

From the given equation, we have

\[
\begin{align*}
y^2 - 4x^2 &= 0 \\
or, \quad (y + 2x)(y - 2x) &= 0
\end{align*}
\]

Clearly, the given equation is satisfied by (i) all those points which satisfy the equation $y + 2x = 0$, and also (ii) by all those points which satisfy the equation $y - 2x = 0$.

Hence, the required graph consists of two straight lines, one being the graph of the equation $y + 2x = 0$, and the other being the graph of the equation $y - 2x = 0$.

Hence, the required graph is as shown below with twice the length of a side of a small square as unit of length.

Example 4. Draw the graph of the equation $2x^2 + xy - y^2 - 11x + 4y + 5 = 0$.

Factorising the left-hand side of the given equation, we have

\[(x + y - 5)(2x - y - 1) = 0.\]
Obviously, the given equation is satisfied (i) by all those points which satisfy the equation \( x + y - 5 = 0 \) as well as (ii) by all those points which satisfy the equation \( 2x - y - 1 = 0 \).

Hence, the required graph consists of two straight lines, one being the graph of the equation \( x + y - 5 = 0 \) and the other being the graph of the equation \( 2x - y - 1 = 0 \), as shown in the diagram with twice the length of a side of a small square as unit of length.

257. Thus, it is clear from the above examples that whenever a quadratic equation can be expressed in the form \( XY = 0 \), where \( X \) and \( Y \) are expressions of the first degree in \( x \) and \( y \), the graph consists of a pair of straight lines, which are respectively the graphs of the equations \( X = 0 \) and \( Y = 0 \).

When, however, a quadratic equation cannot be expressed in the form \( XY = 0 \), its graph is a curve. We shall now proceed to consider a few graphs of this nature.

258. The graph of a quadratic equation in which the coefficients of \( x^2 \) and \( y^2 \) are equal and positive and there is no term involving the product of \( x \) and \( y \), is a Circle. The equation of this type of graph is generally of the following forms:

(i) \( x^2 + y^2 = a^2 \),
(ii) \( (x-h)^2 + (y-k)^2 = a^2 \),
(iii) \( x^2 + y^2 + ax + by + c = 0 \).
Draw the graph of the equation \( x^2 + y^2 = a^2 \).

The process of drawing the graph is being explained, with 6 for the value of \( a \).

\[
x^2 + y^2 = 36, \quad \text{or} \quad y^2 = 36 - x^2.
\]

It is evident that for every value of \( x \), there will be two equal and opposite values of \( y \).

(i) If \( x = 0 \), \( y^2 = 36 \),
\[ y = \pm \sqrt{36} = \pm 6. \]
So the points \((0, 6), (0, -6)\) will be on the required graph.

(ii) If \( x = \pm 6, y = 0 \),
\[ \therefore \quad \text{the points} \ (6, 0) \ \text{and} \ (-6, 0) \ \text{will be on the required graph}. \]

(iii) If \( x = \pm 2, y^2 = 28 \),
\[ \therefore \quad y = \pm 2 \sqrt{7} = \pm 4 \times 1.196\ldots = \pm 5.2. \]
Hence the points \((2, 5.2), (2, -5.2), (-2, 5.2), (-2, -5.2)\) are on the required graph.

(iv) If \( x = \pm 3, y^2 = 27 \),
\[ \therefore \quad y = \pm 3 \sqrt{3} = \pm 3 \times 1.732\ldots = \pm 5.2. \]
Hence the points \((3, 5.2), (3, -5.2), (-3, 5.2)\) and \((-3, -5.2)\) are on the required graph.

(v) If \( x = \pm 4, y^2 = 20 \),
\[ \therefore \quad y = \pm 2 \sqrt{5} = \pm 2 \times 2.236\ldots = \pm 4.5. \]
Hence the points \((4, 4.5), (4, -4.5), (-4, 4.5)\) and \((-4, -4.5)\) are on the required graph.

The corresponding values of \( x \) and \( y \) may be tabulated as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>0</th>
<th>6</th>
<th>-6</th>
<th>2</th>
<th>2</th>
<th>-2</th>
<th>-2</th>
<th>8</th>
<th>8</th>
<th>-8</th>
<th>-8</th>
<th>4</th>
<th>4</th>
<th>-4</th>
<th>-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>6</td>
<td>-6</td>
<td>0</td>
<td>0</td>
<td>5.7</td>
<td>5.7</td>
<td>-5.7</td>
<td>-5.7</td>
<td>5.2</td>
<td>5.2</td>
<td>-5.2</td>
<td>-5.2</td>
<td>4.5</td>
<td>4.5</td>
<td>-4.5</td>
<td>-4.5</td>
</tr>
</tbody>
</table>

Let five times the length of a side of a small square represent the unit of length.

Plotting the points tabulated above and drawing a free-hand and continuous curve, we obtain the required graph as shown in the diagram at page 510.

Note. In the diagram on page 510, if a circle is drawn with the origin \( O \) as centre and radius equal to 6 units of length with the help of a pencil compass, it will be found that the circle almost coincides with the free-hand curve. Thus the accuracy of the free-hand drawing can be verified by drawing a circle.
Take any point $P$ on the circle, and let its co-ordinates be denoted by $x$ and $y$; evidently then $x^2 + y^2 = OP^2 = 36$. But if a point, such as $Q$, be taken anywhere not on the circle, it is easy to see that its co-ordinates will not satisfy the given equation.

Thus, it is shown that the co-ordinates of every point on the circle, and of no other point satisfy the given equation. Hence, the circle drawn is the required graph.

259. Draw the graph of the equation $(x-h)^2 + (y-k)^2 = a^2$.

With different values of $h$, $k$ and $a$ the process of drawing the graph of the equation $(x-h)^2 + (y-k)^2 = a^2$ is being explained. When $h=3$, $k=2$ and $a=5$, the equation becomes $(x-3)^2 + (y-2)^2 = 25$.

\[
(x-3)^2 + (y-2)^2 = 25,
\]

or,

\[
(y-2)^2 = 25 - (x-3)^2;
\]

\[
y-2 = \pm \sqrt{25 - (x-3)^2};
\]

\[
y = 2 \pm \sqrt{25 - (x-3)^2}.
\]
The corresponding values of \( x \) and \( y \) in the equation \( y = 2 \pm \sqrt{25 - (x - 3)^2} \) may be tabulated as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>0</th>
<th>3</th>
<th>3</th>
<th>-1</th>
<th>1</th>
<th>-2</th>
<th>6</th>
<th>6</th>
<th>7</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>6</td>
<td>-2</td>
<td>7</td>
<td>-3</td>
<td>5</td>
<td>-1</td>
<td>2</td>
<td>6</td>
<td>-2</td>
<td>5</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

Taking five times the length of a side of a small square as the unit of length, let us plot the points tabulated above on squared paper. The continuous curve line joining the points is the required graph.

Let \( A \) be the point (3, 2). With centre \( A \) and a radius equal to 5 units of length describe a circle as in the diagram below. Then this circle and the curve passing through the points will be the same. Thus the accuracy of the free-hand drawing can be verified.

Take any point \( P \) on the circle, and let its co-ordinates be denoted by \( x \) and \( y \). Now from the diagram, it is clear that \( AP \) is the hypotenuse of a right-angled triangle of which the sides are \( (x - 3) \) and \( (y - 2) \) units of length respectively.
Hence, \((x - 3)^2 + (y - 2)^2 = AP^2 = 25\), which shows that the co-
ordinates of \(P\) satisfy the given equation. But if a point, such as 
\(Q\), be taken anywhere not on the circle, it is easy to see that its 
co-ordinates will not satisfy the given equation.

Thus, it is clear that the co-ordinates of every point on the 
circle and of no other point, satisfy the given equation. Hence, the 
circle described is the required graph.

Note 1. The graph of \((x + 2)^2 + (y + 5)^2 = 49\). Draw the graph after plotting 
the points as shown at page 511 and verify its accuracy by drawing a circle of which 
the centre is the point \((-2, -5)\), and the radius is equal to 7 units of length.

Note 2. The graph of \(x^2 + y^2 - 8x + 10y + 25 = 0\). The equation \(x^2 + y^2 - 8x + 10y + 25 = 0\) can be easily reduced to the form \((x - 4)^2 + (y + 5)^2 = 16\). Hence, its 
graph is a circle of which the centre is the point \((4, -5)\) and the radius is equal to 
4 units of length.

Example 1. Solve graphically \(x^2 - 6x - 12 = 0\).

The equation may be written in the form

\[(x^2 - 6x + 9) + 4 = 25, \text{ i.e., } (x - 3)^2 + 2 = 25.\]

\[\therefore \text{ the roots of the given equation are the abscissae of the points} \]

where the line \(y = 0\) (i.e., the \(x\)-axis) cuts the graph of the equation 
\((x - 3)^2 + (y - 2)^2 = 25\) [ for, putting \(y = 0\) in the equation of the circle, we 
have \((x - 3)^2 + (y - 2)^2 = 25, \text{ i.e., } (x - 3)^2 + 4 = 25.\].

Hence, drawing the graph of the equation \((x - 3)^2 + (y - 2)^2 = 25\) as in 
Art. 253, we notice from the diagram that these abscissae are 7'6 and 
\(-1'6\) approximately.

\[\therefore \text{ the required roots are 7'6 and } -1'6 \text{ approximately.}\]

Example 2. Trace the graphs of (i) \(x^2 + y^2 = 169\) and (ii) \(x + y = 17\). 
Find the co-ordinates of their points of intersection.

\[x^2 + y^2 = 169, \text{ or, } y^2 = 169 - x^2; \therefore y = \pm \sqrt{169 - x^2}.\]

The corresponding values of \(x\) and \(y\) in the equation \(y = \pm \sqrt{169 - x^2}\) 
may be tabulated as follows:

\[
\begin{array}{cccccccccccc}
\hline
x & 0 & 0 & -13 & 13 & 5 & 5 & -5 & -5 & 19 & 19 & -19 & -19 \\
\hline
y & 18 & -18 & 0 & 0 & 12 & -12 & 12 & -12 & -5 & 5 & 5 & -5 \\
\hline
\end{array}
\]
Similarly the corresponding values of $x$ and $y$ in the equation $x + y = 17$ may be tabulated as follows:

<table>
<thead>
<tr>
<th></th>
<th>17</th>
<th>0</th>
<th>13</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>17</td>
<td>4</td>
<td>13</td>
</tr>
</tbody>
</table>

Taking twice the length of a side of a small square as the unit of length and drawing the graphs of the two equations we shall find that they will intersect at $P(12, 5)$ and $Q(5, 12)$ as in the above diagram.

Note. To solve graphically the equations

\[
\begin{align*}
x^2 + y^2 &= 169 \\
x + y &= 17
\end{align*}
\]

we notice that the co-ordinates of each of the above points $P$ and $Q$ satisfy both the equations and are, therefore, the required solutions.

Thus, the roots are $x = 12 \quad $ and $x = 5 \quad $ and $y = 5 \quad $ and $y = 12$ \]

1—33
Example 3. Show that the graph of $3x + 4y = 25$ touches that of $x^2 + y^2 = 25$, and find the co-ordinates of the point of contact. [C. U. 1911]

$x^2 + y^2 = 25$, or, $y^2 = 25 - x^2$; \[ \therefore \ y = \pm \sqrt{25 - x^2}. \]

The corresponding values of $x$ and $y$ in the equation $y = \pm \sqrt{25 - x^2}$ may be tabulated as follows:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>0</th>
<th>5</th>
<th>-5</th>
<th>3</th>
<th>3</th>
<th>-3</th>
<th>-3</th>
<th>4</th>
<th>4</th>
<th>-4</th>
<th>-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>5</td>
<td>-5</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>-4</td>
<td>4</td>
<td>-4</td>
<td>3</td>
<td>-3</td>
<td>3</td>
<td>-3</td>
</tr>
</tbody>
</table>

$3x + 4y = 25$, 
or, \[ y = \frac{25 - 3x}{4}; \]
the corresponding values of $x$ and $y$ of the equation may be tabulated as follows:

<table>
<thead>
<tr>
<th>$x$</th>
<th>7</th>
<th>5</th>
<th>-1</th>
<th>-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1</td>
<td>2.5</td>
<td>7</td>
<td>8.5</td>
</tr>
</tbody>
</table>
Taking four times the side of a small square as the unit of length and drawing the graphs, we find that they touch at $P(3, 4)$ as in the diagram at page 514.

**EXERCISE 136**

Draw the graphs of the following equations:

1. $x^2 + y^2 = 81$.
2. $(x-5)^2 + (y-6)^2 = 49$.
3. $(x+6)^2 + (y-7)^2 = 100$.
4. $x^2 + y^2 - 9x - 14y + 1 = 0$.
5. $x^2 + y^2 + 14x - 16y + 32 = 0$.
6. $x^2 + y^2 + 12x + 18y + 92 = 0$.
7. $x^2 + y^2 - 10x + 16y - 55 = 0$.

Solve graphically:

8. $x^2 + y^2 = 100 \quad \text{[9. } x^2 + y^2 = 25 \quad \text{]}$.
   
   $x + y = 14 \quad \text{[10. } x + y = -12 \quad \text{]}$.

9. $x^2 + y^2 = 25$.

10. $x^2 + y^2 - 4x - 6y - 12 = 0$.

11. $x^2 - 4x - 12 = 0$.

12. $x^2 - 6x - 12 = 0$.

13. Draw the graphs of $x^2 + y^2 = 36$ and $3x - 4y = 30$. Show that they touch at $(3, 6, -4, 8)$.

14. Draw the graph of $x^2 + y^2 - 4x - 6y - 23 = 0$ and find its tangents parallel to the co-ordinate axes.

15. Draw the graph of $x^2 + y^2 - 10x - 10y + 25 = 0$ and show that it touches the co-ordinate axes. Find the co-ordinates of the points of contact.

16. Draw the graphs of the following equations:

   (1) $x^2 = 16$;
   (2) $x^2 - 5x + 6 = 0$;
   (3) $5x^2 - 3x - 2 = 0$;
   (4) $y^2 - 3y = 0$;
   (5) $xy = 0$;
   (6) $x^2 - 3xy + 2y^2 = 0$;
   (7) $x^2 - y^2 + 4y - 4 = 0$;
   (8) $(x + 3)^2 = 4(y - 5)^2$.

17. Draw the graph of $5x^2 - 24xy - 5y^2 = 0$ and show that they are two perpendicular straight lines.

18. Find the angle between the straight lines which represent the graphs of:

   (i) $xy = 0$;
   (ii) $(x-3)(y-2) = 0$;
   (iii) $(3x - 2y + 5)(2x + 3y + 2) = 0$;
   (iv) $(7x - 6y + 3)(6x + 7y + 8) = 0$.

260. The graph of a quadratic equation in which the coefficients of $x^2$ and $y^2$ are positive and unequal is a curve called an Ellipse. The equation is generally of the form $a^2x^2 + b^2y^2 = c^2$. 
Draw the graph of the equation \(4x^2 + 9y^2 = 36\).

1. When \(x = 0\), we have \(y^2 = 4\), and, therefore, \(y = \pm 2\). Hence, the points \((0, 2)\) and \((0, -2)\) are on the required graph.

2. When \(y = 0\), we have \(x^2 = 4\), and, therefore, \(x = \pm 2\). Hence, the points \((2, 0)\) and \((-2, 0)\) are on the required graph.

3. When \(x = \pm 1\), we have \(9y^2 = 32\), and, therefore, \(y = \pm \sqrt{\frac{32}{9}} = \pm \frac{4\sqrt{2}}{3} = \pm 1.5656... \approx \pm 1.9\) approximately. Hence, the four points \((1, 1.9)\), \((1, -1.9)\), \((-1, 1.9)\) and \((-1, -1.9)\) are on the required graph.

4. When \(x = \pm 2\), we have \(9y^2 = 20\), and, therefore, \(y = \pm \sqrt{\frac{20}{9}} = \pm \sqrt{\frac{2}{3}} = \pm 1.47222... \approx \pm 1.5\) nearly.

Hence, the four points \((2, 1.5)\), \((2, -1.5)\), \((-2, 1.5)\), and \((-2, -1.5)\) are on the required graph.

Corresponding values of \(x\) and \(y\) may be tabulated as follows:

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>0</th>
<th>3</th>
<th>-3</th>
<th>1</th>
<th>1</th>
<th>-1</th>
<th>-1</th>
<th>2</th>
<th>2</th>
<th>-2</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>1.9</td>
<td>-1.9</td>
<td>1.9</td>
<td>-1.9</td>
<td>1.5</td>
<td>-1.5</td>
<td>1.5</td>
<td>-1.5</td>
</tr>
</tbody>
</table>
Let us now plot the twelve points as found above (taking 10 times
the side of a small square as the unit of length) and draw a free-hand
curve through them, as in the diagram at page 516.

The curve so drawn is the required graph.

Note 1. Evidently the curve is symmetrical about the axis of \( x \), i.e., every
chord at right angles to the axis of \( x \) is bisected by it. Similarly, the curve is also
symmetrical about the axis of \( y \).

Note 2. The curve lies entirely within the space enclosed by the four straight
lines \( x = 3, \ x = -3, \ y = 2, \ y = -2 \), since from the given equation it is obvious that \( x \) is
imaginary when, \( y > 2 \) and \( x < -2 \) and \( x \) is imaginary when, \( x > 3 \) and \( x < -3 \).

Example 1. Draw the graph of the expression \( \frac{4}{3} \sqrt{9-x^2} \).

Let \( y = \frac{4}{3} \sqrt{9-x^2} \).

For each value of \( x \), there will be two equal and opposite values
of \( y \). Thus, (1) when \( x = 0, \ y = \pm 4 \); (2) when \( x = \pm 3, \ y = 0 \); (3) when
\( x = \pm 1, \ y = \pm \frac{4}{3} \sqrt{8} = \pm 3'8 \) approximately; (4) when \( x = \pm 2, \ y = \pm \frac{4}{3} \sqrt{5}
= \pm 3'0 \) approximately.
The corresponding values of $x$ and $y$ may be arranged in a tabular form as follows:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>0</th>
<th>3</th>
<th>$-3$</th>
<th>1</th>
<th>1</th>
<th>$-1$</th>
<th>$-1$</th>
<th>2</th>
<th>2</th>
<th>$-2$</th>
<th>$-2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>4</td>
<td>$-4$</td>
<td>0</td>
<td>0</td>
<td>3.8</td>
<td>$-3.8$</td>
<td>3.8</td>
<td>$-3.8$</td>
<td>3</td>
<td>$-3$</td>
<td>3</td>
<td>$-3$</td>
</tr>
</tbody>
</table>

Plotting these twelve points (taking 8 times the side of a small square as the unit of length) and drawing a free-hand curve through them as shown in the diagram on the last page, we obtain the required graph.

**Example 2.** Draw the graph of $4(x-2)^2 + 9(y-3)^2 = 36$.

Re-writing the equation, we have

$$9(y-3)^2 = 36 - 4(x-2)^2,$$

or,

$$y - 3 = \pm \frac{2}{3} \sqrt{9-(x-2)^2}.$$

Hence, for each value of $x-2$, we get two values of $y-3$ from which the corresponding values of $x$ and $y$ may be tabulated as follows:
Plotting these twelve points (taking 10 times the side of a small square as the unit of length) and drawing a free-hand curve through them as shown in the diagram at page 518, we get the required graph.

**Example 3.** Draw the graph of \(4x^2 + 9y^2 - 16x - 54y + 61 = 0\).

The left-hand side of the given equation

\[
4(x^2 - 4x) + 9(y^2 - 6y) + 61 \\
= 4(x - 2)^2 - 4 + 9(y - 3)^2 - 9 + 61 \\
= 4(x - 2)^2 + 9(y - 3)^2 - 36.
\]

\[
\therefore \text{ the equation is } 4(x - 2)^2 + 9(y - 3)^2 - 36 = 0,
\]

or, \(4(x - 2)^2 + 9(y - 3)^2 = 36\).

To draw its graph see example 2 on page 518.

**261. Draw the graph of the equation \(x^2 - y^2 = 1\).**

(1) When \(x = 0\), we have \(y^2 = -1\), and, therefore, \(y\) is imaginary. This shows that the graph does not cut the axis of \(y\).

(2) When \(y = 0\), we have \(x^2 = 1\), and, therefore, \(x = \pm 1\). Hence, the points \((1, 0)\) and \((-1, 0)\) are on the required graph.

(3) When \(x = \pm 2\), we have \(y^2 = 3\), and, therefore, \(y = \pm \sqrt{3} = \pm 1.732 \ldots \approx \pm 1.7\) approximately. Hence, the four points \((2, 1.7), (2, -1.7), (-2, 1.7)\) and \((-2, -1.7)\) are on the required graph.

(4) When \(x = \pm 3\), we have \(y^2 = 8\), and, therefore, \(y = \pm 2 \sqrt{2} = \pm 2 \times 1.414 \ldots = \pm 2.828 \ldots \approx \pm 2.8\) approximately. Hence, the four points \((3, 2.8), (3, -2.8), (-3, 2.8)\) and \((-3, -2.8)\) are on the required graph.

The corresponding values of \(x\) and \(y\) may be tabulated as follows:

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>-1</th>
<th>2</th>
<th>2</th>
<th>-2</th>
<th>-2</th>
<th>3</th>
<th>3</th>
<th>-3</th>
<th>-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>0</td>
<td>0</td>
<td>1.7</td>
<td>-1.7</td>
<td>1.7</td>
<td>-1.7</td>
<td>2.8</td>
<td>-2.8</td>
<td>2.8</td>
<td>-2.8</td>
</tr>
</tbody>
</table>
Let us now plot the ten points as found above (taking 10 times the side of a small square as the unit of length) and draw a free-hand curve through them, as in the diagram below. $A$ and $A'$ are the points of intersection of the right and left branches of the hyperbola respectively, with the $x$-axis.

![Diagram of hyperbola with points labeled]

The curve so drawn is the required graph.

Note 1. The curve so drawn is evidently symmetrical about the axis of $x$ and also about the axis of $y$.

Note 2. The curve consists of two branches, one lying entirely on the right of the line $x=1$ and the other lying entirely on the left of the line $x=-1$.

A curve of this class is called a Hyperbola.

Example 1. Trace the graph of (i) $x^2-y^2=1$, and (ii) $x^2+y^2=1$. Show that they touch each other.

Draw the graph of $x^2-y^2=1$ as above and the graph of the circle $x^2+y^2=1$ on the same scale. It will be found that they touch each other at the points $(1, 0)$ and $(-1, 0)$.

Example 2. Trace the graph of (i) $x^2-y^2=1$ and (ii) $x=2y$. Find the co-ordinates of their points of intersection.
Draw the Hyperbola \( x^2 - y^2 = 1 \) and the straight line \( x = 2y \) on the same scale. Produce the straight line, if necessary, to meet the Hyperbola. They will be found to intersect at two points whose co-ordinates are \((1^{1/2}, 1^6)\) and \((-1^{1/2}, -1^6)\) approximately.

262. Draw the graph of the equation \( y = x^2 \).

Evidently the following points are on the required graph and their co-ordinates may be tabulated as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>-1</th>
<th>1.5</th>
<th>-1.5</th>
<th>2</th>
<th>-2</th>
<th>2.5</th>
<th>-2.5</th>
<th>3</th>
<th>-3</th>
<th>3.5</th>
<th>-3.5</th>
<th>4</th>
<th>-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2.25</td>
<td>2.25</td>
<td>4</td>
<td>4</td>
<td>6.25</td>
<td>6.25</td>
<td>9</td>
<td>9</td>
<td>12.25</td>
<td>12.25</td>
<td>16</td>
<td>16</td>
</tr>
</tbody>
</table>

Let 4 times the side of a small square be the unit of length.

Let us now plot the points found above and draw a curve through them free-hand, as in the following diagram.

The curve so drawn is the required graph.
Note 1. Since, \( y = x^2 \), we have \( x = \pm \sqrt{y} \); \( \therefore \) \( x \) is imaginary when \( y \) is negative. Hence, no point of the curve can have a negative ordinate and, therefore, no part of the curve can lie below the \( x \)-axis. The curve passes through the origin, lies entirely above the \( x \)-axis and extends upwards to infinity.

Note 2. Every chord drawn perpendicular to \( OY \) is bisected by it as can be easily verified. Hence, the curve drawn above is symmetrical about the axis of \( y \). This is also evident from the fact that if the paper be folded about \( OY \), the left-hand portion of the curve entirely coincides with the right-hand portion.

A curve of this class is called a **Parabola**.

The general equation of a parabola is \( y = ax^2 + bx + c \).

In the equation of a parabola either of \( x \) and \( y \) will be of the first degree and there will be no term involving the product of \( x \) and \( y \) (i.e., \( xy \)).

In the above example, \( a = 1 \), \( b = 0 \) and \( c = 0 \).

Note 3. The graph of \( y = -x^2 \). The curve \( y = x^2 \) lies entirely above the axis of \( x \), and extends upwards to infinity. It is easy to see that the graph of the equation \( y = -x^2 \) would be an equal curve being entirely below the axis of \( x \) and extending downwards to infinity.

Note 4. To determine the square root of a number from the graph of \( y = x^2 \). The abscissa of any point on the curve is evidently the square root of the ordinate. Hence, when the graph of the equation \( y = x^2 \) is drawn by measuring the abscissa of any point on the graph we can determine the square root of the number which represents the ordinate. Thus, in the diagram, the ordinates of \( P \) or \( Q \) represent 5. \( \therefore \) the square root of 5 = the abscissa of \( P \) or \( Q = 2.25 \), or, \(-2.25 \) approximately. [ 4 sides of a small square = 1 unit. ]

263. Draw the graph of the equation \( y = 3x^2 \).

Evidently the following points are on the required graph and their co-ordinates may be tabulated as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>( \frac{1}{2} )</th>
<th>(-\frac{1}{2} )</th>
<th>( \frac{1}{3} )</th>
<th>(-\frac{1}{3} )</th>
<th>1</th>
<th>(-1 )</th>
<th>( \frac{1}{12} )</th>
<th>(-\frac{1}{12} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{9} )</td>
<td>( \frac{1}{9} )</td>
<td>8</td>
<td>8</td>
<td>( 4\frac{1}{2} )</td>
<td>( 4\frac{1}{2} )</td>
</tr>
</tbody>
</table>

Taking 12 times the side of a small square as the unit of length, let us plot the points found above and draw a curve through them free-hand, as in the diagram on page 529.
The curve so drawn is the required graph.

Note 1. Since \( y = 3x^2 \), we have \( x^2 = \frac{1}{3}y \). \( \therefore \) \( x \) is imaginary for every negative value of \( y \). Hence, as in the graph of Art. 262, the curve passes through the origin, lies entirely above the \( x \)-axis and extends upwards to infinity.

Again, it may be easily verified that every chord drawn perpendicular to \( OY \) is bisected by it. Hence, the curve is symmetrical about the axis of \( Y \).

Note 2. The graph of \( y = -3x^2 \) can be easily seen to be an equal curve passing through the origin, lying entirely below the \( x \)-axis and extending downwards to infinity.

264. Draw the graph of the equation \( y = -5x^2 \).

Evidently, the following points are on the required graph and their \( \text{coordinates} \) may be tabulated as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>( \frac{1}{2} )</th>
<th>(-\frac{1}{2} )</th>
<th>( \frac{1}{2} )</th>
<th>(-\frac{1}{2} )</th>
<th>( \frac{1}{2} )</th>
<th>(-\frac{1}{2} )</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>(-\frac{1}{4} )</td>
<td>(-\frac{1}{4} )</td>
<td>(-\frac{1}{4} )</td>
<td>(-\frac{1}{4} )</td>
<td>(-\frac{1}{4} )</td>
<td>(-\frac{1}{4} )</td>
<td>(-\frac{3}{4} )</td>
<td>(-\frac{5}{4} )</td>
</tr>
</tbody>
</table>

-1 \(-\frac{3}{3} \) \(-\frac{5}{5} \)
Taking 10 times the side of a small square as the unit of length, let us plot the points found above, and draw a curve through them free-hand, as in the diagram.

The curve so drawn is the required graph.

Note 1. Since \( y = -5x^3 \), we have \( x^3 = -\frac{1}{5}y \). \( \therefore \) \( x \) is imaginary for every positive value of \( y \). Hence, no point on the curve can have a positive ordinate and, therefore, no part of the curve can lie above the \( x \)-axis. The curve passes through the origin, lies entirely below the \( x \)-axis and extends downwards to infinity.

Note 2. It may be easily seen that every chord drawn perpendicular to \( OY' \) is bisected by it. Hence, the curve is symmetrical about the axis of \( y \).

Note 3. The graph of the equation \( y = 5x^3 \) can be easily seen to be an equal curve passing through the origin, lying entirely above the \( x \)-axis and extending upwards to infinity.

265. It is clear, from Arts. 262, 263 and 264, that the graph of any equation of the form \( y = ax^2 \), where \( a \) is any numerical constant, positive or negative, is a curve which (i) is symmetrical about the axis of \( y \), (ii) lies entirely on one side of the axis of \( x \), and (iii) extends up to infinity on that side. A curve of this class is called a Parabola.

If \( a \) be a positive integer, the curve will be as in the figure of Art. 262 but will rise more steeply in the direction of \( OY \). [See the fig. of Art. 263.] If \( a \) be a positive fraction, we shall have a flatter curve, extending more rapidly to the right and left of \( OY \). If \( a \) be negative, as in Art. 264, the curve will lie below the \( x \)-axis and will be steeper or flatter than the graph of \( y = x^2 \), according as \( a \) is greater or less than unity. [See the fig. of Art. 264.]

In every case, the axis of \( x \) is a tangent to the curve at the origin.

266. We shall now discuss the graphs of some quadratic functions of the form \( ax^2 + bx + c \). It will be seen, as in the next article, that the curve is always a parabola, differing in shape and position according to values of \( a, b, c \).

267. Draw the graph of the expression \( 3 - 4x - 2x^2 \).

The required graph is the same as that of the equation \( y = 3 - 4x - 2x^2 \).
It is easy to see that the following points are on the required graph:

- \(x = 0\), \(y = 3\)
- \(x = 1\), \(y = -3\)
- \(x = 1.5\), \(y = -7.5\)
- \(x = -2\), \(y = 3\)
- \(x = -3\), \(y = -9\)
- \(x = -3.5\), \(y = -7.5\)
Take twenty sides of a small square as the unit for measuring \( x \), and two sides of a small square as the unit for measuring \( y \).

Let us now plot the above points and draw a curve through them free-hand, as in the diagram on the last page.

The curve so drawn is the required graph.

Note. The graph of any expression of the form \( ax^2 + bx + c \) is a parabola, provided the numerical value of \( a \) is not zero.

268. Graphical solution of Quadratic Equations.

Example 1. To solve graphically the equation \( 3 - 4x - 2x^2 = 0 \).

Draw the graph of \( y = 3 - 4x - 2x^2 \) as in the last article.

From the figure it is evident that \( y = 0 \), when \( x \) is approximately equal to \( 0.6 \) or \(-2.6\). Hence, \( 3 - 4x - 2x^2 = 0 \), when \( x = 0.6 \) or \(-2.6\) approximately, in other words, the roots of the equation \( 3 - 4x - 2x^2 = 0 \) are \( 0.6 \) and \(-2.6\) approximately. From this it is clear that the roots of the equation \( 3 - 4x - 2x^2 = 0 \) are the abscissae of the points where the graph of the expression \( 3 - 4x - 2x^2 \) cuts the axis of \( x \).

Example 2. Trace the graph of \( y = x^2 - x \) from \( x = -1 \) to \( x = 2 \) and therefrom obtain an approximate solution of the equation

\[ 1 = x^2 - x. \]

The following points evidently lie on the graph:

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>-0.5</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>2</td>
<td>2.5</td>
<td>0</td>
<td>-0.5</td>
<td>0</td>
<td>1.5</td>
<td>2</td>
</tr>
</tbody>
</table>

[ C. U. 1917 ]
Taking 16 sides of a small square as the unit of length, the graph will be as shown in the diagram on page 526.

If we now put \( y = 1 \), the equation \( y = x^2 - x \) becomes \( 1 = x^2 - x \). Hence, the roots of the equation \( 1 = x^2 - x \) are the abscissae of the points \( P \) and \( Q \) of the graph of \( y = x^2 - x \), at which the ordinate is 1. \( P \) and \( Q \) are evidently the points where the line \( y = 1 \) meets the graph. From the figure, we find that the abscissae of \( P \) and \( Q \) are 1.\( \frac{1}{6} \) and \( -1.6 \) respectively, which are, therefore, the required solutions.

Example 3. Trace the graphs of (i) \( y = 3x^2 \) and (ii) \( y = 2x + 1 \), and determine the points where they meet.

[ C. U. 1915 ]

Deduce the roots of the equation \( 3x^2 = 2x + 1 \).

Evidently the corresponding values of \( x \) and \( y \) on \( y = 3x^2 \) may be tabulated as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>( \frac{1}{3} )</th>
<th>( -\frac{1}{3} )</th>
<th>( \frac{1}{2} )</th>
<th>( -\frac{1}{2} )</th>
<th>1</th>
<th>( -1 )</th>
<th>( \frac{1}{2} )</th>
<th>( -\frac{1}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>3</td>
<td>3</td>
<td>( 5\frac{1}{2} )</td>
<td>( 5\frac{1}{2} )</td>
</tr>
</tbody>
</table>
Also, the points \(x = 0\), \(x = \frac{1}{2}\) and \(x = \frac{3}{2}\) lie on the straight line \(y = 2x + 1\).

Taking twelve times the side of a small square as the unit of length the graphs will be as shown in the diagram on the last page.

Let the straight line meet the parabola at \(P\) and \(Q\) whose co-ordinates are found from the diagram to be \((1, 3)\) and \((-\frac{1}{2}, \frac{3}{2})\) respectively.

The abscissæ of the points common to the graphs of \(y = 3x^2\) and \(y = 2x + 1\) are evidently the roots of \(3x^2 = 2x + 1\). But, from the figure, these abscissæ are 1 and \(-\frac{1}{2}\), which are, therefore, the required roots of \(3x^2 = 2x + 1\).

269. Draw the graph of \(y^2 = x\).

We have \(y = \pm \sqrt{x}\). The corresponding values of \(x\) and \(y\) may be tabulated as follows:

\[
\begin{array}{cccccccc}
  x & 0 & .25 & .25 & 1 & 2.25 & 2.25 & 4 & 6.25 & 6.25 \\
  y & 0 & .5 & -.5 & 1 & -1 & 1.5 & -1.5 & 2 & -2 & 2.5 & -2.5 \\
\end{array}
\]

Let eight sides of a small square be the unit of length. Now plotting the points found above and drawing a curve through them free-hand, the graph will be as in the diagram.
Note 1. Since for every point of the graph, \( y = \pm \sqrt{x} \) and is, therefore, imaginary when \( x \) is negative, it follows that no point of the graph can have a negative abscissa, i.e., no part of the graph lies on the negative side of the \( x \)-axis. This graph, therefore, lies on the positive side of the \( x \)-axis and extends to infinity on that side. It is easy to see that the curve is symmetrical about the \( x \)-axis.

Note 2. The graph of \( y^2 = -x \) is evidently an equal curve turned in the opposite direction on the negative side of the \( x \)-axis.

270. Maximum and minimum values of quadratic expressions.

Example 1. Show graphically that the expression \( 3 - 4x - 2x^2 \) is positive for all values of \( x \) between \(-2\) and \(6 \) and find its maximum value.

Let \( y = 3 - 4x - 2x^2 \).

Drawing the graph of \( y = 3 - 4x - 2x^2 \) as in Art. 267, we find that for all values of \( x \) between \(-2\) and \(6 \) the curve lies above the \( x \)-axis and 

\[ \therefore \text{the ordinates are positive, and for values of } x \text{ greater than } 6 \text{ and less than } -2, \text{ the curve is below the axis of } x \text{ and } \therefore \text{the ordinates are negative. But the ordinate } (y) = 3 - 4x - 2x^2 \].

Hence, \( 3 - 4x - 2x^2 \) is positive for all values of \( x \) between \(-2\) and \(6 \).

Also, we notice from the figure that the ordinate is greatest at the point \( P (-1, 5) \), its greatest value being 5.

\[ \therefore \text{the maximum value required } = 5. \]

Example 2. Show graphically that the expression \( x^2 - x \) is negative for all values of \( x \) between \( x = 0 \) and \( x = 1 \). Find its minimum value.

Let \( y = x^2 - x \).

Drawing the graph of \( y = x^2 - x \) as in Art. 268, Example 2 (see the diagram on page 526), we find that for all values of \( x \) between \( x = 0 \) and \( x = 1 \) the curve is below the \( x \)-axis and \( \therefore \) the ordinates are negative.

But the ordinate \( (y) = x^2 - x \).

Hence, \( x^2 - x \) is negative for all values of \( x \) between \( x = 0 \) and \( x = 1 \). Also, it is evident from the figure that \( y \) (i.e., \( x^2 - x \)) has the minimum value \(-\frac{1}{4} \) at the point \( A \).

271. Draw the graph of the equation \( xy = 1 \).

It is easy to see that the following points are on the required graph:

\[ x = -1 \quad \{ x = 2 \quad \{ x = 4 \quad \{ x = 5 \}
\]
\[ y = 10\} \quad y = 5\} \quad y = 25\} \quad y = 2\} \]

\[ x = -8 \quad \{ x = -1 \quad \{ x = 2 \}
\]
\[ y = 1.25\} \quad y = 1\} \quad y = 5\} \]

1—34
Evidently also the following points are on the required graph:

\[
\begin{align*}
  x &= -1, \quad y = -10, \\
  x &= -2, \quad y = -5, \\
  x &= -4, \quad y = -2.25, \\
  x &= -5, \quad y = -2, \\
  x &= -8, \quad y = -1.25, \\
  x &= -1, \quad y = -1, \\
  x &= -2, \quad y = -0.5.
\end{align*}
\]

Let two centimetres be the unit for measuring \( x \) and 2 millimetres the unit for measuring \( y \).

Let us now plot the points and draw a curve through them freehand, as in the above diagram.

The curve so drawn is the required graph.

Note 1. As \( x \) diminishes from 1 to zero, \( y \) increases from 1 to infinity; and as \( x \) diminishes from zero to \(-1\), \( y \) increases from negative infinity to \(-1\).
Note 2. As $x$ increases from 1 to infinity, $y$ diminishes from 1 to zero; and as $x$ diminishes from $-1$ to negative infinity, $y$ increases from $-1$ to zero.

Note 3. The graph consists of two branches, one lying between $OX$ and $OY$ and the other between $OX'$ and $OY'$.

Note 4. The more we move towards the right or left of $O$, the nearer does the curve approach the axis of $x$; whilst the more we move upwards and downwards from $O$, the nearer does the curve approach the axis of $y$. But in no case does the curve meet the axis except at an infinite distance from $O$. Hence, each of the axes is said to be an Asymptote to the curve.

Note 5. A curve of this kind is called a Rectangular Hyperbola.

Example. Draw the graphs of (i) $xy=8$ and (ii) $x+y=9$. Find the co-ordinates of their points of intersection.

Drawing the graph of $xy=8$ by the above method and the graph of the straight line $x+y=9$ in the same figure on the same scale, as in the above diagram it will be found that they intersect at two points $P$ and $Q$ whose co-ordinates are

$$\begin{align*}
\{x=8\} \quad \text{and} \quad \{x=1\} \\
y=1 \quad \text{and} \quad y=8
\end{align*}$$

respectively.

EXERCISE 137

Draw the graphs of the following equations:

1. $x^2+4y^2=4$.  
2. $4x^2+9y^2=1$.  
3. $25x^2+y^2=25$.  

4. \(16x^2 + 9y^2 = 1\).
5. \(x^2 - 4y^2 = 4\).
6. \(y^2 - x^2 = 1\).
7. \(4x^2 - y^2 = 16\).
8. \(y^2 - 9x^2 = 9\).
9. In one and the same diagram draw the graphs of \(4x^2 - 9y^2 = 0\) and \(4x^2 - 9y^2 = 36\).
10. In one and the same diagram draw the graphs of \(9y^2 - 4x^2 = 0\) and \(9y^2 - 4x^2 = 36\).

11. Draw the graph of the equation \(5y^2 = x^2 - 10\), taking the unit for measuring \(y\) five times as large as that for measuring \(x\).

12. Draw the graph of the equation \(x^2 - 4x + 2y = 0\), taking the unit for measuring \(y\) twice as large as that for measuring \(x\).

13. Draw the graph of the equation \(y^2 + x = 0\), taking the unit for measuring \(x\) equal to half that for measuring \(y\).

14. Draw the graph of the equation \(3y = x^2\), taking the same unit for measuring both \(x\) and \(y\).

15. Find graphically, correct to the first figure after the decimal point, the square roots of:
   (i) 3 ;
   (ii) 5 ;
   (iii) 7.

16. Find graphically, the minimum values of the expressions:
   (i) \(x^2 + 6x + 10\);
   (ii) \(4x^2 + 4x + 5\);
   (iii) \(\frac{1}{2}x^2 + 4x + 1\);
   (iv) \(2x^2 - 6x + 7\).

17. Find graphically, the maximum values of the expressions:
   (i) \(4x - x^2\);
   (ii) \(3 + 6x - 9x^2\);
   (iii) \(12 - 3x - \frac{2x^2}{4}\);
   (iv) \(1 + 2x - 2x^2\).

18. Draw the graphs of the equations (i) \(xy = 4\) and (ii) \(x + y = 5\), and find where they intersect.

19. Show graphically that (i) the expression \(4x - x^2\) is positive for all values of \(x\) between 0 and 4; (ii) the expression \(x^2 + 6x + 12\) is positive for all values of \(x\) and (iii) \(x^3 - 4x - 5\) is negative for all values of \(x\) between -1 and 5.

20. Draw the graphs of (i) \(xy = -8\), and (ii) \(x + y = 2\) and find where they intersect.

Solve graphically:
21. \(x^2 = 4x - 3\).
22. \(3x^2 = x + 2\).
23. \(2x^2 - 7x + 5 = 0\).
24. \(7x^2 - 2x = 5\).
25. \(x^2 = 9\) \(\frac{y^2}{2} = 2y\) \(x = 2x\) \(x = -2y\).
CHAPTER XXXVII
ARITHMETICAL PROGRESSION

272. Definition. Quantities are said to be in Arithmetical Progression when they increase continually by a common quantity (called the common difference).

Thus, each of the following series of quantities is in Arithmetical Progression:

\[2, \ 5, \ 8, \ 11, \ 14, \ \&c.\]
\[a, \ a+b, \ a+2b, \ a+3b, \ \&c.\]
\[9, \ 5, \ 1, \ -3, \ -7, \ \&c.\]
\[a, \ a-b, \ a-2b, \ a-3b, \ \&c.\]

In the first of the above examples the quantities increase by 3, whereas in the second the quantities decrease by 4; so the common differences of these two cases are said to be 3 and \(-4\) respectively. Similarly, in the third example the common difference is \(b\) and in the fourth it is \(-b\).

N. B. Arithmetical Progression is briefly written as A.P.

273. The common difference of the terms of an A. P. is found by subtracting any term of the series from the term following it.

Thus, in the series \(a, \ a+b, \ a+2b, \ a+3b, \ldots\), the common difference
\(= (a+b) - a = (a+2b) - (a+b) = (a+3b) - (a+2b) = \cdots = b.\)

274. To find the \(n\)th term of an A. P.

If \(a\) be the first term and \(b\), the common difference of a series of numbers in Arithmetical Progression, we have the 2nd term \(= a+b\), the 3rd term \(= a+2b\), the 4th term \(= a+3b\), \ldots the 10th term \(= a+9b\), \ldots the 21st term \(= a+20b\); and so on. Hence, the \(n\)th term \(= a + (n-1)b\).

Example 1. Find the 19th term of the series \(10, \ 8, \ 6, \ 4, \ \&c.\)

The first term \(= 10\), and the common difference \(= -2\).

Hence, the 19th term \(= 10 + 18(-2) = 10 - 36 = -26\).

Example 2. What term of the series \(5, \ 7, \ 9, \ 11, \ \&c.\) is 25?

Let the \(r\)th term of the given series be the required term; then, we must have
\[25 = 5 + (r-1)2\]
\[-3 + 2r, \ \text{whence} \ r = 11.\]

Thus, the 11th term of the given series \(= 25\).
275. Given any two terms of an A. P., to find it completely.

**Example 1.** The 7th and 13th terms of an A. P. are 34 and 64 respectively. Find the series.

Let \( a \) = the first term,
and \( b \) = the common difference of the A. P.

\[ \therefore \text{the 7th term} = a + (7-1)b = a + 6b = 34, \quad \ldots \quad (1) \]

and the 13th term \( = a + (13-1)b = a + 12b = 64. \quad \ldots \quad (2) \]

From (1) and (2), by subtraction,

\[ 6b = 30, \quad i.e., \quad b = 5. \]

Now from (1), \( a + 6 \times 5 = 34, \quad \text{or,} \quad a = 34 - 30 = 4. \]

Hence, the first term and the common difference of the required series are 4 and 5 respectively.

\[ \therefore \text{the series is } 4, 9, 14, 19, 24, \ldots \]

**Example 2.** The \( p \)th and \( q \)th terms of an A. P. are \( c \) and \( d \) respectively. Find the series completely.

Let \( a \) = the first term,
and \( b \) = the common difference of the A. P.

\[ \therefore \text{the } p \text{th term} = a + (p-1)b = c, \quad \ldots \quad \ldots \quad (1) \]

and the \( q \)th term \( = a + (q-1)b = d. \quad \ldots \quad \ldots \quad (2) \]

Solving equations (1) and (2), \( a \) and \( b \) can be obtained. Thus, by subtracting (2) from (1) we have,

\[ (p-q)b = c - d, \quad \therefore \quad b = \frac{c - d}{p-q} \]

Also, from (1), \( a + (p-1)b = a + (p-1) \cdot \frac{c - d}{p-q} = c. \)

\[ \therefore \quad a = c - \frac{(p-1)(c - d)}{p-q} = \frac{d(p-1) - c(q-1)}{p-q} \]

Hence, \( a \) and \( b \) being known, the whole series may be written down.

**EXERCISE 138**

1. Find the 8th, 20th and \((n-3)\)th terms of the series:
   
   (i) 2, 4, 6, 8, &c.  \quad (ii) 1, 3, 5, 7, &c.  \quad (iii) \( \frac{1}{2} \), \( \frac{3}{2} \), \( \frac{5}{2} \), \( -\frac{1}{2} \), &c.
   
   (iv) \( \frac{4}{5} \), \( \frac{1}{5} \), \( \frac{3}{5} \), &c.  \quad (v) 5, 11, 17, ...

2. What terms of the series 9, 11, 13, 15, &c. are 65, 99 and \( 6n - 13 \)?
3. The first term of a given series is 3 and the 7th term 39, find the common difference.

4. If there be 60 terms in A. P. of which the first term is 8 and the last term 185; find the 31st term.

5. The 3rd and 13th terms of a series in A. P. are -40 and 0. Find the series and determine its 20th term.

6. The 5th and 31st terms of an A. P. are 1 and -77. Obtain its 1st and 18th terms.

7. Find the 1st term and the common difference of a series whose 8th and 102nd terms are 23 and 305 respectively.

8. The $p$th term of an A. P. is $c$ and its $q$th term is $d$. Find the $r$th term.

9. If every term of an A. P. be increased or diminished by the same quantity, the resulting terms will also be in A. P.

10. Prove that if each term of an A. P. be multiplied or divided by the same quantity, the resulting series will also be in A. P.

11. If $a$ be the first term and $l$ the last term of a series of numbers in A. P., show that the 5th term from the beginning + the 5th term from the end $= a + l$.

12. In the preceding example, show that the $r$th term from the beginning + the $r$th term from the end $= a + l$.

13. Is 302 a term of the series 3, 8, 13, 18, &c.?

[Here, the common difference = 5. If possible, let 302 = the $r$th term of the series, $r$ being evidently an integer.

$\therefore 302 = 3 + (r - 1)5$, or, $r - 1 = \frac{302 - 3}{5}$, or, $r = \frac{304}{5}$.

The value of $r$ being fractional is inadmissible.

$\therefore 302$ is not a term of the series.]

14. The $p$th term of an A. P. is $q$ and the $q$th term is $p$. Show that the $m$th term is $p + q - m$.

276. To find the sum of $n$ terms of an Arithmetic series of which the first term is $a$ and the common difference, $b$.

Let $S$ denote the required sum, and $l$, the last term (i.e., the $n$th term).

Then, $S = a + (a + b) + (a + 2b) + (a + 3b) + &c. + [a + (n - 1)b]$.

And, by writing the series in the reverse order, we have also

$S = l + (l - b) + (l - 2b) + (l - 3b) + &c. + [l - (n - 1)b]$. 
Therefore, by addition,
\[ 2S = (a + l) + (a + l) + (a + l) + \&c. \quad \text{to } n \text{ terms} = n(a + l). \]
\[ \therefore \quad S = \frac{n}{2} (a + l) \quad \ldots \quad \ldots \quad \ldots \quad (1) \]

Thus, the sum of \( n \) terms in A. P. is \( n \) times the semi-sum of the first and last terms, or, in other words, \( n \) times the average of the first and last terms.

Also, since \( l = a + (n - 1)b \),
\[ \therefore \quad S = \frac{n}{2} \left[ a + \left( a + (n - 1)b \right) \right] = \frac{n}{2} \left[ 2a + (n - 1)b \right] \quad \ldots \quad (2) \]

**N. B.** The formula (1) and (2) should be carefully remembered so that they might readily be applied in any suitable case.

**Example 1.** Find the sum of 20 terms of the series 5, 4\( \frac{1}{2} \), 3\( \frac{1}{2} \), \&c.
The first term = 5, and the common diff. = \( \frac{3}{2} - 5 = -\frac{7}{2} \).
Hence, the required sum = \( \frac{3}{2} \times 20 + (20 - 1) \times \left( -\frac{7}{2} \right) \)
\[ = 10 \times \left( 10 - \frac{7}{2} \times 2 \right) = 10 \times (-\frac{1}{2}) = -26\frac{1}{2}. \]

**Example 2.** Find the value of \( 1 + 2 + 3 + 4 + \&c. \) to 100 terms.
The last term of the series evidently = 100.
Hence, the required sum = \( \frac{1}{2} \times 2 \times (100) = 50 \times 101 = 5050. \)

**Example 3.** Find, without assuming any formula, the sum of \( 1 + 4 + 7 + 10 + \cdots + 37. \)
\[ \text{Evidently, the common difference = 3, and the number of terms in the series = 13.} \]
Let \( S \) denote the required sum.
\[ \therefore \quad S = 1 + 4 + 7 + \cdots + 31 + 34 + 37. \]
Also, re-writing the series in the reverse order,
\[ S = 37 + 34 + 31 + \cdots + 7 + 4 + 1. \]
Adding together the two series,
\[ 2S = 38 + 38 + 38 + \cdots \text{ to 13 terms} = 38 \times 13. \]
\[ \therefore \quad S = \frac{38 \times 13}{2} = 19 \times 13 = 247. \]

**Example 4.** Find, without assuming any formula, the sum of the series \( 1 + 3 + 5 + 7 + \cdots \) to \( n \) terms.
\[ \text{Evidently, the common difference = 2, and the } n^{\text{th}} \text{ term = } 1 + (n - 1) \times 2 = 2n - 1. \]
Let \( S = \) the sum required.
\[ \therefore \quad S = 1 + 3 + 5 + \cdots + (2n - 5) + (2n - 3) + (2n - 1). \]
Re-writing the series in the reverse order,

\[ S = (2n - 1) + (2n - 3) + (2n - 5) + \cdots + 5 + 3 + 1. \]

Adding the two series,

\[ 2S = 2n + 2n + 2n + \cdots \text{ to } n \text{ terms} = 2n^2. \]

\[ \therefore \quad S = n^2. \]

EXERCISE 139

Find the sum of the following series:

1. \(1 + 2 + 3 + 4 + \cdots \cdots \& \text{c. to 25 terms.}\)
2. \(1 + 3 + 5 + 7 + \cdots \cdots \& \text{c. to 30 terms.}\)
3. \(-3, 3, 9, 15, \ldots \ldots \text{ to 14 terms.}\)
4. \(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \cdots \text{ to 20 terms.}\)
5. \(\frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots \cdots \text{ to 30 terms.}\)
6. \(1 \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{4} + \cdots \cdots \text{ to 16 terms.}\)
7. \(3 + 4 + 8 + 9 + 13 + 14 + 18 + 19 + \cdots \cdots \text{ to 20 terms.} \quad \text{[C. U. F. A. 1881]}\)

\[ \text{[The given series} = (3 + 4) + (8 + 9) + (13 + 14) + (18 + 19) + \cdots \]

\[ = 7 + 17 + 27 + 37 + \cdots \cdots \text{ to 10 terms} \]

\[ = \frac{(14 + (10 - 1) \times 10)}{2} \times 10 = 520. \]

8. \(5 + 4\frac{1}{2} + 4\frac{3}{2} + \cdots \cdots \& \text{c. to 21 terms.}\)
9. \(13 + 12\frac{1}{2} + 11\frac{3}{2} + \cdots \cdots \& \text{c. to 40 terms.}\)
10. \(2 + 7 + 12 + \cdots \cdots \& \text{c. to 101 terms.}\)
11. \(\frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} + \cdots \cdots \& \text{c. to } n \text{ terms.}\)
12. \(\frac{a-b}{a+b} + \frac{3a-2b}{a+b} + \frac{5a-3b}{a+b} + \cdots \cdots \& \text{c. to } n \text{ terms.}\)
13. \(1 + 5 + 3 + 9 + 5 + 13 + 7 + 17 + \cdots \cdots \text{ to 30 terms.}\)
14. \(\left(2 - \frac{1}{n}\right) + \left(2 - \frac{3}{n}\right) + \left(2 - \frac{5}{n}\right) + \cdots \cdots \text{ to } n \text{ terms.}\)
15. \((a + b)^2 + (a^2 + b^2) + (a - b)^2 + \cdots \cdots \text{ to } n \text{ terms.}\)

Find the sum of the following series without applying any formula:

16. \(3 + 5 + 7 + \cdots \cdots \text{ to 29 terms.}\)
17. \(-10 - 6 - 2 + 2 + \cdots \cdots \text{ to 22 terms.}\)
18. \((x - y) + (2x - 3y) + (3x - 5y) + \cdots \cdots \text{ to } n \text{ terms.}\)
19. \(5 + 8 + 11 + \cdots \cdots + 155. \quad 20. \quad 8 + 3 - 2 - 7 - 12 - \cdots \cdots \text{ to } n \text{ terms.}\)
277. Applications of the formulae (1) and (2) of the preceding article. The following examples illustrate some important applications of those formulae.

Example 1. The first term of a series in A. P. is 17, the last term \(-12\frac{3}{8}\) and the sum \(25\frac{7}{16}\); find the common difference.

Let \(n\) = the number of terms; then, we must have

\[
25\frac{7}{16} = \frac{n}{2} \left(17 + \left(-12\frac{3}{8}\right)\right) = \frac{n}{2} \left(17 - 12\frac{3}{8}\right) = \frac{n}{2} \times 4\frac{5}{8},
\]

or, \[
\frac{407}{16} = 37n \quad \therefore \quad n = \frac{407}{37} = 11.
\]

If, then, \(b\) be the required common difference, we must have

\[-12\frac{3}{8} (= \text{the 11th term}) = 17 + 10b.\]

\[\therefore \quad 10b = -12\frac{3}{8} - 17 = -29\frac{5}{8} = -29\frac{5}{8}.\]

\[\therefore \quad b = \frac{-235}{8 \times 10} = \frac{-5 \times 47}{5 \times 2 \times 8} = -\frac{47}{16}.\]

Example 2. The sum of a series in A. P. is 72, the first term 17, and the common difference \(-2\); find the number of terms, and explain the double answer.

Let \(n\) = the number of terms.

Then, we must have

\[
72 = \frac{n}{2} \left[2 \times 17 + (n - 1) \times (-2)\right]
\]

\[= \frac{n}{2} \left[34 - 2(n - 1)\right] = \frac{n}{2} (36 - 2n) = 18n - n^2.
\]

\[\therefore \quad n^2 - 18n + 72 = 0, \quad \text{or,} \quad (n - 6)(n - 12) = 0.
\]

\[\therefore \quad n = 6, \text{ or, } 12.
\]

The double answer shows that there are two sets of numbers, satisfying the conditions of the problem, and this can be easily verified. For the series to 6 terms is 17, 15, 13, 11, 9, 7; and to 12 terms it is 17, 15, 13, 11, 9, 7, 5, 3, 1, -1, -3, -5; now since the sum of the last 6 terms of the latter set of numbers = 0; evidently, therefore, the sum of 6 terms of the series, is exactly the same as that of 12 terms.
Example 3. How many terms of the series \(-8, -6, -4, \&c.\) amount to 52?

Let \(n\) = the required number.

Then, we must have

\[
52 = \frac{n}{2} \left\{ 2 \times (-8) + (n - 1) \times 2 \right\}
\]

\[
= \frac{n}{2} (2n - 18) = n^2 - 9n.
\]

\[
\therefore \quad n^2 - 9n - 52 = 0;
\]

or, \((n - 13)(n + 4) = 0;\)

\[
\therefore \quad n = 13, \text{ or, } -4.
\]

Hence, since the number of terms can only be a positive integer, we must reject the negative value and take 13 to be the answer to the question.

Example 4. The sum of \(p\) terms of an A. P. is \(q\) and the sum of \(q\) terms is \(p\); find the sum of \(p + q\) terms.

Let \(a\) be the first term, and \(b\) the common difference; then, since the sum of \(p\) terms = \(q\), we must have

\[
q = \frac{p}{2} \left\{ 2a + (p - 1)b \right\},
\]

or, \(2q = p \cdot 2a + p(p - 1)b. \ldots \quad (1)\)

Similarly, \(2p = q \cdot 2a + q(q - 1)b. \ldots \quad (2)\)

Subtracting (2) from (1), we have

\[
2(q - p) = (p - q) \cdot 2a + \{(p^2 - q^2) - (p - q)\}b
\]

\[
= (p - q) \cdot 2a + (p - q)(p + q - 1)b.
\]

\[
\therefore \quad -2 = 2a + (p + q - 1)b.
\]

Hence, the sum of \((p + q)\) terms

\[
= \frac{p + q}{2} \left\{ 2a + (p + q - 1)b \right\}
\]

\[
= \frac{p + q}{2} \times (-2) = -(p + q).
\]

EXERCISE 140

1. The first term of an A. P. is 5, the number of terms 30, and their sum 1455; find the common difference.

2. The first term of a series being 2, and the 5th term being 7, find how many terms must be taken so that the sum may be 63.

3. What is the common difference when the first term is 1, the last 50, and the sum 204?

4. How many terms of the series 19, 17, 15, \&c., amount to 91?
5. The sum of a certain number of terms of the series 21, 19, 17, &c. is 120. Find the last term and the number of terms.

6. How many terms of the series 54, 51, 48, &c., must be taken to make 513? Explain the double answer.

7. If the sum of 8 terms of an A. P. is 64, and the sum of 19 terms is 361, find the sum of \( n \) terms.

8. Find the series of which the \( n \)th term is \( \frac{3+n}{4} \); and also find the sum of the series to 105 terms.

9. Find the series whose \( r \)th term is \( 2r-1 \); find the sum of the series to \( n \) terms.

10. The sum of \( n \) terms of an A. P. is \( 3n^2 - n \), and the common difference 6; find the first term.

11. The sum of \( n \) terms of an A. P. is 40, the common difference 2, and the last term 13; find \( n \).

12. Prove that the sum of the latter half of \( 2n \) terms of any arithmetical series is \( \frac{n}{2} \) of the sum of \( 3n \) terms of the same series.

13. If \( 2n+1 \) terms of the series 1, 3, 5, 7, 9, &c., be taken, then the sum of the alternate terms 1, 5, 9, &c., will be to the sum of the remaining terms 3, 7, 11, &c., as \( n+1 \) is to \( n \).

14. Prove that (i) \( b = \frac{l^2 - a^2}{2s-(l+a)} \), and (ii) \( s = \frac{l+a(l-a+b)}{2b} \).

278. Arithmetic means.

Definitions: (1) When three quantities are in Arithmetical Progression, the middle one is said to be the Arithmetic mean between the other two.

Thus, 5 is the Arithmetic mean between 3 and 7.

(2) If \( A \) and \( B \) be any two quantities and \( x_1, x_2, x_3, x_4, \&c., x_{n-1}, x_n \), a number of others such that \( A, x_1, x_2, x_3, \&c., x_{n-1}, x_n, B \) are in Arithmetical Progression, then \( x_1, x_2, x_3, \&c. \) are called the Arithmetic means between \( A \) and \( B \).

Thus, 3, 4, 5, 6, 7 are Arithmetic means between 2 and 8, and so are the numbers \( 3\frac{1}{2}, 5 \) and \( 6\frac{1}{2} \); for both the series 2, 3, 4, 5, 6, 7, 8 and 2, 3\frac{1}{2}, 5, 6\frac{1}{2}, 8 \) are in A. P.

Note. It is evident from the above example that between any two quantities there may be an unlimited number of different sets of Arithmetic means.

279. To insert a given number of Arithmetic means between two given quantities.

Let \( a \) and \( c \) be the two given quantities, and \( n \) the number of Arithmetic means to be inserted.
Then, we have to find out \( n \) quantities \( x_1, x_2, x_3, \ldots, x_{n-2}, x_{n-1}, x_n \) such that \( a, x_1, x_2, \ldots, x_{n-1}, x_n, c \) may be in A.P. Evidently the series \( [a, x_1, x_2, x_3, \ldots, x_{n-1}, x_n, c] \) consists of \( n+2 \) terms of which \( a \) is the first term and \( c \) the last.

Hence, if \( b \) be the common difference, we must have

\[
c = a + (n + 1)b,
\]

whence

\[
b = \frac{c - a}{n + 1}.
\]

Hence,

\[
x_1 = a + b = a + \frac{c - a}{n + 1},
\]

\[
x_2 = a + 2b = a + \frac{2(c - a)}{n + 1},
\]

\[
&c.
\]

\[
x_n = a + nb = a + \frac{n(c - a)}{n + 1},
\]

**Example 1.** Find the Arithmetic mean between any two quantities \( a \) and \( b \).

Let \( x \) be the quantity sought.

Then, \( a, x, b \) are in A.P.; and \( :. \) we must have \( x - a = b - x \).

whence \( x = \frac{a + b}{2} \).

**Example 2.** Insert 4 Arithmetic means between 3 and 18.

Let \( x_1, x_2, x_3, x_4 \) be the required means.

Then, \( 3, x_1, x_2, x_3, x_4, 18 \) are in A.P.

Hence, if \( b \) be the common difference, we must have \( 18 = 3 + 5b \). \( :. \) \( b = 3 \).

Hence,

\[
x_1 = 3 + b = 6
\]

\[
x_2 = 3 + 2b = 9
\]

\[
x_3 = 3 + 3b = 12
\]

\[
x_4 = 3 + 4b = 15
\]

Thus, the required means are 6, 9, 12 and 15.

**EXERCISE 141**

1. Find the Arithmetic means between (i) 5 and 8; (ii) \(-5\) and 21; (iii) \(m - n\) and \(m + n\); (iv) \((a + x)^9\) and \((a - x)^9\).

2. Insert 2 Arithmetic means between (i) 8 and 12; (ii) \(-6\) and 14.

3. Insert 3 Arithmetic means between 117 and 477.

4. Insert 4 Arithmetic means between 2 and \(-18\).

5. Insert 17 Arithmetic means between \(3\frac{1}{2}\) and \(-41\frac{1}{2}\).

6. There are \( n \) Arithmetic means between 1 and 31, such that the 7th mean : \((n - 1)\)th mean = \(5 : 9\); required \( n \).
280. The natural numbers. The numbers 1, 2, 3, &c. are called
the natural numbers.

(i) To find the sum of the first \( n \) natural numbers.

Let \( S \) denote the sum; then

\[
S = 1 + 2 + 3 + \cdots + n
\]

\[
= \frac{n}{2} (1 + n) = \frac{n(n + 1)}{2}.
\]

\[\text{... (A)}\]

(ii) To find the sum of the first \( n \) odd natural numbers.

Let \( S \) denote the sum; then

\[
S = 1 + 3 + 5 + 7 + \cdots \text{ to } n \text{ terms}
\]

\[
= \frac{n}{2} \{2 + (n - 1) \times 2\}
\]

\[
= \frac{n}{2} \times 2n = n^2.
\]

\[\text{... ... (B)}\]

(iii) To find the sum of the squares of the first \( n \) natural
numbers.

Let \( S \) denote the sum; then

\[
S = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2.
\]

We have, \( n^3 - (n - 1)^3 = 3n^2 - 3n + 1 \).

Hence, putting 1, 2, 3, &c., for \( n \), we have

\[
1^3 - 0^3 = 3.1^2 - 3.1 + 1,
\]

\[
2^3 - 1^3 = 3.2^2 - 3.2 + 1,
\]

\[
3^3 - 2^3 = 3.3^2 - 3.3 + 1,
\]

\[
4^3 - 3^3 = 3.4^2 - 3.4 + 1,
\]

\[
\vdots \quad \vdots \quad \vdots \quad \vdots \]

\[
(n - 1)^3 - (n - 2)^3 = 3.(n - 1)^2 - 3.(n - 1) + 1,
\]

\[
n^3 - (n - 1)^3 = 3n^2 - 3n + 1.
\]

Hence, by addition,

\[
n^3 = 3(1^2 + 2^2 + 3^2 + \cdots + n^2) - 3(1 + 2 + 3 + \cdots + n) + n
\]

\[
= 3S - \frac{3n(n + 1)}{2} + n;
\]

\[
\therefore \quad 3S = n^3 - n + \frac{3n(n + 1)}{2} = n(n + 1)((n - 1) + 3);\]

\[
\therefore \quad S = \frac{n(n + 1)(2n + 1)}{6}. \quad \text{... (C)}
\]
(iv) To find the sum of the cubes of the first \( n \) natural numbers.

Let \( S \) denote the sum; then
\[
S = 1^3 + 2^3 + 3^3 + \cdots + n^3.
\]
We have, \( n^4 - (n - 1)^4 = 4n^3 - 6n^2 + 4n - 1 \).
Hence, putting 1, 2, 3, &c., for \( n \), we have
\[
\begin{align*}
1^4 - 0^4 &= 4 \cdot 1^3 - 6 \cdot 1^2 + 4 \cdot 1 - 1, \\
2^4 - 1^4 &= 4 \cdot 2^3 - 6 \cdot 2^2 + 4 \cdot 2 - 1, \\
3^4 - 2^4 &= 4 \cdot 3^3 - 6 \cdot 3^2 + 4 \cdot 3 - 1, \\
&\cdots \\
(n-1)^4 - (n-2)^4 &= 4 \cdot (n-1)^3 - 6 \cdot (n-1)^2 + 4 \cdot (n-1) - 1, \\
(n - 1)^4 - (n - 2)^4 &= 4n^3 - 6n^2 + 4n - 1.
\end{align*}
\]
Hence, by addition,
\[
n^4 = 4(1^3 + 2^3 + 3^3 + \&c. + n^3) - 6(1^2 + 2^2 + 3^2 + \&c. + n^2) + 4(1 + 2 + 3 + \&c. + n) - n
\]
\[
= 4S - 6 \cdot \frac{n(n+1)(2n+1)}{6} = 4 \cdot \frac{n(n+1)}{2} - n;
\]
\[
\therefore \quad 4S = n^4 + n + n(n+1)(2n+1) - 2n(n+1)
\]
\[
= n(n+1)(n^2 - n + 1) + (2n+1) - 2 = n(n+1)(n^2 + n);
\]
\[
\therefore \quad S = \frac{n^2(n+1)^2}{4}\left(\frac{n(n+1)}{2}\right) = \frac{(n+1)^3}{3}.
\]

Thus, the sum of the cubes of the first \( n \) natural numbers is equal to the square of the sum of these numbers.

**Example 1.** Sum the series 1.2 + 2.3 + 3.4 + &c. to \( n \) terms.

The \( n \)th term of the series evidently = \( n(n+1) = n^2 + n \).
Hence, putting \( n = 1 \), the 1st term = \( 1^2 + 1 \),
\[
\therefore \text{ } n = 2, \text{ } \text{2nd term} = 2^2 + 2,
\]
\[
\therefore \text{ } n = 3, \text{ } \text{3rd term} = 3^2 + 3,
\]
\[
\text{and so on.}
\]
Hence, if \( S \) denote the sum of the given series, we have
\[
S = (1^2 + 1) + (2^2 + 2) + (3^2 + 3) + \&c. \text{ to } n \text{ terms}
\]
\[
= (1^2 + 2^2 + 3^2 + \&c. + n^2) + (1 + 2 + 3 + \&c. + n)
\]
\[
= n(n+1)(2n+1) + \frac{n(n+1)}{6} + \frac{n(n+1)}{2}
\]
\[
= \frac{n(n+1)(2n+1)}{3} + 1 = \frac{n(n+1)(n+2)}{3}.
\]
Example 2. Sum the series \(1^2 + 3^2 + 5^2 + 7^2 + \&c.\) to \(n\) terms.

Since evidently each term of the given series is equal to the square of the corresponding term of the series 1, 3, 5, 7, \&c., \(\ldots\) the \(n\)th term of the given series is the square of the \(n\)th term of the series 1, 3, 5, 7, \&c.; and \(\ldots\) the \(n\)th term = \(1 + (n - 1) \times 2\)² = \((2n - 1)\)² = \(4n^2 - 4n + 1\).

Hence, putting \(n = 1, 2, 3, \&c.,\) we have

the 1st term = \(4.1^2 - 4.1 + 1,\)

2nd \(\ast\) = \(4.2^2 - 4.2 + 1,\)

3rd \(\ast\) = \(4.3^2 - 4.3 + 1,\)

\(\ldots\) \(\ldots\) \(\ldots\)

and so on.

Hence, if \(S\) denote the sum of the given series, we must have

\[
S = 4\left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + n\right) - 4\left(\frac{1+2+3+\&c. + n}{12}\right)
\]

\[
= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + \frac{n(n^2-1) + 3}{3} = \frac{n}{3}(4n^2 - 1)
\]

Example 3. Sum the series:

\(1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \&c.\) \(\text{to} \ n \text{ terms}.

The \(n\)th term of the given series

\[
= 1^2 + 2^2 + 3^2 + \&c. + n^2
\]

\[
= \frac{n(n+1)(2n+1)}{6} - \frac{n(2n^2 + 3n + 1)}{6} = \frac{1}{3}n^3 + \frac{1}{3}n^2 + \frac{1}{6}n.
\]

Hence, the 1st term = \(\frac{1}{3}.1^3 + \frac{1}{3}.1^2 + \frac{1}{6}.1,\)

2nd \(\ast\) = \(\frac{1}{3}.2^3 + \frac{1}{3}.2^2 + \frac{1}{6}.2,\)

3rd \(\ast\) = \(\frac{1}{3}.3^3 + \frac{1}{3}.3^2 + \frac{1}{6}.3,\)

\(\ldots\) \(\ldots\) \(\ldots\)

and so on.

Hence, if \(S\) denote the required sum, we must have

\[
S = \frac{1}{3}\left(1^3 + 2^3 + 3^3 + \&c. + n^3\right)
\]

\[
+ \frac{1}{3}(1^2 + 2^2 + 3^2 + \&c. + n^2) + \frac{1}{6}(1 + 2 + 3 + \&c. + n)
\]

\[
= \frac{1}{3} \cdot \frac{n^2(n+1)^2}{4} + \frac{1}{2} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{1}{6} \cdot \frac{n(n+1)}{2}
\]

\[
= \frac{n(n+1)}{12} \left| n(n+1) + (2n+1) + 1 \right|
\]

\[
= \frac{n(n+1)}{12} \left( n^2 + 3n + 2 \right) = \frac{n(n+1)^2(n+2)}{12}.
\]
Example 4. Sum the series
\[ 3.7 + 5.10 + 7.13 + 9.16 + \cdots n \text{ terms.} \]

The \( n \)th term of the series evidently = \((2n + 1)(3n + 4)\)
\[ = 6n^2 + 11n + 4. \]

Hence, putting \( n = 1 \), the 1st term = \( 6.1^2 + 11.1 + 4 \),
\[ = 6.2^2 + 11.2 + 4, \]
\[ = 6.3^2 + 11.3 + 4, \]
and so on.

Hence, if \( S \) denote the sum of the given series, we have
\[ S = 6(1^2 + 2^2 + 3^2 + \cdots n \text{ terms}) + 11(1 + 2 + 3 + 4 + \cdots n \text{ terms}) + 4n \]
\[ = \frac{6}{2} n(n+1)(2n+1) + \frac{11}{2} n(n+1) + 4n \]
\[ = n(n+1)(2n+1) + \frac{11}{2} n(n+1) + 4n \]
\[ = n\left(2n^2 + 3n + 1 + \frac{11}{2} n + \frac{11}{2} + 4\right) \]
\[ = n\left(2n^2 + \frac{17}{2} n + \frac{21}{2}\right) = \frac{n}{2}(4n^2 + 17n + 21). \]

EXERCISE 142

Sum the series:
1. \( 2^a + 5^a + 8^a + \&c. \) to \( n \) terms.
2. \( 1.2^a + 2.3^a + 3.4^a + \&c. \) to \( n \) terms.
3. \( 1.3 + 3.5 + 5.7 + 7.9 + \&c. \) to \( n \) terms.
4. \( 2.3 + 3.4 + 4.5 + \&c. \) to \( n \) terms. [E. B. S. B. 1949]
5. \( 3 \times 8 + 6 \times 11 + 9 \times 14 + \&c. \) to \( n \)th term. [W. B. S. F. 1954 (Suppl.)]
6. \( 2.3.4 + 3.4.5 + 4.5.6 + 5.6.7 + \cdots n \text{ terms.} \)
7. \( 1 \times 3^a + 2 \times 4^a + 3 \times 5^a + \cdots \) 100th term. [W. B. S. F. 1954]
8. \( 1^a + 3^a + 5^a + \&c. \) to \( n \) terms.
9. \( 1 + (1 + 2) + (1 + 2 + 3) + \&c. \) to \( n \) terms.
10. \( (1) + (1 + 3) + (1 + 3 + 5) + \&c. \) to \( n \) terms.
11. \( 1.2.3 + 2.3.4 + 3.4.5 + \&c. \) to \( n \) terms.
12. \( 2.3.4 + 3.4.5 + 4.5.7 + \&c. \) to \( n \) terms.
13. \( 1 - 2 + 3 - 4 + 5 - 6 + \&c. \) to \( n \) terms.
14. \( 1^a - 2^a + 3^a - 4^a + 5^a - 6^a + \&c. \) to \( n \) terms.

1—35
281. Miscellaneous Examples and Problems.

Example 1. Prove that if the number of terms of an A. P. be odd, twice the middle term is equal to the sum of the first and last terms.

Since the number of terms is odd, let it be denoted by \(2n + 1\).

Evidently, the middle term is one which has \(n\) terms on either side of it; hence, it is the \((n+1)\)th term from the beginning and also the \((n+1)\)th term from the end.

Hence, putting \(M\) for the middle term, we must have

\[
M = a + (n + 1 - 1)b = a + nb \quad \cdots \quad \cdots (1)
\]

and also \(M = l - (n + 1 - 1)b = l - nb\). \(\cdots \quad \cdots (2)\)

Hence, by addition, \(2M = a + l\).

Example 2. Prove that the sum of an odd number of terms in A. P. is equal to the middle term multiplied by the number of terms.

Let \(2n + 1\) = the number of terms.

Then, the sum of the terms

\[
\frac{2n+1}{2}(a + l) = \frac{2n+1}{2} \times 2M \quad \text{[last example]}
\]

\[
= (2n + 1) \times M.
\]

Example 3. Find the first five terms of the series of which the sum to \(n\) terms = \(5n^2 + 3n\).

Let \(t_1, t_2, t_3, \&c., t_n\) denote respectively the 1st, 2nd, 3rd, \&c., \(n\)th terms of the series;

and let \(s_1, s_2, s_3, \&c., s_n\) denote respectively the sums of 1, 2, 3, \&c., \(n\) terms of the series.

Evidently then \(s_1 = t_1 ; s_2 = t_1 + t_2 ; s_3 = t_1 + t_2 + t_3 ; \&c., \) and so on.

Now, by the question, we have \(s_n = 5n^2 + 3n\), \(i.e.,\) the sum of any number of terms = 5 times the square of that number + 3 times that number.

Hence, putting \(n = 1, 2, 3, 4, 5, \&c.,\) we have \(s_1 = 5 + 3 = 8,\)

\[
s_2 = 20 + 6 = 26,
\]

\[
s_3 = 45 + 9 = 54,
\]

\[
s_4 = 80 + 12 = 92,
\]

\[
s_5 = 125 + 15 = 140, \text{and so on.}
\]

Hence, \(t_1 = t_2 = 8,\)

\(t_3 = s_2 - s_1 = 26 - 8 = 18,\)

\(t_4 = s_3 - s_2 = 54 - 26 = 28,\)

\(t_5 = s_4 - s_3 = 92 - 54 = 38,\)

\(t_6 = s_5 - s_4 = 140 - 92 = 48, \text{and so on.}\)

Thus, the first five terms of the series are 8, 18, 28, 38 and 48.
Example 4. Sum the series: \(1 + 5 + 12 + 22 + 35 + \&c.\) to \(n\) terms.

[The peculiarity of the series is that the successive differences of the terms are in A. P.]

Let \(S\) denote the required sum and let \(t_n\) denote the \(n\)th term of the series. Then, we have

\[
S = 1 + 5 + 12 + 22 + \cdots + t_n;
\]

also \(S = 0 + 1 + 5 + 12 + \cdots + t_{n-1} + t_n.\)

Hence, by subtraction,

\[
0 = 1 + 4 + 7 + 10 + \&c. + (t_n - t_{n-1}) - t_n
\]

\[
= \{1 + 4 + 7 + 10 + \&c. \text{ to } n \text{ terms}\} - t_n.
\]

\[
\therefore \quad t_n = \frac{n}{2} \{2 + (n - 1)3\} = \frac{n(3n - 1)}{2},
\]

i.e., the \(n\)th term of the given series = \(\frac{n}{2} n^2 - \frac{1}{2} n.\)

Hence, the 1st term = \(\frac{1}{2} \cdot 1^2 - \frac{1}{2} \cdot 1,\)

2nd = \(\frac{1}{2} \cdot 2^2 - \frac{1}{2} \cdot 2,\)

3rd = \(\frac{1}{2} \cdot 3^2 - \frac{1}{2} \cdot 3,\) and so on.

Hence, \(S = \frac{1}{2}(1^2 + 2^2 + 3^2 + \&c. + n^2) - \frac{1}{2}(1 + 2 + 3 + \&c. + n)\)

\[
= \frac{3}{2} \cdot \frac{n(n + 1)(2n + 1)}{6} - \frac{1}{2} \cdot \frac{n(n + 1)}{2} = \frac{n(n + 1)}{4} \cdot 2n = \frac{n^3(n + 1)}{2}.
\]

Example 5. Sum the series \(\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \&c.\) to \(n\) terms.

Let \(S\) denote the sum to \(n\) terms.

Now, we have

\[
t_1 = \frac{1}{1.2} = 1 - \frac{1}{2},
\]

\[
t_2 = \frac{1}{2.3} = \frac{1}{2} - \frac{1}{3},
\]

\[
t_3 = \frac{1}{3.4} = \frac{1}{3} - \frac{1}{4},
\]

\&c., \&c., \&c.,

\[
t_n = \frac{1}{n(n + 1)} = \frac{1}{n} - \frac{1}{n + 1}.
\]

Hence, \(S = 1 - \frac{1}{n + 1} = \frac{n}{n + 1}.\)

Example 6. Divide 15 into three parts which are in A. P. and whose product = 120.

Let \(a - \beta, a\) and \(a + \beta\) be the numbers;

then, we have

\[
(a - \beta)a(a + \beta) = 120 \quad \ldots \quad (1)
\]

and \((a - \beta) + a + (a + \beta) = 15. \quad \ldots \quad (2)\)
From (2), \[ 3a = 15 \] \[ \therefore a = 5. \]
From (1), \[ a(a^2 - \beta^2) = 120. \]
\[ \therefore 5(25 - \beta^2) = 120. \] \[ \therefore 25 - \beta^2 = 24. \] \[ \therefore \beta^2 = 1. \] \[ \therefore \beta = \pm 1. \]

Hence, the numbers are 4, 5, 6.

Example 7. If \( a^2, b^2, c^2 \) be in A. P., then
\[ \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \] are in A. P.

Evidently \[ \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \] are in A. P.,
\[ \text{if } \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a} ; \]
i.e., \[ \text{if } \frac{b-a}{(c+a)(b+c)} = \frac{a-b}{(a+b)(c+a)} \]
i.e., \[ \text{if } (b-a)(b+a) = (a-b)(c+b) , \]
i.e., \[ \text{if } b^2 - a^2 = c^2 - b^2 ; \]
but, this is true by hypothesis.
\[ \therefore \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \] are in A. P.

Example 8. If \( a, b, c \) be respectively the \( p \)th, \( q \)th and \( r \)th terms of an A. P., prove that \( a(q-r) + b(r-p) + c(p-q) = 0 \).

Let \( a \) denote the first term and \( \beta \) the common difference of the A. P., of which \( a, b, c \) are the \( p \)th, \( q \)th and \( r \)th terms; then, we must have
\[
\begin{align*}
a &= a + (p-1)\beta \quad \therefore (1) \\
b &= a + (q-1)\beta \quad \therefore (2) \\
c &= a + (r-1)\beta \quad \therefore (3)
\end{align*}
\]

Now, we have to eliminate \( a \) and \( \beta \) from these three equations.
Subtracting (2) from (1), and (3) from (2), we have
\[ a-b=(p-q)\beta, \]
\[ b-c=(q-r)\beta. \]
Hence, \( (a-b)(q-r) = (b-c)(p-q) \),
or, \( a(q-r) + b(r-p) + c(p-q) = 0 \).

Example 9. A person lends Rs. 1000 to a friend agreeing to charge no interest and also to recover the amount by monthly instalments.
decreasing successively by Rs. 2. In how many months will the loan be paid up, if the first instalment be Rs. 64? [C. U. 1930]

Let \( n \) = the number of months required,
the successive instalments are evidently in A. P.
whose 1st term = 64, 
and whose common difference = -2.

Since, the sum of the \( n \) instalments = Rs. 1000, the sum of the 1st \( n \) terms of this A. P. = 1000,

\[
\frac{n}{2} [2 \times 64 + (n - 1)(-2)] = 1000,
\]
or,
\[
(65n - n^2) = 1000,
\]
or,
\[
n^2 - 65n + 1000 = 0,
\]
or,
\[
(n - 25)(n - 40) = 0.
\]
Hence, \( n = 25 \), or, 40
But \( n \) cannot be 40, since in that case the 40th instalment = the 40th term of the A. P.
\[-64 + (-2)(40 - 1) = -14,
\]
which is inadmissible, as no instalment can be negative.

\( \therefore \) \( n \) must be 25.

EXERCISE 149

1. The \((n + 1)\)th term of a series in A. P. is \( \frac{ma - nb}{a - b} \); required the sum of the series to \((2n + 1)\) terms.

2. Find the first five terms of the series of which the sum to \( n \) terms is \( 2n^2 + 7n \).

3. The sum to \( n \) terms of an A. P. is \( 3n^2 + 10n \); find the first term and the common difference.

4. Find the 35th term of the series of which the sum to \( n \) terms is \( n^2 + n \).

5. Sum the series: \( 1 + 3 + 6 + 10 + 15 + \&c. \) to \( n \) terms.

6. Sum the series: \( 2 + 5 + 10 + 17 + \&c. \) to \( n \) terms.

7. Sum the series: \( 2 + 7 + 14 + 23 + 34 + \&c. \) to \( n \) terms.

8. Sum the series:

- \((i)\) \( \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \&c. \) to \( n \) terms.

- \((ii)\) \( \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \&c. \) to \( n \) terms. [W. B. S. E. 1953]

- \((iii)\) \( \frac{1}{a(a + b)} + \frac{1}{(a + b)(a + 2b)} + \frac{1}{(a + 2b)(a + 3b)} + \&c. \) to \( n \) terms.
9. Find 4 numbers in A. P., such that their sum shall be 56, and the sum of their squares 864.

[ Let $a - 3\beta$, $a - \beta$, $a + \beta$ and $a + 3\beta$ be the numbers. ]

10. Divide 36 into three parts which are in A. P., and whose product = 1536.

11. The sum of three numbers in A. P. is 15, and the sum of the squares of the two extremes is 58. What are the numbers?

12. There are four numbers in A. P., the sum of the two extremes is 8, and the product of the means is 15. What are the numbers?

13. Find six numbers in A. P., such that the sum of the two extremes may be 16 and the product of the two middle terms 63.

[ Let $a - 5\beta$, $a - 3\beta$, $a - \beta$, $a + \beta$, $a + 3\beta$, $a + 5\beta$ be the numbers. ]

14. If $(b-c)^a$, $(c-a)^a$, $(a-b)^a$ are in A. P., show that

$$\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$$

are in A. P.

15. If $a$, $b$, $c$ be in A. P., show that

(1) $\frac{1}{bc}$, $\frac{1}{ca}$, $\frac{1}{ab}$ are in A. P.

(2) $b+c$, $c+a$, $a+b$ are in A. P.

(3) $a^a(b+c)$, $b^a(c+a)$, $c^a(a+b)$ are in A. P.

(4) $\frac{1}{a}\left(\frac{1}{b} + \frac{1}{c}\right)$, $\frac{1}{b}\left(\frac{1}{c} + \frac{1}{a}\right)$, $\frac{1}{c}\left(\frac{1}{a} + \frac{1}{b}\right)$ are in A. P.

(5) $a\left(\frac{1}{b} + \frac{1}{c}\right)$, $b\left(\frac{1}{c} + \frac{1}{a}\right)$, $c\left(\frac{1}{a} + \frac{1}{b}\right)$ are in A. P.

(6) $(a + 2b - c)(2b + c - a)(c + a - b) = 4abc$. [ D. B. 1931 ]

16. If $a$, $b$ and $c$ be respectively the sums of $p$, $q$ and $r$ terms of an A. P., prove that

$$\frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = 0.$$

17. The $p$th term of an A. P. is $a$ and the $q$th term, $b$. Show that the sum of the first $(p+q)$ terms is

$$\frac{p+q}{2}\left(a + b + \frac{a-b}{p-q}\right).$$

[ M. U. 1887 ]

[ See Example 2, Art. 276 ]

18. There are $n$ Arithmetic means between 3 and 54, such that the 8th mean : (n - 2)th mean = 3 : 5; find $n$. 
19. If $S_1$, $S_2$, $S_3$ be the sums of $n$ terms of three Arithmetic series the first term of each being 1 and the respective common difference 1, 2, 3, prove that $S_1 + S_2 = 2S_3$.

20. If there be $r$ Arithmetic Progressions, each beginning from unity, whose common differences are 1, 2, 3, &c., $r$, show that the sum of their terms is $\frac{1}{2}(n-1)r^2 + (n+1)r$.


[The $r^{th}$ term of the series $= (n - (r - 1)).r = (n + 1)r - r^2$. Hence, the required sum $= (n + 1)(1 + 2 + 3 + \cdots + n) - (1^2 + 2^2 + 3^2 + \cdots + n^2) = &c.$]

22. On the ground are placed $r$ stones; the distance between the first and second is one metre, between the 2nd and 3rd three metres, between the 3rd and 4th five metres, and so on. How far will a person have to travel who shall bring them, one by one, to a basket placed at the first stone?

23. A class consists of a number of boys whose ages are in A. P., the common difference being four months. If the youngest boy is just eight years old, and if the sum of the ages is 168 years, find the number of boys in the class. [C. U. Entr. Paper, 1872]

24. The interior angles of a rectilineal figure are in A. P. If the least angle is 42° and the common difference is 33°, find the number of sides.

25. If sums of the first $p$, $q$ and $r$ terms of an A. P. are $x$, $y$ and $z$ respectively, prove that $xqr(q - r) + yrp(r - p) + zpq(p - q) = 0$.

CHAPTER XXXVIII

GEOMETRICAL PROGRESSION

282. Definition. Quantities are said to be in Geometrical Progression when each is equal to the product of the preceding and some constant factor.

The constant factor is called the common ratio of the series, and it is found by dividing any term by that which immediately precedes it.

Thus, each of the following series forms a Geometrical Progression:

<table>
<thead>
<tr>
<th>1,</th>
<th>2,</th>
<th>4,</th>
<th>8,</th>
<th>16,</th>
<th>&amp;c.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,</td>
<td>$\frac{1}{2}$,</td>
<td>$\frac{1}{4}$,</td>
<td>$\frac{1}{8}$,</td>
<td>$\frac{1}{16}$,</td>
<td>&amp;c.</td>
</tr>
<tr>
<td>1,</td>
<td>$-\frac{1}{2}$,</td>
<td>$\frac{1}{2}$,</td>
<td>$-\frac{1}{4}$,</td>
<td>$\frac{1}{16}$,</td>
<td>&amp;c.</td>
</tr>
<tr>
<td>$a$,</td>
<td>$ar$,</td>
<td>$ar^2$,</td>
<td>$ar^3$,</td>
<td>$ar^4$,</td>
<td>&amp;c.</td>
</tr>
</tbody>
</table>
In the first example the common ratio is 2, in the second \( \frac{1}{3} \), in the third \( -\frac{1}{4} \), and in the fourth \( r \).

**N. B.** 'Geometrical Progression' is briefly written as G. P.

283. To find the \( n \)th term of a G. P.

If \( a \) be the first term and \( r \) the common ratio of a Geometric series, we have the 2nd term = \( a \cdot r \), the 3rd term = \( a \cdot r^2 \), the 4th term = \( a \cdot r^3 \), ..., the 10th term = \( a \cdot r^9 \), ..., the 21st term = \( a \cdot r^{20} \), and so on. Hence, the \( n \)th term = \( a \cdot r^{n-1} \).

**Example.** Find the 6th term of the series 2, 6, 18, 54, &c.

Here, \( a = 2 \) and the common ratio = \( \frac{3}{2} = 3 \);

\[ \therefore \text{the 6th term} = 2 \times (3)^{6-1} = 486. \]

284. Given any two terms of a G. P., to find the series completely.

**Example 1.** Find the G. P. whose 5th term is 81 and whose 8th term is 2187.

Let \( a \) = the 1st term, and \( r \) = the common ratio.

\[ \therefore 81 = a \cdot r^{5-1} = ar^4, \quad \ldots \quad \ldots \quad (1) \]

and \[ 2187 = a \cdot r^{8-1} = ar^7, \quad \ldots \quad \ldots \quad (2) \]

Dividing (2) by (1), \[ r^3 = \frac{2187}{81} = 27. \quad \therefore r = 3. \]

Hence, \[ ar^4 = a \cdot 3^4 = 81, \]

or, \[ a = \frac{81}{3^4} = 1. \]

Thus, the series is 1, 3, 9, 27, &c.

**Example 2.** If \( c \) and \( d \) be the \( p \)th and \( q \)th terms respectively of a G. P., to determine it completely.

Let \( a \) = the 1st term, and \( r \) = the common ratio.

\[ \therefore c = \text{the } p \text{th term of the G. P.} \]

\[ = ar^{p-1}. \quad \ldots \quad \ldots \quad (1) \]

Similarly, \[ d = ar^{q-1}. \quad \ldots \quad \ldots \quad (2) \]

By division, \[ r^{q-p} = \frac{d}{c}; \quad \therefore r = \left(\frac{d}{c}\right)^{\frac{1}{q-p}}. \]

Substituting for \( r \) in (1), we have

\[ a = \frac{c}{r^{p-1}} = \frac{c}{\left(\frac{d}{c}\right)^{\frac{p-1}{q-p}}} = \left(\frac{c^{q-1}}{d^{q-p}}\right)^{\frac{1}{q-p}}. \]

Hence, the 1st term and the common ratio being known, the complete series may be written down.
EXERCISE 144

1. Find the 8th term of the series 4, 12, 36, &c.
2. Find the 6th term of the series $3^2, 2^4, 1^2, &c.$
3. Find the 9th term of the series 1, 4, 16, 64, &c.
4. Find the 6th term of the series 1, -3, 9, -27, &c.
5. Find the 5th term and the $(n-1)$th term of the series $3, -1, 3, &c.$
6. Find the 7th term of the series $-21, 14, -9^\frac{1}{2}, &c.$
7. Find the $n$th term of the series $\sqrt{3} + \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} + \cdots$
8. The first two terms of a series in G. P., are 125 and 25. What are the 6th and 7th terms?
9. Find the series (i) whose 6th and 11th terms are respectively 192 and 6144; (ii) whose 2nd and 8th terms are 9 and $\frac{1}{3}$ respectively; (iii) whose 5th and 8th terms are 8 and $-\frac{16}{3}$ respectively.
10. The $p$th and the $q$th terms of a G. P., are $c$ and $d$ respectively. Find the $n$th term.
11. If every term of a G. P. is multiplied or divided by the same quantity, the resulting series is also a G. P.
12. In a G. P., if the $(p+q)$th term $= m$ and the $(p-q)$th term $= n$, find the $p$th and $q$th terms.

[B. U. 1888]
13. In a G. P., prove that the product of any pair of terms equidistant from the beginning and the end is constant.

14. There are $2n$ terms in a Geometric series. Prove that the product of the first and last terms is equal to the product of the two middle terms.

285. To find the sum of a number of terms in Geometrical Progression.

Let $a$ be the first term, $r$ the common ratio, $n$ the number of terms and $S$ the sum required; then

\[ S = a + ar + ar^2 + ar^3 + \&c. + ar^{n-1}. \]

\[ \therefore \quad Sr = ar + ar^2 + ar^3 + \&c. + ar^{n-1} + ar^n. \]

Hence, by subtraction,

\[ Sr - S = ar^n - a. \quad \therefore \quad S(r-1) = a(r^n - 1). \]

\[ \therefore \quad S = \frac{a(r^n - 1)}{r - 1} \quad \cdots \quad \cdots \quad (1) \]

or, \[ S = \frac{a(1-r^n)}{1-r} \quad \cdots \quad \cdots \quad (2) \]
**Cor.** If \( l \) denote the last (or the \( n \)th) term of the series, we have
\[
l = ar^{n-1}; \quad \text{hence, from (1), } S = \frac{r^l - a}{r - 1}.
\] ... (3)

**Note.** The formula (2) may conveniently be used in all cases except when \( r \) is positive and greater than 1.

**Example 1.** Find the sum of \( \frac{1}{2} \) \(-\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots \) \&c. to 7 terms.

The common ratio \(- \frac{1}{2} + \frac{1}{4} = - \frac{1}{2} \times \frac{1}{2} = - \frac{1}{4}.

Hence, by formula (2), the sum \[
= \frac{\frac{1}{2} \left\{ 1 - \left(- \frac{1}{2}\right)^7 \right\}}{1 + \frac{1}{2}} = \frac{\frac{1}{2} \left\{ 1 + \frac{1}{2} \cdot \frac{1}{2} \cdot 2 \right\}}{1 + \frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{2} = \frac{1}{8} \cdot \frac{1}{8} = \frac{1}{64}.
\]

**Example 2.** Find the sum of \( 3 + 4\frac{1}{2} + 6\frac{3}{4} + \cdots \) \&c. to 5 terms.

The common ratio \( 3 + 4\frac{1}{2} = \frac{1}{2} x \frac{1}{2} = \frac{3}{2}.

Hence, if \( S \) denote the required sum, we have by formula (1),
\[
S = \frac{3 \cdot \left\{ \left(\frac{3}{2}\right)^{\frac{5}{8}} - 1 \right\}}{\frac{3}{2} - 1} = 3 \cdot \frac{\left(\frac{3}{2}\right)^{\frac{5}{8}} - 1}{\frac{1}{2}} = 3 \times \frac{3}{2} \times 2 = \frac{3}{2} \times 2 = 39\frac{3}{8}.
\]

**Example 3.** Find, without the help of any formula, the sum of the series \( 1 + 5 + 25 + \cdots \) \&c. to 10 terms.

The common ratio = 5.

\[
\therefore \quad \text{the 10th term} = 1 \cdot 5^9 = 5^9.
\]

Suppose \( S \) is the required sum.

\[
\therefore \quad S = 1 + 5 + 5^2 + \cdots + 5^9.
\]

\[
\therefore \quad 5S = 5 + 5^2 + \cdots + 5^9 + 5^{10}
\]

Subtracting (1) from (2),
\[
4S = 5^{10} - 1.
\]

\[
\therefore \quad S = \frac{1}{4} (5^{10} - 1) = \frac{1}{4} (9765625 - 1) = \frac{1}{4} \times 9765624 = 2441406.
\]

**EXERCISE 145**

1. Sum \( 1 + 3 + 9 + 27 + \cdots \) \&c. to 12 terms.
2. Sum \( 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots \) \&c. to \( n \) terms. \[ C. U. 1939 \text{ (Suppl.)} \]
3. Sum \( 81 - 27 + 9 - \cdots \) \&c. to 8 terms.
4. Sum \( 2 - 4 + 8 - \cdots \) \&c. to 10 terms.
5. Sum \( \frac{1}{3} - \frac{1}{3} + \frac{1}{3} - \cdots \) \&c. to 5 terms.
6. Sum \( 2 - 4 + 8 - \cdots \) \&c. to 2\( r \) terms.
7. Sum \( 2\frac{1}{2} - 1 + \frac{1}{2} - \cdots \) \&c. to \( n \) terms.
8. Find without applying any formula the sum of

(i) The series \(1 + \frac{1}{2} + \frac{1}{2^2} + \&c.\) to \(n\) terms.

(ii) The series \(5 + 15 + 45 + \&c.\) to \(8\) terms.

9. Show that the sum of \(n\) terms of a G. P. beginning with the\n
\(p\)th term, is \(r^{p-q}\) times the sum of an equal number of terms of the same

series beginning with the \(q\)th term.

286. If \(n\) be an integer and \(r\) a given proper fraction, to prove that \(r^n\) diminishes as \(n\) increases.

Let \(r = \frac{1}{2}.\) Now, since \(\frac{1}{2}\) of any number is undoubtedly less than that number,

\(\left(\frac{1}{2}\right)^2\) is less than \(\frac{1}{2},\) because \(\left(\frac{1}{2}\right)^2 = \frac{1}{4}\) of \(\frac{1}{2} ;\)

\(\left(\frac{1}{2}\right)^2\) is less than \(\left(\frac{1}{2}\right)^3\), because \(\left(\frac{1}{2}\right)^3 = \frac{1}{8}\) of \(\left(\frac{1}{2}\right)^2 ;\)

\(\left(\frac{1}{2}\right)^4\) is less than \(\left(\frac{1}{2}\right)^5\), because \(\left(\frac{1}{2}\right)^5 = \frac{1}{32}\) of \(\left(\frac{1}{2}\right)^4 ;\)

and so on.

Hence, it is clear that in the series \(\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \left(\frac{1}{2}\right)^4, \ldots\ldots\) each term

is less than the preceding one; which is briefly expressed by saying that \n
\(\left(\frac{1}{2}\right)^n\) diminishes as \(n\) increases.

Similarly, the proposition may be proved for any other value of \(r\)

which is less than 1.

Hence, generally speaking, if \(r\) has a given value less than 1,

\(r^n\) diminishes as \(n\) increases.

Note. From the above it is quite clear that if \(r\) be a proper fraction, \(r^n\) is very

small when \(n\) is infinitely large.

237. The sum of a Geometrical series continued to infinity.

Let us consider the series \(a, ar, ar^2, ar^3, \&c.\)

If \(S\) denote the sum to \(n\) terms, we have

\[ S = a\left(1-r^n\right) \begin{array}{c} 1-r \end{array} \begin{array}{c} 1-r \end{array} = \frac{a}{1-r} - \frac{ar^n}{1-r} \]

If then \(r\) be a proper fraction, the larger \(n\) is, the smaller will be

\(r^n\) and \(\ldots\ldots\frac{ar^n}{1-r}\); hence by sufficiently increasing the value of \(n\) we can

make \(\frac{ar^n}{1-r}\) less than any assigned quantity, however small; and there-

fore by sufficiently increasing the value of \(n\), the sum of \(n\) terms of the

series can be made to differ from \(\frac{a}{1-r}\) by as small a quantity as we please.
This statement is usually put thus: the sum of an infinite number of terms of the Geometrical Progression is \( \frac{a}{1 - r} \), or more briefly, the sum to infinity is \( \frac{a}{1 - r} \).

Let us apply all these remarks to a particular example.

Consider the series 1, \( \frac{1}{2} \), \( \frac{1}{3} \), \( \frac{1}{4} \), &c.

Here, \( a = 1, \ r = \frac{1}{2} \); hence the sum to \( n \) terms
\[
\frac{1}{1 - \frac{1}{2}} \left(1 - \frac{1}{2^n}\right) - 2 \left(1 - \frac{1}{2^n}\right) = 2 - \frac{1}{2^n}.
\]

Now, by taking \( n \) large enough, \( 2^{n-1} \) can be made as large as we please, and therefore, \( \frac{1}{2^{n-1}} \) as small as we please.

Hence, we may say that by taking \( n \) large enough, the sum of \( n \) terms of the series can be made to differ from 2 by as small a quantity as we please; or briefly, the sum of an infinite number of terms of this series is 2.

**N. B.** It must be borne in mind that the sum of \( n \) terms of a Geometrical Progression approaches a fixed limit as \( n \) increases indefinitely only when \( r \) is less than unity. If \( r \) be greater than unity there is no such fixed limit.

**Example 1.** Prove that in a decreasing Geometrical Progression continued to infinity each term bears a constant ratio to the sum of all which follow it.

Let the series be \( a, \ ar, \ ar^2, \ ar^3, \ &c. \), where \( r \) is less than unity.

Then, the \( n \)th term = \( ar^{n-1} \) and the sum of all the terms which follow this
\[
= ar^n(1 + r + r^2 + r^3 + &c. \ to \ infinity)
\]
\[
= ar^n \cdot \frac{1}{1 - r}.
\]

Hence, the ratio of the \( n \)th term to the sum of all which follow it
\[
= \left( \frac{ar^{n-1} + ar^n}{1 - r} \right) = \frac{1 - r}{r}.
\]

Now, this is constant whatever value \( n \) may have, which proves the proposition.

**Example 2.** Sum to infinity \( \frac{1}{3} - \frac{1}{2} + \frac{1}{3} - \&c. \)

Here, \( a = \frac{1}{3}, \) and \( r = -\frac{1}{2} + \frac{1}{3} = -\frac{1}{6}. \)

Hence, the required sum = \( \frac{\frac{1}{3}}{1 + \frac{1}{6}} - \frac{1}{6} \times \frac{1}{3} - \frac{1}{2}. \)
EXERCISE 146

Sum to infinity each of the following series:
1. \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c. \)
2. \( 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \&c. \)
3. \( \frac{1}{5} + \frac{1}{10} + \frac{1}{20} + \&c. \)
4. \( -\frac{1}{6} + \frac{1}{12} - \&c. \)
5. \( \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \&c. \)
6. \( \frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \&c. \) [Split this up into two series.]
7. \( \frac{4}{7} + \frac{5}{7^2} + \frac{4}{7^3} + \frac{5}{7^4} + \&c. \)
8. \( \sqrt{3} + \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} + \&c. \)
9. \( (\sqrt{2}+1)+1+(\sqrt{2}-1)+\&c. \)

10. Find the common ratio of a G. P., continued to infinity in which each term is ten times the sum of all the terms which follow it.


Thus, for example, \( \text{‘264 = ‘23434343…} \)

\[
\begin{align*}
= & \quad 2 \\
+ & \quad 034 \\
+ & \quad 00034 \\
+ & \quad 0000034 \\
+ & \quad \&c., \&c.
\end{align*}
\]

\[= \frac{2}{10} + \frac{34}{10^2} + \frac{34}{10^3} + \&c. \]

Here the terms after \( \frac{1}{10^3} \) constitute a G. P. of which the first term is \( \frac{34}{10^3} \) and the common ratio \( \frac{1}{10^3} \).

Hence, we may take \( \text{‘234 = } \frac{2}{10} + \frac{34}{10^2} + \left(1 - \frac{1}{10^3}\right) = \frac{2}{10} + \frac{34}{990} = \frac{232}{990} \), which agrees with the value found by the usual Arithmetical rule.

289. Geometric means. Definition 1. When three quantities are in Geometrical Progression the middle one is called the Geometric mean between the other two.

Definition 2. When any number of quantities \( x_1, x_2, x_3, \&c. \), are such that \( a, x_1, x_2, x_3, \&c. \), \( b \) are in G. P., then \( x_1, x_2, x_3, \&c. \), are called Geometric means between \( a \) and \( b \).

(i) To find the Geometric means between two given quantities.

Let \( a \) and \( b \) be the two given quantities; \( G \) the Geometric mean.

Then since, \( a, G, b \) are in G. P., we must have, \( \frac{G}{a} = \frac{b}{G} \), each being equal to the common ratio. \( \therefore \) \( G^2 = ab \), and \( \therefore G = \sqrt{ab} \).

(ii) To insert a given number of Geometric means between two given quantities.
Let \( a \) and \( b \) be the two given quantities; and \( x_1, x_2, x_3, \ldots, x_n \), the \( n \) means to be inserted.

Then \( a, x_1, x_2, \ldots, x_n, b \) are in G. P.

Let \( r \) denote the common ratio of the series;

then \( b = \text{the} (n+2) \text{th term} = a \cdot r^{n+1} \).

\[
\therefore r^{n+1} = \frac{b}{a}, \quad \text{and} \quad \therefore r = \left( \frac{b}{a} \right)^{\frac{1}{n+1}}.
\]

Hence, \( x_1 = a \cdot \left( \frac{b}{a} \right)^{\frac{1}{n+1}} \); \( x_2 = a \cdot \left( \frac{b}{a} \right)^{\frac{2}{n+1}} \); \( x_3 = a \cdot \left( \frac{b}{a} \right)^{\frac{3}{n+1}} \); and so on.

**Example.** Insert 3 Geometric means between \( \frac{1}{3} \) and 128.

Let \( x_1, x_2, x_3 \) be the means.

Then, \( \frac{1}{3}, x_1, x_2, x_3, 128 \) are in G. P.

Hence, if \( r \) be the common ratio of the series, we must have \( 128 = \text{the} 5 \text{th term} = \frac{1}{3} \cdot r^4 \).

\[
\therefore r^4 = 256, \quad \text{whence} \quad r = 4.
\]

Hence,

\[
\begin{align*}
x_1 &= \frac{1}{3} \cdot 4 = 2 \\
x_2 &= \frac{1}{3} \cdot 4^2 = 8 \\
x_3 &= \frac{1}{3} \cdot 4^3 = 32
\end{align*}
\]

**290.** *The Arithmetic mean of any two positive quantities is greater than their Geometric mean.***

Let \( a \) and \( b \) be two positive quantities.

\[
\therefore \text{their Arithmetic mean} = \frac{a + b}{2}, \quad \text{and Geometric mean} = \sqrt{ab}.
\]

Now,

\[
\frac{a + b}{2} - \sqrt{ab} = \frac{1}{2}[a - 2 \sqrt{a \cdot b} + b] = \frac{1}{2} (\sqrt{a} - \sqrt{b})^2
\]

= a positive quantity.

\[
\therefore \frac{a + b}{2} > \sqrt{ab}.
\]

**EXERCISE 147**

1. Insert 2 Geometric means between 3 and 24.
2. Insert 3 Geometric means between 2\( \frac{1}{2} \) and \( \frac{1}{2} \).
3. Insert 4 Geometric means between \( \frac{1}{3} \) and \(-5\frac{1}{2} \).
4. Insert 5 Geometric means between 3\( \frac{1}{2} \) and 40\( \frac{1}{2} \).
5. What are the three Geometric means between 25 and 164025?  
[Pat. U. 1919]

6. If $a$, $b$ and $c$ be in G. P., and $x$, $y$ be the Arithmetic means between $a$, $b$ and $b$, $c$ respectively, prove that
\[
\frac{a}{x} + \frac{c}{y} = 2 \quad \text{and} \quad \frac{1}{x} + \frac{1}{y} = \frac{2}{b}.
\]
[P. U. 1892]

7. The Arithmetic mean of $a$ and $b$ is to their Geometric mean as $m$ to $n$; show that $a : b = m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$.  
[A. U. 1889]

8. If the Arithmetic and Geometric means between two quantities be respectively $A$ and $B$, prove that the quantities are
\[
A + \sqrt{A^2 - B^2} \text{ and } A - \sqrt{A^2 - B^2}.
\]
[Let the numbers be $a$ and $b$. Suppose $a > b$.

\[a + b = 2A, \quad \ldots \quad \ldots \quad \ldots \quad (1)\]

\[\sqrt{ab} = B.\]

Now,  
\[(a - b)^2 = (a + b)^2 - 4ab = 4(A^2 - B^2),\]

or,  
\[a - b = 2\sqrt{A^2 - B^2}, \quad \ldots \quad \ldots \quad \ldots \quad (3)\]

(taking the positive root, since, $a > b$,  
i.e., $a - b$ is positive.)

Adding (1) and (2),  
\[2a = 2A + 2\sqrt{A^2 - B^2}, \text{ or, } a = A + \sqrt{A^2 - B^2}.\]

Also, subtracting (2) from (1),  
\[b = A - \sqrt{A^2 - B^2}.\]

291. Miscellaneous Series and Examples.

Example 1. If $x < 1$, sum the series
\[1 + 2x + 3x^2 + 4x^3 + \&c., \text{ to infinity}.\]

Let $S$ denote the required sum; then
\[S = 1 + 2x + 3x^2 + 4x^3 + \&c.,\]

and  
\[Sx = x + 2x^2 + 3x^3 + \&c.\]

Hence, by subtraction,
\[S(1-x) = 1 + x + x^2 + x^3 + \&c., \text{ to infinity}\]
\[= \frac{1}{1-x}.
\]
\[\therefore \quad S = \frac{1}{(1-x)^2}.\]
Example 2. Sum to \( n \) terms \( 5 + 55 + 555 + \&c. \)

Let \( S \) denote the required sum; then

\[
S = 5 + 55 + 555 + \&c. \quad \text{to } n \text{ terms}
\]

\[
= 5[1 + 11 + 111 + \&c. \quad \text{to } n \text{ terms}]
\]

\[
= \frac{5}{9}[1 + 11 + 111 + \&c. \quad \text{to } n \text{ terms}]
\]

\[
= \frac{5}{9}[9 + 99 + 999 + \&c. \quad \text{to } n \text{ terms}]
\]

\[
= \frac{5}{9}[10(10^n - 1) + (10^n - 1) + \&c. \quad \text{to } n \text{ terms}]
\]

\[
= \frac{5}{9}[10 + 10^n + \&c. \quad \text{to } n \text{ terms} - n]
\]

\[
S = \frac{5}{9} \left( \frac{10(10^n - 1) - n}{10 - 1} \right) = \frac{50}{81}(10^n - 1) - \frac{5n}{9}.
\]

Example 3. Sum to \( n \) terms \( 1 + 5 + 13 + 29 + \&c. \)

Let \( t_n \) denote the \( n \)th term of the series, and \( S \) the required sum; then

\[
S = 1 + 5 + 13 + 29 + \cdots + t_n;
\]

and \( S = 0 + 1 + 5 + 13 + \cdots + t_{n-1} + t_n \).

Therefore, by subtraction,

\[
0 = (1 + 4 + 8 + 16 + \&c. \quad \text{to } n \text{ terms}) - t_n
\]

\[
= 1 + 4(2^{n-1} - 1)
\]

\[
= 1 + 2^n(2^{n-1} - 1) = 2^{n+1} - 3.
\]

Hence, the 1st term = \( 2^1 - 3 \),

2nd = \( 2^2 - 3 \),

3rd = \( 2^3 - 3 \),

and so on.

Hence,

\[
S = (2^1 - 3) + (2^2 - 3) + (2^3 - 3) + \&c. + (2^{n+1} - 3)
\]

\[
= (2^1 + 2^2 + 2^3 + \&c. \quad \text{to } n \text{ terms}) - 3n
\]

\[
= \frac{2^n(2^{n+1} - 1)}{2 - 1} - 3n
\]

\[
= 4(2^n - 1) - 3n.
\]

Example 4. If \( a, b, c, d \) be in G. P., show that

\[
(a - d)^2 = (b - c)^2 + (c - a)^2 + (d - b)^2.
\]

We have \( \frac{b}{a} = \frac{c}{b} = \frac{d}{c} \), each of them being equal to the common ratio; \( \therefore b^2 = ac, c^2 = bd, \) and \( bc = ad \).

\[
\therefore (a)
\]
Hence, \((b-c)^2 + (c-a)^2 + (d-b)^2\)
\[= (b^2 + c^2 - 2bc) + (c^2 + a^2 - 2ca) + (d^2 + b^2 - 2bd)\]
\[= 2(b^2 - ac) + 2(c^2 - bd) + a^2 + d^2 - 2bc\]
\[= 2 \times 0 + 2 \times 0 + a^2 + d^2 - 2ad. \quad \text{[ by } a \text{]}\]
\[= (a-d)^2.\]

**Example 5.** If \(a, b, c, d\) be in G. P., show that
\[a^2 - b^2, \quad b^2 - c^2, \quad c^2 - d^2\]
are in G. P., if
\[(a^2 - b^2)(c^2 - d^2) = (b^2 - c^2)^2.\]

Now, since \(a, b, c, d\) are in G. P., we have \(\frac{b}{a} = \frac{c}{b} = \frac{d}{c}\).

\[\therefore \quad ac = b^2, \quad bd = c^2 \quad \text{and} \quad ad = bc.\]

Hence,
\[(a^2 - b^2)(c^2 - d^2) = a^2 c^2 - b^2 c^2 - a^2 d^2 + b^2 d^2\]
\[= b^4 - b^2 c^2 - b^2 d^2 + b^4\]
\[= b^4 - 2b^2 c^2 + c^4 = (b^2 - c^2)^2.\]

\[\therefore \quad a^2 - b^2, \quad b^2 - c^2, \quad c^2 - d^2\]
are in G. P.

**Example 6.** If \(p, q, r\) be in A. P., prove that the \(p\)th, \(q\)th, and \(r\)th terms of any Geometric series form a Geometric series.

[ W. B. S. F. 1952 ]

Suppose the 1st term of the Geometric series = \(a\) and the common ratio = \(R\).

\[\therefore \quad \text{the } p\text{th term} = aR^{p-1}, \quad \text{the } q\text{th term} = aR^{q-1}, \quad \text{and} \quad \text{the } r\text{th term} = aR^{r-1},\]
\[aR^{p-1}, \quad aR^{q-1} \quad \text{and} \quad aR^{r-1} \quad \text{will be in G. P., if} \quad (aR^{p-1})^2 = aR^{q-1} \cdot aR^{r-1},\]
\[i.e., \quad a^2 R^{2p-2} = a^2 R^{p+r-2}, \quad i.e., \quad \text{if} \quad 2q = p + r - 2,\]
or, \(2q = p + r, \) if \(p, q, r\) are in A. P.

**Example 7.** The continued product of three numbers in G. P. is 216, and the sum of the products of them in pairs is 156; find the numbers.

Let \(\frac{a}{r}, a, ar\) be the numbers;

then by the conditions given, we must have
\[\frac{a}{r} \cdot a \cdot ar = 216 \quad \ldots \quad (1)\]
and \[\frac{a}{r} \cdot a + \frac{a}{r} \cdot ar + a \cdot ar = 156 \quad \ldots \quad (2)\]

From (1), \[a^2 = 216. \quad \therefore \quad a = 6.\]

\[\therefore \quad \text{the numbers are} \quad a = 6, \quad ar = 36, \quad aR = 18.\]
Hence, from (2), \[ \frac{1}{r} + 1 + r = \frac{156}{36} = \frac{13}{3} \cdot \therefore 3(1 + r + r^2) = 13r, \]
or, \[ (3r^2 - 10r + 3) = 0, \quad \text{or,} \quad (r - 3)(3r - 1) = 0. \]
\[ r = 3, \quad \text{or}, \quad \frac{1}{3}. \]
Hence, the numbers are 2, 6, 18.

**EXERCISE 148**

1. Find by the method of summation of infinite Geometric series the values of:
   
   (i) 0.27; \quad (ii) 1.115 ; \quad (iii) 21501 ; \quad (iv) 142857

2. Sum \[ 1 + 3x + 5x^2 + 7x^3 + \&c. \text{ to infinity}. \]

3. Sum \[ 1.2x + 2.4x^2 + 3.8x^3 + \&c. \text{ to infinity}. \]

4. Sum \[ 1.3x + 4.9x^2 + 7.27x^3 + \&c. \text{ to infinity}. \]

5. Sum \[ a + 2a^2 + 3a^3 + 4a^4 + \&c. \text{ to } n \text{ terms}. \]

6. Sum \[ 1 - 3x + 5x^2 - 7x^3 + \&c. \text{ to infinity}. \]

7. Sum \[ \frac{1}{3} + \frac{5}{9} + \&c. \text{ to infinity}. \]

8. Sum \[ 1 + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \&c. \text{ to } n \text{ terms}. \]

9. Find the nth term, and the sum to n terms of the series:
   
   1.1, 2.3, 4.5, 8.7, \&c.

10. Sum \[ 1 + \frac{2}{5} + \frac{3}{5^2} + \frac{4}{5^3} + \&c. \text{ to } n \text{ terms}. \]

11. Sum to n terms 4 + 44 + 444 + \&c.

12. Sum the series \[ 9 + 99 + 999 + \&c. \text{ to } n \text{ terms}. \]

13. Sum the series \[ 1 + 3 + 7 + 15 + \&c. \text{ to } n \text{ terms}. \]

14. Sum to n terms \[ -6 - 4 + 0 + 8 + 24 + \&c. \]

15. Find the sum of \[ 6 + 9 + 21 + 69 + 261 + \&c. \text{ to } n \text{ terms}. \]

16. Find the sum of \[ (1 + (1 + 3) + (1 + 3 + 3^2) + (1 + 3 + 3^2 + 3^3) + \cdots ) \text{ to } n \text{ terms}. \]

[ O. U. 1931 ]

17. If \( a, b, c, d \) be in G. P., show that
   
   \[ (a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2. \]

   [ We have \( \frac{a}{b} = \frac{b}{c} = \frac{c}{d} = k \text{ (say)} ; \]
   
   thus, \( a = bk, b = ck, c = dk, \)
   
   hence \( a^2 + b^2 + c^2 - k^2(b^2 + c^2 + d^2), \)
   
   and also \( a^2 + b^2 + c^2 = k(ab + bc + cd). \)]
18. If \(a, \, b, \, c, \, d\) are in G. P., prove that
(i) \((b + c)(b + d) = (c + a)(c + d)\);
(ii) \((a + d)(b + c) - (a + c)(b + d) = (b - c)^2\);
(iii) \(a^2 + b^2, \, b^2 + c^2, \, c^2 + d^2\) are in G. P. [C. U. 1919]

19. Three numbers whose sum is 15 are in A. P.; if 1, 4 and 19 be added to them respectively, the results are in G. P. Determine the numbers. [Let \(a - \beta, \, a, \, a + \beta\) be the numbers.]

20. Three numbers whose product is 512 are in G. P.; if 8 be added to the first and 6 to the second, the numbers are in A. P. Find the numbers.

21. The sum of three quantities in G. P. is 24\(\frac{3}{2}\), and their product is 64; find them.

22. If \(a, \, b, \, c\) be respectively the \(p\)th, \(q\)th and \(r\)th terms of a Geometric series, prove that \(a^r b^q c^p = 1\).

23. If \(a, \, b, \, c\) be in A. P. and \(x, \, y, \, z\) in G. P., prove that \(x^a \cdot y^b \cdot z^c = 1\).

24. If \(a, \, b, \, c\) be in G. P., prove that \(\frac{1}{a + b}, \, \frac{1}{2b}, \, \frac{1}{b + c}\) are in A. P. [D. B. 1946; G. U. 1948]

25. The 1st term and \(n\)th term of a Geometric series are \(a\) and \(l\) respectively and the product of the first \(n\) terms of the series is \(P\). Prove that \(P = (al)^{\frac{n}{2}}\). [C. U. 1918; D. B. 1948]

26. If \(S\) be the sum, \(P\) the product and \(R\) the sum of the reciprocals of \(n\) terms in G. P., prove that \(P^a = \left(\frac{S}{R}\right)^n\).

27. Find the sum of \(n\) terms of the series, the \(r\)th term of which is \((2r + 1)2^r\).

28. If \(A = 1 + r^a + r^{2a} + \cdots\) to infinity and \(B = 1 + r^b + r^{2b} + \cdots\) to infinity, prove that \(r = \left(\frac{A - 1}{A}\right)^{\frac{1}{a}} = \left(\frac{B - 1}{B}\right)^{\frac{1}{b}}\).

29. If there be \(n\) terms in G. P., prove that the \(n\)th root of their product is equal to the square root of the product of the first and last terms.

30. If \(n\) Geometrical means be found between two quantities \(a\) and \(c\), show that their product will be \((ac)^{\frac{n}{2}}\).

31. If \(a, \, b, \, c, \, d\) are in G. P., show that the reciprocals of \(a^2 - b^2, \, b^2 - c^2, \, c^2 - d^2\) are also in G. P.
32. If \( S_1, S_2, S_3, \&c., S_n \) are the sums of infinite Geometric series, whose first terms are 1, 2, 3, \&c., \( n \), and whose common ratios are \( \frac{1}{n+1}, \frac{1}{3}, \frac{1}{4}, \&c. \), respectively, prove that

\[
S_1 + S_2 + S_3 + \&c. + S_n = \frac{n}{2} (n+3).
\]

33. Find the sum of the infinite series—

\[
1 + (1+a)r + (1+a+a^2)r^2 + (1+a+a^2+a^3)r^3 + \&c., \quad r \text{ and } a \text{ being proper fractions.}
\]

CHAPTER XXXIX

VARIATION

292. Definition. One quantity is said to vary directly as another when the two quantities are so related that if one of them be changed, the other is changed in the same ratio; or, in other words, if \( a, a' \) be any two values of a quantity \( A \), and \( b, b' \) the corresponding values of a second quantity \( B \), then \( A \) is said to vary directly as \( B \) when

\[
a : a' = b : b'.
\]

For instance, suppose the measure of the area of a triangle is \( a \), when that of the base is \( b \); now if the height remaining unchanged, the base is increased to \( 2b \), then as we know from Geometry the area will become \( 2a \); if the base becomes \( 3b \), the area will be \( 3a \); and so on. Thus, the height remaining the same if the base is doubled, trebled, quadrupled, \&c., the area also becomes doubled, trebled, quadrupled, \&c., (i.e., the area changes in the same ratio as the base) and so we say that if the height of a triangle remains unaltered, the area varies directly as the base.

Note 1. The word directly is often omitted, so that when we say \( A \) varies as \( B \), it is implied that \( A \) varies directly as \( B \).

Note 2. The symbol \( \propto \) is used to express variation; thus, \( A \propto B \) stands for "\( A \) varies as \( B \)."

293. If \( A \) varies as \( B \), then the numerical measure of \( \text{any} \) value of \( A \) and that of the corresponding value of \( B \) are in a constant ratio.

Let \( a_1, a_2, a_3, \&c. \), be the measures of a series of values of \( A \), and let \( b_1, b_2, b_3, \&c. \), be the measures of the corresponding values of \( B \).
Then, by definition, \( \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{a_4}{b_4} \), and so on.

Hence, \( \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{a_4}{b_4} = \&c. \), which proves the proposition.

Note. Putting \( m \) for each of the above ratios, we have \( a_1 = mb_1 \), \( a_2 = mb_2 \), \( a_3 = mb_3 \), and so on. Thus, when \( A \) varies as \( B \), the numerical measure of any value of \( A \) is equal to that of the corresponding value of \( B \) multiplied by a constant. This result is briefly expressed as follows: "If \( A \propto B \), then \( A = mB \), where \( m \) is a constant."

294. Definition. (1) One quantity \( A \) is said to vary inversely as another quantity \( B \), when \( A \) varies directly as the reciprocal of \( B \).

Thus, if \( A \) varies inversely as \( B \), \( A = \frac{m}{B} \), where \( m \) is constant.

Illustration: If 20 men do a certain work in 4 hours, 10 men would do it in 8 hours; 40 men in 2 hours; and so on. Thus, when the number of men diminishes, the time proportionally increases and vice versa. This is expressed by saying that if the amount of work to be done remains constant, the number of men varies inversely as the time.

(2) One quantity is said to vary jointly as a number of others when it varies directly as their product. Thus, if \( A \) varies jointly as \( B \) and \( C \), \( A = mBC \), where \( m \) is constant.

Illustration: The monthly income of a day labourer varies jointly as his daily earning and the number of days he works in a month.

(3) \( A \) is said to vary directly as \( B \) and inversely as \( C \) when \( A \) varies jointly as \( B \) and the reciprocal of \( C \), that is, when \( A = \frac{B}{C} \), where \( m \) is constant.

Illustration. The time of travelling a distance varies directly as the distance and inversely as the speed of travelling.

295. An Important Theorem.

If \( A \) varies as \( B \) when \( C \) is constant, \( a_1 \) \&c. \( A \) varies as \( C \) when \( B \) is constant, then will \( A \) vary as \( BC \) when both \( B \) and \( C \) vary.

Suppose \( a_1 \) is the value of \( A \) \&c. \( a_2 \) is that of \( B \), and \( c_1 \) that of \( C \). Suppose also that \( a_3 \) is the value of \( A \) when \( b_2 \) is that of \( B \) and \( c_2 \) that of \( C \). Then the proposition will be proved if we can show that \( a_1 : a_2 = b_1c_1 : b_2c_2 \).

Now, the change of \( A \) from \( a_1 \) to \( a_2 \) is due to two causes, namely,

(1) the change of \( B \) from \( b_1 \) to \( b_2 \) and (2) the change of \( C \) from \( c_1 \) to \( c_2 \).
Hence, it is clear that if one only of these causes be present (i.e., if either $B$ or $C$ alone undergoes the supposed change), $A$ will change from $a_1$ to some value which is different from $a_2$. Let, therefore, $a'$ be the value of $A$ when $b_2$ is that of $B$, and $c_1$ that of $C$.

Thus, we have the value of $A$

\[-a_1 \text{ when those of } B \text{ and } C \text{ are respectively } b_1 \text{ and } c_1 \quad \cdots (1)\]

\[-a' \text{ when those of } B \text{ and } C \text{ are respectively } b_2 \text{ and } c_1 \quad \cdots (2)\]

\[-a_2 \text{ when those of } B \text{ and } C \text{ are respectively } b_2 \text{ and } c_2 \quad \cdots (3)\]

Hence, from (1) and (2), we see that $A$ changes from $a_1$ to $a'$, when $B$ changes from $b_1$ to $b_2$, $C$ remaining constant (i.e., retaining the value $c_1$), and, therefore, by hypothesis,

\[
\frac{a_1}{a'} = \frac{b_1}{b_2}, \quad \cdots \quad (a)
\]

and from (2) and (3), we see that $A$ changes from $a'$ to $a_2$ when $C$ changes from $c_1$ to $c_2$, $B$ remaining constant (i.e., retaining the value $b_2$), and, therefore, by hypothesis,

\[
\frac{a'}{a_2} = \frac{c_1}{c_2}, \quad \cdots \quad (b)
\]

Hence, from (a) and (b),

\[
\frac{a_1}{a_2} \times \frac{a'}{a_2} = \frac{b_1}{b_2} \times \frac{c_1}{c_2} \quad \text{or} \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} \times \frac{c_1}{c_2}
\]

which proves the proposition.

Illustration: (1) Suppose that a number of plants have to be watered; the quantity of water supplied for watering evidently varies directly as the number of men employed if the time for watering remains unchanged; and also it varies directly as the number of hours for which the men can work, if the number of men engaged remain the same; hence, if the number of men and the number of hours be both variable, the quantity of water will vary as the product of the number of men and the number of hours.

(2) The area of a triangle varies directly as the base when the height is constant, and it also varies directly as the height when the base is constant; hence when both the base and the height are variable, the area varies as the product of the numbers which express the base and the height.

Cor. If there be any number of quantities $B$, $C$, $D$, &c., each of which varies as another quantity $A$, when the rest are constant; then if they are all variable, $A$ varies as their product.

296. Some results worth remembering.

(1) If $A \propto B$ and $B \propto C$, then $A \propto C$.

For, let $A = mB$, and $B = nC$, where $m$ and $n$ are constants; then $A = mnC$; and hence as $mn$ is constant, $A \propto C$. 

(2) If \( A \propto C \), and \( B \propto C \), then \( A + B \propto C \), and \( \sqrt{AB} \propto C \).

For, let \( A = mc \), and \( B = nc \), where \( m \) and \( n \) are constants; then 
\[ A + B = (m + n)c, \text{ and } A - B = (m - n)c. \quad \therefore (A \pm B) \propto C. \]

Also \( \sqrt{AB} = \sqrt{mn}c^2 = C \times \sqrt{mn}. \quad \therefore \sqrt{AB} \propto C. \)

(3) If \( A \propto BC \), then \( B \propto \frac{A}{C} \), and \( C \propto \frac{A}{B} \).

For, let \( A = mBC \), then 
\[ B = \frac{1}{m} \times \frac{A}{C}. \quad \therefore B \propto \frac{A}{C}. \]

Similarly, \( C \propto \frac{A}{B} \).

(4) If \( A \propto B \), and \( C \propto D \), then \( AC \propto BD \).

For, let \( A = mb \), and \( C = nD \), then \( AC = mnBD \); \( \therefore AC \propto BD. \)

(5) If \( A \propto B \), then \( A^n \propto B^n \).

For, let \( A = mb \), then \( A^n = m^nB^n \); \( \therefore A^n \propto B^n \).

(6) If \( A \propto B \), then \( AP \propto BP \), where \( P \) is any quantity variable or constant.

For, let \( A = mb \), then \( AP = mBP \); \( \therefore AP \propto BP \).

297. Examples. Application of the principles explained in some of the preceding articles will be illustrated by the following examples.

Example 1. If \( y \) varies as \( x \), and \( y = 5 \) when \( x = 12 \), find the value of \( y \) when \( x = 18 \).

By supposition, \( y = mx \), where \( m \) is constant.

Putting \( y = 5 \), \( x = 12 \), we have \( 5 = m \times 12 \). \( \therefore m = \frac{5}{12} \).

Hence, \( x \) and \( y \) are connected by the relation \( y = \frac{5}{12}x \).

Hence, when \( x = 18 \), we have \( y = \frac{5}{12} \times 18 = \frac{16}{2} = 7 \frac{1}{2} \).

Example 2. If \( z \) varies as \( px + y \), and if \( z = 3 \) when \( x = 1 \), \( y = 2 \), and \( z = 5 \) when \( x = 2 \) and \( y = 3 \), find \( p \).

By supposition, \( z = m(px + y) \), where \( m \) is constant.

Putting \( z = 3 \), \( x = 1 \), \( y = 2 \), we have \( 3 = m(2p + 2) \). 

Again putting \( z = 5 \), \( x = 2 \), \( y = 3 \), we have \( 5 = m(4p + 3) \). 

Hence, from (1) and (2), by division, \( \frac{3}{5} = \frac{2p + 2}{2p + 3} \); whence \( p = 1 \).

Example 3. If \( y \) is the sum of 3 quantities, of which the 1st \( = a^x \), the 2nd \( = x \), and the 3rd is constant; and when \( x = 1, 2, 3 \); \( y = 3, 11, 18 \) respectively, find the equation between \( x \) and \( y \).

By supposition, \( y = mx^3 + nx + y \), where \( m, n, p \) are constants. 
Now, since \( y = 6 \), when \( x = 1 \), we have

\[
6 = m + n + p. \quad \ldots \quad \ldots \quad (1)
\]

Similarly, \( 11 = 4m + 2n + p \), \( \ldots \quad \ldots \quad (2) \)
and \( 18 = 9m + 3n + p \), \( \ldots \quad \ldots \quad (3) \)

From (1) and (2), by subtraction, \( 3m + n = 5 \). \( \ldots \quad \ldots \quad (4) \)
Similarly, from (2) and (3), \( 5m + n = 7 \). \( \ldots \quad \ldots \quad (5) \)

Now, subtracting (4) from (5), we have

\( 2m = 2 \). \quad \therefore \quad m = 1 \)

hence, from (4), \( n = 2 \). \quad \therefore \quad \text{from (1), } p = 3.

Hence, the equation between \( x \) and \( y \) is \( y = x^2 + 2x + 3 \).

**Example 4.** If (i) \( a + b = a - b \), prove that \( a^2 + b^2 = ab \); and (ii) \( a = b \), prove that \( a^2 - b^2 = ab \).

(i) By supposition, \( a + b = m(a - b) \), where \( m \) is constant.

Hence, \( a^2 + b^2 = m^2(a - b)^2 \),
or, \( a^2 + b^2 + 2ab = m^2(a^2 + b^2 - 2ab) \).
\[
\therefore \quad (m^2 - 1)(a^2 + b^2) = 2ab(1 + m^2).
\]
\[
\therefore \quad a^2 + b^2 = \frac{2(m^2 + 1)}{m^2 - 1} \cdot ab.
\]

But \( \frac{2(m^2 + 1)}{m^2 - 1} \) is constant. \( \therefore \quad a^2 + b^2 = ab \).

(ii) Since \( a = mb \),

multiplying both sides by \( a \), we have \( a^2 = m \cdot ab \) \( \ldots \quad \ldots \quad (1) \)
and also multiplying both sides by \( \frac{b}{m} \), we have \( b^2 = \frac{ab}{m} \) \( \ldots \quad \ldots \quad (2) \)

Subtracting (2) from (1),

\[
a^2 - b^2 = \left( m - \frac{1}{m} \right) \cdot ab, \quad \text{where} \quad \left( m - \frac{1}{m} \right) \text{ is constant,}
\]
\[
\therefore \quad a^2 - b^2 = ab.
\]

**Example 5.** The wages of 5 men for 6 weeks being \( £14 \), 5s., how many weeks will 4 men work for \( £19 \) ?

Let \( x \) denote the wages (in pounds), earned by \( y \) men in \( z \) week.

Then, evidently \( x \propto y \), when \( z \) is constant ;
and also \( x \propto z \), when \( y \) is constant.
\[
\therefore \quad \text{when } y \text{ and } z \text{ are both variable,}
\]
\[
x \propto yz
\]

i.e., \( x = m \cdot yz \), when \( m \) is constant.
Now, since \( z = 14 \frac{1}{2} \), when \( y = 5 \) and \( z = 6 \).

\[ 14 \frac{1}{2} = m \times 5 \times 6. \]

**... (1)**

Also, if \( s_1 \) denote the required number of weeks, then, since the corresponding values of \( x \) and \( y \) are respectively 19 and 4, we have

\[ 19 = m \times 4 \times s_1. \]

**... (2)**

Hence, dividing (1) by (2),

\[ \frac{3}{4} = \frac{5 \times 6}{4 \times s_1}, \text{ whence } s_1 = 10; \]

**i.e.,** the required time = 10 weeks.

**Example 6.** Assuming that the quantity of work done varies as the cube root of the number of agents when the time is the same, and varies as the square root of the time when the number of agents is the same; find how long 3 men would take to do one-fifth of the work which 24 men can do in 25 hours.

Let \( x \) denote the quantity of work done by \( y \) men in \( z \) hours.

Then by supposition,

\[ x \propto y^\frac{1}{3} \text{ when } z \text{ and } z^\frac{1}{2} \text{ is constant,} \]

and also, \( x \propto z^\frac{1}{2} \text{ when } y \text{ and } y^\frac{1}{3} \text{ is constant.} \)

Hence, when both \( y \) and \( z \) and \( y^\frac{1}{3} \) and \( z^\frac{1}{2} \) are variable,

\[ x \propto y^\frac{1}{3} z^\frac{1}{2}, \]

**i.e.,** \( x = k \cdot y^\frac{1}{3} z^\frac{1}{2} \), when \( k \) is constant.

Now, since by the problem,

\[ x = 1, \text{ when } y = 24 \text{ and } z = 25 \]

**... (1)**

Also, if \( s_1 \) be the required number of hours, since the corresponding values of \( x \) and \( y \) are respectively \( \frac{1}{5} \) and 3, we have

\[ \frac{1}{5} = k \cdot \frac{\sqrt[3]{3}}{3} \cdot \sqrt[3]{s_1}. \]

**... (2)**

Hence, dividing (1) by (2),

\[ 5 = \frac{\sqrt[3]{24} \times 5}{\frac{\sqrt[3]{3}}{3} \times \sqrt[3]{s_1} = \frac{\sqrt[3]{8} \times 5}{\sqrt[3]{s_1}}.} \]

**... (1)**

\[ \sqrt[3]{s_1} = 2 \text{ and } \sqrt[3]{s_1} = 4; \]

**i.e.,** the required time = 4 hours.

**Example 7.** A sphere of metal is known to have a hollow space about its centre in the form of a concentric sphere, and its weight is \( \frac{1}{3} \) of the weight of a solid sphere of the same substance and radius; compare the inner and outer radii, having given that the weights of spheres of the same substance \( \propto (\text{radii})^3 \).
Let $R$ be the outer radius and $W$ the weight of a solid sphere of the given metal of radius $R$; also let $r$ be the inner radius (i.e., radius of the spherical cavity), and $w$ the weight of a solid sphere of the given metal of radius $r$.

Then, by hypothesis,

$$W = KR^n, \text{ and } w = Kr^n, \text{ where } K \text{ is constant.}$$

Now, since $(W - w)$ is the weight of the given sphere, we have, by the question, $W - w = \frac{2}{3}W$, hence, we must have

$$K(R^n - r^n) = \frac{2}{3}KR^n.$$

$$\frac{2}{3}R^n = r^n, \text{ whence } \frac{r}{R} = \frac{1}{2}.$$

**Example 8.** A point moves with a speed which is different in different kilometres, but invariable in the same kilometre, and its speed in any kilometre varies inversely as the number of kilometres travelled before it commences this kilometre. If the second kilometre be described in 2 hours, find the time occupied in describing the $n$th kilometre.

Evidently, the time of describing any kilometre varies inversely as the speed in that kilometre; hence, if $v_n$ denote the speed in $n$th kilometre and $t_n$ the number of hours required to describe the $n$th kilometre, we must have

$$t_n = \frac{m}{v_n}, \text{ where } m \text{ is constant.}$$

Also, by hypothesis, $v_n = \frac{K}{n-1}$, where $K$ is constant;

hence, $t_n = \frac{m}{K(\frac{1}{n-1})}$.

Evidently, then $t_n$ is known if $\frac{m}{K}$ is known; and since the time of describing the 2nd kilometre is two hours (i.e., $t_n = 2$, when $n = 2$), we have

$$2 = \frac{m}{K} \cdot 1.$$

$$\therefore \quad \frac{m}{K} = m.$$

Hence, $t_n = 2(n-1)$,

i.e., the $n$th kilometre is described in $2(n-1)$ hours.

**Example 9.** A locomotive engine without a train can go 24 kilometres an hour, and its speed is diminished by a quantity which varies as the square root of the number of waggons attached. With four waggons its speed is 20 kilometres an hour. Find the greatest number of waggons with which the engine can move.
Let \( x \) be the number of waggon attached.

Then the number of kilometres travelled by the train per hour \( i.e., \) its speed \( = 24 - m\sqrt{x} \), where \( m \) is a constant.

Now, since the speed is 20 kilometres per hour when \( x = 4 \), we must have

\[
20 = 24 - m\sqrt{4} = 24 - 2m.
\]

\[
\therefore m = 2
\]

Hence, the speed of the engine with \( x \) waggon \( = 24 - 2\sqrt{x} \); evidently, therefore, the speed diminishes as \( x \) increases.

Now, let us see for what value of \( x \) the speed is reduced to nothing. If \( x_1 \) be this value, we must have

\[
0 = 24 - 2\sqrt{x_1}.
\]

\[
\therefore \sqrt{x_1} = 12, \text{ and } x_1 = 144.
\]

Thus, when 144 waggon are attached, the engine just fails to move the train.

Hence, the greatest number of waggon with which the engine can move = 143.

Example 10. If \( x, y, z \) be variable quantities such that \( y + z - x \) is constant, and that \( (x + y - z)(x + z - y) \) varies as \( yz \), prove that \( x + y + z \) varies as \( yz \).

By supposition, we have \( y + z - x = k \quad (1) \)

and \( (x + y - z)(x + z - y) = myz \quad (2) \)

where \( k \) and \( m \) are constants.

Now, from (2), we have \( x^2 - (y - z)^2 = myz. \)

\[
\therefore x^2 - (y + z)^2 = (m - 4)yz,
\]

or,

\[
(x + y + z)(x - y - z) = (m - 4)yz.
\]

Hence, from (1),

\[
(x + y + z)(-k) = (m - 4)yz.
\]

\[
\therefore x + y + z = \left(\frac{4 - m}{k}\right)yz, \text{ i.e., } = (a \text{ constant}) \times yz.
\]

Hence, \( x + y + z \propto yz \).

EXERCISE 149

1. If \( y \propto x, \) and \( y = 5 \) when \( x = 15 \), find the equation between \( x \) and \( y. \)

2. If \( y \propto x, \) and \( y = 10 \) when \( x = 25 \), find \( y \) when \( x = 35. \)
3. If $P$ varies inversely as $Q$, and $Q=10$ when $P=2$, what will $P$ become when $Q=8$?

4. If $P \propto QR$, and the three corresponding values of $P$, $Q$, $R$ be 6, 9, 10 respectively, find the value of $P$ when $Q=3$ and $R=3$.

5. If the square of $z$ vary as the cube of $y$, and $z=2$, when $y=3$, find the equation between $x$ and $y$.

6. Given that $y$ varies as the sum of two quantities, one of which varies as $x$ directly, the other as $x$ inversely and that $y=4$, when $x=1$, and $y=5$ when $x=2$, find the equation between $x$ and $y$.

7. If $xy \propto x^2 + y^2$, and $y=4$ when $x=3$, find the equation between $x$ and $y$.

8. Given that $y$ is equal to the sum of two quantities, one of which varies as $x$, and the other varies inversely as $x^2$, and when $x=1, 2, y=6, 5$ respectively, find the equation between $x$ and $y$.

9. If the sum of 3 quantities of which the 1st is constant, the 2nd $\propto x$, and the 3rd $\propto x^2$, also when $x=3, 5, 7, y=0, -12, -32$ respectively, find the equation between $x$ and $y$.

10. Given that $y^2 \propto a^2 - x^2$ and when $x=\sqrt{a^2 - b^2}$, $y=\frac{b^2}{a}$, find the equation between $x$ and $y$.

11. If $y=r+s$, whilst $r \propto x$, and $s \propto \frac{1}{\sqrt{x}}$; and if, when $x=4$, $y=5$, and when $x=9$, $y=10$, show that $6y=5(x + \sqrt{x})$.

12. Assuming that the time of oscillation of a pendulum varies as the square root of its length; if the length of a pendulum which oscillates once in a second be 39'2 inches, find the length of one which oscillates 56 times in a minute.

13. If 13 men earn £7 in 15 days of 8 hours each, what will be the wages of 52 men for 12$\frac{1}{2}$ days of 9 hours each?

14. Given that the volume of a sphere varies as the cube of its radius, prove that the volume of a sphere whose radius is 5 centimetres is equal to the sum of the volumes of three spheres whose radii are 3, 4, 5 centimetres.

15. The volume of a pyramid varies jointly as its height and the area of its base; and when the area of the base is 60 square metres and the height 14 metres, the volume is 280 cubic metres. What is the area of the base of a pyramid whose volume is 390 cubic metres and whose height is 26 metres?

16. Given that the area of a circle varies as the square of its radius, and that the area of a circle is 154 square centimetres, when the radius is 7 centimetres; find the area of a circle whose radius is 10'5 centimetres.
17. If the volume of a cone whose height is 12 centimetres and base 30 square centimetres be 120 cubic centimetres, find the volume of another whose height is 20 centimetres and base 144 square centimetres; the volume of a cone varying as the height and base jointly.

18. The volume of a circular cylinder varies as the square of the radius of the base when the height is the same and as the height when the base is the same. The volume is 88 cubic metres when the height is 7 metres, and the radius of the base is 2 metres; what will be the height of a cylinder on a base of a radius 9 metres, when the volume is 396 cubic metres?

19. Two circular gold plates, each one centimetre thick, the diameters of which are 6 centimetres and 8 centimetres respectively, are melted and formed into a single circular plate one centimetre thick. Find its diameter, having given that the area of a circle varies as the square of its diameter.

20. Given that the illumination from a source of light varies inversely as the square of the distance, how much farther from a candle must a book, which is now three inches off, be removed, so as to receive just half as much light?

21. A solid spherical mass of glass, 1 decimetre in diameter, is blown into a shell bounded by two concentric spheres, the diameter of the outer one being 3 decimetres. Calculate the thickness of the shell. (The volume of a sphere varies directly as the cube of its diameter.)

22. When a body falls from rest, its distance from the starting point varies as the square of the time it has been falling; if a body falls through 402½ feet in 5 seconds, how far does it fall in 10 seconds? Also how far does it fall in the 10th second?

23. If 10 men can reap a field of 7½ hectares, in 3 days of 12 hours each, how long will it take 8 men to reap 9 hectares, working 16 hours a day?

24. The square of the time of a planet’s revolution varies as the cube of its distance from the Sun; find the time of Venus’s revolution assuming the distance of the Earth and Venus from the Sun to be 91½ and 16 millions of miles respectively.

[If $P$ be the time of revolution measured in days, and $D$ the distance in millions of miles, we have $P^2 = KD^3$, where $K$ is a constant, &c.]

25. The value of a silver coin varies directly as the square of its diameter while its thickness remains the same and directly as its thickness while its diameter remains the same. The silver coins have their diameters in the ratio of 4 : 3; find the ratio of their thickness if the value of the first be four times the value of the second.

[B. U. P. E. 1885]
26. The value of diamonds \( \propto \) the square of their weights, and the square of the value of rubies \( \propto \) the cube of their weights. A diamond of \( a \) carats is worth \( m \) times the value of a ruby of \( b \) carats, and both together are worth \( \mathcal{L}c \). Required the value of a diamond and of a ruby, each weighing \( n \) carats.

27. If \( a \propto b \) and \( b \propto c \), show that \( (a^2 + b^2)^\frac{3}{2} \propto c^2 \).

28. If \( x+y \propto x-y \), show that \( x^2 + y^2 \propto xy \) and \( x^2 + y^2 \propto xy(x \pm y) \).

29. Given that \( x+y \propto z + \frac{1}{z} \), and that \( x-y \propto z - \frac{1}{z} \), find the relation between \( x \) and \( z \), provided that \( z=2 \), when \( x=3 \), and \( y=1 \).

[ B. U. P. E. 1888 ]

30. If \( x \propto \frac{1}{y} \), prove that \( x+y \) is least when \( x=y \).

[ We have \( xy = \text{a constant} \).]

31. The consumption of coal by a locomotive varies as the square of the velocity; when the speed is 16 miles an hour the consumption of coal per hour is 2 tons; if the price of coal be 10s. per ton and the other expenses of the engine be 11s. 3d. an hour, find the least cost of a journey of 100 miles.

[ Apply the preceding example.]

32. If \( z \propto y \), and \( y \propto x \), show that

\[
x + y + z \propto (yz)^\frac{3}{2} + (zx)^\frac{3}{2} + (xy)^\frac{3}{2}
\]