CHAPTER XV
LOWEST COMMON MULTIPLE
(By factorisation)

1st. Definitions. One expression is said to be a multiple of another when the former is exactly divisible by the latter.

One expression is said to be a common multiple of two or more others when it is exactly divisible by each of these latter.

Of the different common multiples of two or more expressions that which consists of the least number of elementary factors is called the Lowest Common Multiple of those expressions. In other words, a common multiple of two or more expressions is said to be their Lowest Common Multiple when it is the product of just as many elementary factors as it must necessarily have and no more.

Thus, the common multiples of $a$ and $b$ are $ab$, $2ab$, $a^2b$, $ab^2$, $a^2b^2$, &c.; but of these $ab$ consists of the least number of elementary factors, and hence, it is called the lowest common multiple of the quantities $a$ and $b$.

Cor. Hence, every common multiple of two or more expressions is divisible by their Lowest Common Multiple.

Note. The letters L.C.M. are usually written for 'Lowest Common Multiple'.

104. L.C.M. of simple expressions or such compound expressions as can be easily resolved into their elementary factors.

In such cases the L.C.M. can be written down by inspection. The following examples will illustrate the process:

Example 1. Find the L.C.M. of $4a^2bc$ and $6ab^2d$.

The 1st expression $= 2^2 \times a^2 \times b \times c$.
The 2nd expression $= 2 \times 3 \times a \times b^2 \times d$.

Hence, $2^2 \times 3 \times a^2 \times b^2 \times c \times d$ must necessarily be a factor of every common multiple of them.

Hence, the required L.C.M.

$= 2^2 \times 3 \times a^2 \times b^2 \times c \times d$

$= 12a^2b^2cd$.

Example 2. Find the L.C.M. of $24x^2yz$, $18xy^2z^2$ and $27x^4y^2z^3$.

The 1st expression $= 2^3 \times 3 \times x^2 \times y \times z$.
The 2nd expression $= 2 \times 3^2 \times x \times y^2 \times z^2$.
The 3rd expression $= 3^3 \times x^4 \times y^2 \times z^3$. 
Hence, \( 2^a \times 3^a \times x^a \times y^a \times z^a \) must necessarily be a factor of every common multiple of them.

Hence, the required L.C.M.
\[
= 2^a \times 3^a \times x^a \times y^a \times z^a = 2^a 3^a y^a z^a.
\]

**Example 3.** Find the L.C.M. of

\[4a^a(x^a + a^2),~ 6a^a(x^a - a^2)\text{ and } 9a^a(x^a - a^3)\]

The 1st expression = \(2^a \times x^a \times (x + a)^3\).

The 2nd expression = \(2^a \times 3^a \times x^a \times (x + a)(x - a)\).

The 3rd expression = \(3^a \times x^a \times (x - a)(x^a + ax + a^2)\).

Hence, \(2^a \times 3^a \times x^a \times (x + a)^3(x - a)(x^a + ax + a^2)\) must necessarily be a factor of every common multiple of them.

Hence, the required L.C.M.
\[
= 2^a \times 3^a \times a^3 \times x^a \times (x + a)^3(x - a)(x^a + ax + a^2)
= 36a^a x^a (x + a)^3(x - a^a).
\]

**Note.** The L.C.M. of two or more algebraic expressions having monomial and compound expressions as factors is the product of the L.C.M. of the monomial expressions and that of the compound ones.

In the preceding example, L.C.M. of \(4x^a\), \(6a^a x\) and \(9x^a\) is \(36a^a x^a\) and that of \((x + a)^3\), \((x^a - a^3)\) and \((x^a - a^3)\) is \((x + a)^3(x - a^a)\). Therefore, the required L.C.M. is \((36a^a x^a) \times (x + a)^3(x - a^a) = 36a^a x^a (x + a)^3(x - a^a)\).

**Example 4.** Find the L.C.M. of

\[x^a - 3x^2 + 2,~ x^a + 2x^2 - 3x\text{ and } x^4 + x^a - 6x^a\]

The 1st expression = \((x - 1)(x - 2)\).

The 2nd expression = \(x^a(x^a + 2x - 3) = x(x - 1)(x + 3)\).

The 3rd expression = \(x^a(x^a + x - 6) = x^a(x - 2)(x + 3)\).

Hence, \(x^a(x - 1)(x - 2)(x + 3)\) must necessarily be a factor of every common multiple of the given expressions.

Hence, the required L.C.M. = \(x^a(x - 1)(x - 2)(x + 3)\).

**Example 5.** Find the L.C.M. of \(x^a - 3x^a + 3x^a - 1\), \(x^a - x^a - x^a + 1\) and \(x^a - 3x^a - 2x^2 + 3x - 1\).

\[x^a - 3x^a + 3x^a - 1 = (x - 1)^a.
\]

\[x^a - x^a - x^a + 1 = x^a(x - 1) - (x - 1)
\]

\[= (x - 1)(x^a - 1) = (x - 1)^a(x + 1).
\]

\[x^a - 2x^2 + 2x - 1 = (x^a - 1) - 2x(x^a - 1)
\]

\[= (x^a - 1)(x^a + 1) - 2x = (x^a - 1)(x - 1)^2
\]

\[= (x - 1)^a(x + 1).
\]
Hence, \((x-1)^n(x+1)\) must necessarily be a factor of every common multiple of the given expressions.

Hence, the required L.C.M. = \((x-1)^n(x+1)\).

**EXERCISE 53**

Find the L.C.M. of:

1. \(a^2b\) and \(ab^3\).
2. \(a^3b^2\) and \(a^3bc\).
3. \(6x^2y^4\) and \(10xy^n\).
4. \(4m^2n^3\) and \(14m^n3^n\).
5. \(8x^3y^5z\) and \(12x^3y^5z^2\).
6. \(4a^2bc, 10ab^2c\), and \(14abc^2\).
7. \(8a^2b^2c, 12ab^3c^2\), and \(20a^2bc^3\).
8. \(6x^2y, 9x^2y^2z, 12a^2xy^5\), and \(15axz^2\).
9. \(a^3b - ab^2\) and \(a^3b^2 + a^2b^3\).
10. \(4(x-y)^2, 6(x^2 - y^2)\), and \(8(x+y)^2\).
11. \(x^2 - 4x + 3\) and \(x^2 - 5x + 6\).
12. \(a^2 + 2a^2x - 3ax^2\) and \(a^2 + a^2x - 6a^2x^2\).
13. \(a^3(a^2 - 4)\) and \(a^4 + 2a^3 - 8a^2\).
14. \(4a^2x^2, 2a(a^2 - a^2)\), and \(6a^2x(a^3 + a^3)\).
15. \(12(x^3 + 3x - 10)\) and \(16(x^3 + 4x - 12)\).
16. \(x^2 + 2x - 15, x^2 + 9x + 20\), and \(x^2 + 4x - 21\).
17. \(12a^4 - 27a^3b^2, 2a^2 + ab - 3b^3\), and \(2a^2 - ab - 3b^3\).
18. \(8a^3 + 27b^3, 8a^3 - 27b^3\), and \(16a^4 + 36a^3b^2 + 81b^6\).
19. \(8x^4 - 50x^3y^2, 12x^3 + 24x^2y - 15xy^3\), and \(16x^3 - 48xy + 20y^3\).
20. \(4x^3 - 12ax + 9a^2\), \(6x^3 - 7ax - 3a^2\), and \(6x^3 - 11ax + 3a^2\).
21. \(2x^3 + 6x + 9, 4x^3 - 12x^2 + 18x\), and \(4x^3 + 81\).
22. \(9a^2 - 6ax + x^2, 6a^2 + 10ax - 4x^2\), and \(9a^2 - 21ax + 6x^2\).
23. \(8x^n - 12x^3 + 6x - 1, 8x^n - 4x^2 - 2x + 1\), and \(2x^2 + 5x - 3\).
24. \(x^n - 6xy + 9y^2, x^2 - 7xy + 12y^2, x^2 + 2xy - 15y^2\), and \(x^2 + xy - 20y^3\).
25. \(6x^3 - x - 1, 3x^2 + 7x + 2\), and \(2x^3 + 3x - 3\).  \[\text{C. U. 1889}\]
26. \(1 + 4x + 4x^2 - 16x^4\) and \(1 + 2x - 8x^2 - 16x^4\).  \[\text{C. U. 1871}\]
27. \(9x^4 - 28x^3 + 3, 27x^4 - 12x^3 + 1, 27x^4 + 6x^3 - 1\), and \(x^4 - 6x^3 + 9\).  \[\text{C. U. 1886}\]
CHAPTER XVI
EASY FRACTIONS

105. Definition. The algebraical fraction \( \frac{a}{b} \), where \( a \) and \( b \) may have any numerical values, is defined to be a quantity which, when multiplied by \( b \), becomes equal to \( a \). In other words, \( \frac{a}{b} \) is defined to be equivalent to \( a + b \). In \( \frac{a}{b} \), \( a \) is called the numerator and \( b \) the denominator.

Note. Thus an algebraical fraction is no other than the quotient of one expression by another, expressed by placing the dividend over the divisor with a horizontal line between them; and the dividend and the divisor so placed are respectively called the numerator and the denominator of the fraction.

106. The value of a fraction is not altered if both its numerator and denominator are multiplied or divided by any the same quantity.

If \( a, b \) and \( m \) stand for any quantities whatever, to prove that

\[
\frac{a}{b} = \frac{am}{bm}
\]

Let \( x = \frac{a}{b} \),

then \( x \times b = \frac{a}{b} \times b = a \) [by definition];

\[
\therefore \ x \times b \times m = a \times m, \quad \text{or,} \quad x \times bm = am.
\]

Hence, \( x = am + bm \), i.e., \( \frac{a}{b} = \frac{am}{bm} \).

Conversely, we have \( \frac{am}{bm} = \frac{a}{b} \); i.e., \( \frac{am}{bm} = \frac{am + m}{bm + m} \).

Thus, the proposition is established.

Cor. \( \frac{a}{b} = \frac{a \times (-1)}{b \times (-1)} = \frac{-a}{-b} \). Thus, the value of a fraction is not altered if the signs of both the numerator and the denominator be changed.

107. Reduction of a fraction to its lowest terms. A fraction is said to be in its lowest terms, when its numerator and denominator have no common factor.
Hence, to reduce a fraction to its lowest terms, or more briefly to simplify it, is no other than to find an equivalent fraction whose numerator and denominator have no common factor, and this is evidently done by dividing the numerator and the denominator of the fraction by their highest common factor.

Note. In all cases where the numerator and the denominator can be factorised by inspection, the reduction is at once effected by simply removing the common factors.

Example 1. Reduce \( \frac{4a^2b^3c^3}{10ab^4c^5} \) to its lowest terms.

\[
\frac{4a^2b^3c^3}{10ab^4c^5} = \frac{2 \times 2 \times a^2 \times b^3 \times c^3}{2 \times 5 \times a \times b^4 \times c^2} = \frac{2a}{5b}.
\]

Example 2. Simplify \( \frac{a^2b^3(a^2 - b^2)}{3ab^4(a^2 + b^2)} \).

\[
\frac{a^2b^3(a^2 - b^2)}{3ab^4(a^2 + b^2)} = \frac{a^2b^3(a + b)(a - b)}{3ab^4(a^2 - ab + b^2)} = \frac{a(a - b)}{b(a^2 - ab + b^2)}.
\]

Example 3. Reduce \( \frac{x^2 + 3x - 40}{x^2 + 4x - 32} \) to its lowest terms.

The numerator \( = (x + 8)(x - 5) \).
The denominator \( = (x + 8)(x - 4) \).

Hence, the given fraction \( = \frac{(x + 8)(x - 5)}{(x + 8)(x - 4)} = \frac{x - 5}{x - 4} \).

Example 4. Simplify \( \frac{2a^3 + 3ax - 2ab - 3bx}{3a^3 - 2ax - 3ab + 2bx} \).

The numerator \( = 2a(a - b) + 3x(a - b) = (a - b)(2a + 3x) \).
The denominator \( = 3a(a - b) - 2x(a - b) \)

\( = (a - b)(3a - 2x) \).

Hence, the given expression \( = \frac{(a - b)(2a + 3x)}{(a - b)(3a - 2x)} = \frac{2a + 3x}{3a - 2x} \).

**EXERCISE 54**

Reduce to lowest terms:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>1.</td>
<td>( \frac{2a^2b^3}{4a^2b^4} )</td>
<td>2.</td>
<td>( \frac{6ax^4}{8xy^4} )</td>
<td>3.</td>
</tr>
<tr>
<td>4.</td>
<td>( \frac{15x^2y^3z}{25y^4z^3} )</td>
<td>5.</td>
<td>( \frac{16a^3bc^3d^2}{27a^3bc^4d^4} )</td>
<td>6.</td>
</tr>
<tr>
<td>7.</td>
<td>( \frac{70a^5b^5c^6d^7}{105b^6c^7a^3b^3} )</td>
<td>8.</td>
<td>( \frac{39m^3n^2p^2q^3}{66p^3m^3n^4q^5} )</td>
<td>9.</td>
</tr>
<tr>
<td>10.</td>
<td>( \frac{x^2 - 5x}{9x - x^3} )</td>
<td>11.</td>
<td>( \frac{4x^3 - 9a^3}{4x^3 + 6ax} )</td>
<td>12.</td>
</tr>
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</table>
13. \( \frac{3ax - 12a^2}{x^2 - 16a^2} \)  \( x^2 + 2x - 8 \)  \( x^2 + x - 12 \)

14. \( \frac{2x^4 - 4a^3x^3}{x^2 - 4a^3x^2 + 4a^4} \)  \( \frac{a^4 - a^2b + ab^2}{a^2 + b^2} \)  \( \frac{x^2 - 6xy + 5y^2}{x^2 + 2xy - 35y^2} \)

17. \( \frac{a^3 + 2x - 15}{x^2 + 9x + 20} \)  \( \frac{a^2 - 3ab - 4b^2}{a^2 - 4ab - 5b^2} \)

18. \( \frac{4x^2 + 8x}{x^2 + 5x + 6} \)

19. \( \frac{2x^2 - x - 6}{3x^2 - 2x - 8} \)  \( \frac{3x^3 + 16ax + 5a^2}{3x^3 + 22ax + 7a^2} \)  \( \frac{2x^3 + 2ax - 20a^2}{3x^3 + 5ax - 28a^2} \)

21. \( \frac{x^3 - 8x^2 - 65}{x^2 + x - 20} \)  \( \frac{6x^2 - 7x - 20}{9x^2 + 6x - 8} \)

22. \( \frac{1 - 7x + 12x^2}{1 - 8x + 15x^2} \)  \( \frac{1 - 9a^2 + 14a^4}{1 - 4a^2 - 21a^4} \)

24. \( \frac{3a^3x + 9a^2x^2 + 27ax^3}{a^3 - 27x^3} \)  \( \frac{3x^3 - 5ax + 2a^2}{3x^3 + ax - 2a^2} \)

26. \( \frac{6x^3 + 7x - 20}{9x^3 + 6x - 8} \)

28. \( \frac{10 - 17ax + 3a^2x^2}{5 - 26ax + 5a^2x^2} \)  \( \frac{6ac + 10bc + 9ac + 15bx}{6c^2 + 9cx - 2c - 3x} \)

31. \( \frac{8bx + 12ab + 6xy + 9ay}{12bx + 8ab + 9xy + 6ay} \)

32. \( \frac{2a^3 + ab - b^2}{a^3 + a^2b - a - b} \)

33. \( \frac{a^2 - b^a - 2bc - c^2}{a^2 + 2ab + b^2 - c^2} \)

108. Reduction of two or more fractions to a common denominator.

Let \( \frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \) &c., stand for any number of fractions.

Let \( L \) denote the L.C.M. of the denominators, i.e., of \( b, d, f, \) &c. Then, since the value of a fraction is not altered when its numerator and denominator are both multiplied by the same quantity, we must have

\[
\frac{a}{b} = \frac{a \times (L+b)}{b 	imes (L+b)} = \frac{a \times (L+b)}{L} ;
\]

\[
\frac{c}{d} = \frac{c \times (L+d)}{d \times (L+d)} = \frac{c \times (L+d)}{L} ;
\]

\[
\frac{e}{f} = \frac{e \times (L+f)}{f \times (L+f)} = \frac{e \times (L+f)}{L} ;
\]

and so on.

Thus, the fractions in the third column are respectively equivalent to the given fractions and they have all got the same denominator, namely, \( L \).
Hence, we have the following rule for reducing fractions to a common denominator: Find the L.C.M. of the denominators, and multiply the numerator and the denominator of each fraction by the quotient obtained by dividing the L.C.M., thus found, by the denominator of that fraction.

Example 1. Reduce \( \frac{x}{a+b} \), \( \frac{x^2}{a(a-b)} \) and \( \frac{x^3}{b(a^2-b^2)} \) to a common denominator.

The L.C.M. of the denominators = \( ab(a^2-b^2) \); and the quotients obtained by dividing it by the denominators are respectively \( ab(a-b) \), \( b(a+b) \) and \( a \).

Hence, we have \( \frac{x}{a+b} = \frac{x \times ab(a-b)}{(a+b) \times ab(a-b)} = \frac{xab(a-b)}{ab(a^2-b^2)} \);
\( \frac{x^2}{a(a-b)} = \frac{x^2 \times b(a+b)}{a(a-b) \times b(a+b)} = \frac{x^2b(a+b)}{ab(a^2-b^2)} \);
\( \frac{x^3}{b(a^2-b^2)} = \frac{x^3 \times a}{b(a^2-b^2) \times a} = \frac{x^3a}{ab(a^2-b^2)} \).

Example 2. Reduce \( \frac{x-1}{x^2-5x+6} \), \( \frac{x-2}{x^2-4x+3} \) and \( \frac{x-3}{x^2-3x+2} \) to a common denominator.

The denominators are respectively \( (x-2)(x-3) \), \( (x-1)(x-3) \) and \( (x-1)(x-2) \).

Hence, their L.C.M. = \( (x-1)(x-2)(x-3) \), and the quotients obtained by dividing it by the denominators are respectively \( x-1 \), \( x-2 \) and \( x-3 \).

Hence, we have
\( \frac{x-1}{x^2-5x+6} = \frac{(x-1)(x-1)}{(x^2-5x+6)(x-1)} = \frac{x^2-2x+1}{x^2-6x+9+11x-6} \);
\( \frac{x-2}{x^2-4x+3} = \frac{(x-2)(x-2)}{(x^2-4x+3)(x-2)} = \frac{x^2-4x+4}{x^2-6x+9+11x-6} \);
\( \frac{x-3}{x^2-3x+2} = \frac{(x-3)(x-3)}{(x^2-3x+2)(x-3)} = \frac{x^2-6x+9}{x^2-6x+9+11x-6} \).

EXERCISE 55

Reduce to a common denominator:

1. \( \frac{a}{2b}, \frac{3c}{4d}, \frac{e}{f} \)
2. \( \frac{x^2}{2bc}, \frac{y^2}{3ca}, \frac{z^2}{4ab} \)
3. \( \frac{ab}{4xy^2}, \frac{bc}{6x^2y}, \frac{ca}{10x^2} \)
4. \( \frac{a}{a-b}, \frac{b}{a+b}, \frac{c}{a(a+b)} \)
5. \( \frac{x^2}{a^2 + 2ab} \cdot \frac{y^2}{a - 2b} \)
6. \( \frac{2a}{a - b} \cdot \frac{a - c}{b - a} \cdot \frac{a - c}{a^2 - b^2} \)
7. \( \frac{2a}{a - b} \cdot \frac{3b}{b - a} \cdot \frac{4c}{a + b} \)
8. \( \frac{2x}{a^2(a + x)} \cdot \frac{3y}{b^2(a - x)} \cdot \frac{4z}{c^2(a^2 - x^2)} \)
9. \( \frac{a^2 b}{2xy - 3y^2} \cdot \frac{b^2}{2x^2 + 3xy} \cdot \frac{c^2}{4x^2 y - 9xy^2} \)
10. \( \frac{a^2}{x^2 + x + 1} \cdot \frac{b^2}{x^2 - x + 1} \)
11. \( \frac{a - 2b}{x^2 - x - 2} \cdot \frac{b^2}{x^2 + x - 6} \)
12. \( \frac{a}{a(a^2 - 2ab + 4b^2)} \cdot \frac{b^2}{a^2 + 3b^2} \)
13. \( \frac{a}{a - 3b} \cdot \frac{b}{a^2 + 3ab + 9b^2} \cdot \frac{c}{a^2 - 27b^2} \)
14. \( \frac{a}{b(x - b - c)} \cdot \frac{b}{a(a - 5 + c)} \cdot \frac{c}{a^2 + c^2 - 2ab} \)
15. \( \frac{c - a}{(a - b)(b - c)} \cdot \frac{b - a}{(a - c)(b - c)} \cdot \frac{b - c}{(c - a)(a - b)} \)


From Cor. 3, Art. 47, we know that
\[ a(b + c + d + e) = ab + ac + ad + ae, \]
where \( a, b, c, d, e \) are any quantities whatever.

Hence, conversely,
\[ \frac{ab + ac + ad + ae}{a} = b + c + d + e = \frac{ab}{a} + \frac{ac}{a} + \frac{ad}{a} + \frac{ae}{a} \]

Hence, putting \( p, q, r, s \) respectively for \( ab, ac, ad, ae \), we have
\[ \frac{p + q + r + s}{a} = \frac{p}{a} + \frac{q}{a} + \frac{r}{a} + \frac{s}{a}, \]
where \( p, q, r, s \) and \( a \) are any quantities whatever.

Thus, the sum of any number of fractions which have a common denominator is a fraction whose denominator is the same and whose numerator is the sum of the numerators of the given fractions.

Hence, to obtain the sum of any number of fractions which have not the same denominator, we must first reduce them to equivalent fractions having a common denominator and then proceed as above.

Note. To subtract a fraction from another fraction we are to follow the above rule; but the numerator of the fraction to be subtracted is to be taken with minus sign. Thus,
\[ \frac{a}{b} - \frac{x}{b} = \frac{a - x}{b} = \frac{a + (-x)}{b} = \frac{a - x}{b} \]
Example 1. Find the value of \( \frac{a}{a-b} + \frac{b}{b-a} \).

Since, \( \frac{b}{b-a} = \frac{b \times (-1)}{(b-a) \times (-1)} = \frac{-b}{a-b} \),

we have \( \frac{a}{a-b} + \frac{b}{b-a} = \frac{a}{a-b} + \frac{-b}{a-b} = \frac{a + (-b)}{a-b} = \frac{a-b}{a-b} = 1 \).

Example 2. Find the value of \( \frac{x}{x+a} + \frac{a}{x-a} \).

Since the L.C.M. of the denominators = \( x^2 - a^2 \),

we have \( \frac{x}{x+a} = \frac{x(x-a)}{x^2 - a^2} \) and \( \frac{a}{x-a} = \frac{a(x+a)}{x^2 - a^2} \).

Hence, the required value \( = \frac{x(x-a) + a(x+a)}{x^2 - a^2} = \frac{x^2 + a^2}{x^2 - a^2} \).

Example 3. Find the value of \( \frac{a+b}{a^2-b^2} + \frac{a-b}{a^2-b^2} - \frac{a^2+b^2}{a^2-b^2} \).

In the first and second terms the numerator and the denominator have a common factor. So they are to be reduced to their lowest terms. This is not essential. But this will make the operation easy.

\( \frac{a+b}{a^2-b^2} = \frac{1}{a-b} ; \quad \frac{a-b}{a^2-b^2} = \frac{1}{a+b+b^2} \).

\( \therefore \) the given expression \( = \frac{1}{a-b} + \frac{1}{a^2+ab+b^2} - \frac{a^2+b^2}{a^2-b^2} \).

\( = (a^2+ab+b^2) + (a-b) - (a^2+b^2) = ab + a - b \).

Example 4. Find the value of \( \frac{1}{a+b} + \frac{b}{a^2-b^2} - \frac{a}{a^2+b^2} \).

In the present example it is not convenient to reduce all the fractions to a common denominator at once. We can proceed best as follows:

We have \( \frac{1}{a+b} + \frac{b}{a^2-b^2} = \frac{(a-b)+b}{a^2-b^2} = \frac{a}{a^2-b^2} \).
Hence, the required value is \[ \frac{a}{a^2 - b^2} - \frac{a}{a^2 + b^2} = \frac{a(a^2 + b^2) - a(a^2 - b^2)}{a^2 - b^2} = \frac{2ab^2}{a^2 - b^2}. \]

**Example 5.** Simplify \( \frac{1}{x - 2} - \frac{1}{x + 2} - \frac{4}{x^2 + 4} + \frac{32}{x^4 + 16} \).

We have
\[
\frac{1}{x - 2} - \frac{1}{x + 2} = \frac{(x + 2) - (x - 2)}{x^2 - 4} = \frac{4}{x^2 - 4};
\]
\[
\frac{4}{x^2 - 4} - \frac{4}{x^2 + 4} = \frac{4(x^2 + 4) - 4(x^2 - 4)}{x^4 - 16} = \frac{32}{x^4 - 16};
\]
\[
\frac{32}{x^4 - 16} + \frac{32}{x^4 + 16} = \frac{32(x^4 + 16) + 32(x^4 - 16)}{x^8 - 256} = \frac{64x^4}{x^8 - 256},
\]
which is the required result.

**Example 6.** Simplify \( \frac{1}{a + b} - \frac{1}{a + 2b} - \frac{1}{a + 3b} + \frac{1}{a + 4b} \).

The given expression is \( \left\{ \frac{1}{a + b} - \frac{1}{a + 2b} \right\} - \left\{ \frac{1}{a + 3b} - \frac{1}{a + 4b} \right\} \).

Now, we have
\[
\frac{1}{a + b} - \frac{1}{a + 2b} = \frac{(a + 2b) - (a + b)}{(a + b)(a + 2b)} = \frac{b}{(a + b)(a + 2b)};
\]
and
\[
\frac{1}{a + 3b} - \frac{1}{a + 4b} = \frac{(a + 4b) - (a + 3b)}{(a + 3b)(a + 4b)} = \frac{b}{(a + 3b)(a + 4b)}.
\]

Lastly,
\[
\frac{b}{(a + b)(a + 2b)} - \frac{b}{(a + 3b)(a + 4b)} = \frac{b(a + 3b)(a + 4b) - b(a + b)(a + 2b)}{(a + b)(a + 2b)(a + 3b)(a + 4b)};
\]
of which the numerator is
\[
b(a^2 + 7ab + 12b^2) - b(a^2 + 3ab + 2b^2) = b(4ab + 10b^2) = 2b^2(2a + 5b).
\]

Hence, the reqd. result is
\[
\frac{2b^2(2a + 5b)}{(a + b)(a + 2b)(a + 3b)(a + 4b)}.
\]

**EXERCISE 56**

Find the value of:

1. \( \frac{a + b}{a} + \frac{a - b}{b} \)
2. \( \frac{x - y + y - z + z - x}{xy} + \frac{yz}{yz} + \frac{zx}{zx} \)
3. \( \frac{a}{a - x} + \frac{x}{x - a} \)
4. \( \frac{a + b}{a - b} - \frac{a - b}{a + b} \)
5. \( \frac{a^2 + b^2}{a^2 - b^2} - \frac{a - b}{2(a + b)} \)
6. \( \frac{4x^2 + 9y^2}{4x^2 - 9y^2} - \frac{2x - 3y}{2x + 3y} \)
110. Multiplication of Fractions.

Let \( \frac{a}{b} \) and \( \frac{c}{d} \) be any two fractions; to find the value of \( \frac{a}{b} \times \frac{c}{d} \).

Let \( x = \frac{a}{b} \times \frac{c}{d} \).
Then, we have \( x \times b \times d = \frac{a}{b} \times \frac{c}{d} \times b \times d = \frac{a}{b} \times b \times \frac{c}{d} \times d \)

\[-\left( \frac{a}{b} \times b \right) \times \left( \frac{c}{d} \times d \right) = a \times c; \]

or, \( x \times bd = ac \), \( \therefore \quad x = \frac{ac}{bd} \); \( \text{i.e.,} \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \).

Hence, \( \frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ac}{bd} \times \frac{e}{f} = \frac{ace}{bdf}; \)

\( \frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} \times \frac{g}{h} = \frac{ace}{bdf} \times \frac{g}{h} = \frac{aceg}{bfh} \); and so on.

Thus, the product of any number of fractions is a fraction whose numerator is the product of the numerators of the given fractions and whose denominator is the product of their denominators.

In practice, factors which are common to numerator and denominator are cancelled.

**Cor.** Since, \( c = \frac{c}{1} \), we have \( \frac{a}{b} \times c = \frac{a}{b} \times \frac{c}{1} = \frac{ac}{b} \).

**Example 1.** Multiply together \( \frac{x^2}{yz}, \frac{y^2}{zx}, \text{ and } \frac{z^2}{xy} \).

The required product \( = \frac{x^2 \times y^2 \times z^2}{yz \times zx \times xy} = \frac{x^2 \times y^2 \times z^2}{y^2 \times z^2 \times x^2} = 1 \).

The result is obtained by removing like factors from numerator and denominator.

**N. B.** 'When all the factors of numerator and denominator cancel each other, it is a common mistake with the beginners to give the result as 0. A little reflection will show that the result of such a multiplication can never be zero.'

**Example 2.** Multiply \( \frac{x(a-x)}{a^2+2ax+x^2} \) by \( \frac{a(a+x)}{a^2-2ax+x^2} \).

The required product \( = \frac{x(a-x) \times a(a+x)}{(a^2+2ax+x^2)(a^2-2ax+x^2)} = \frac{ax(a-x)(a+x)}{(a+x)^2(a-x)^2} = \frac{ax}{(a+x)(a-x)} = \frac{ax}{a^2-x^2} \).

**Example 3.** Multiply together \( \frac{1-x^2}{1+y^2}, \frac{1-y^2}{x+x^2}, \text{ and } 1+\frac{x}{1-x} \).
Since, \(1 + \frac{x}{1-x} = \frac{1 - x + x}{1-x} = \frac{1}{1-x}\),

the required product \(= \frac{(1+x)(1-x)}{1+y} \times \frac{(1+y)(1-y)}{x(1+x)} \times \frac{1}{1-x} \times \frac{1}{x} \times \frac{(1+y)(1-x)}{(1+y)x(1+x)(1-x)} \times \frac{1}{1-y}\).

**EXERCISE 57**

Multiply together:

1. \(\frac{2a^3}{3ab}, \frac{9b^2}{16ac} \text{ and } \frac{8c^3}{9bc}\).
2. \(\frac{4a^3b^3}{3c^3}, \frac{9c^2}{16a^3} \text{ and } \frac{4d^3}{27b^1}\).
3. \(\frac{x^3}{yz}, \frac{y^3}{x} \text{ and } \frac{z^3}{xy}\).
4. \(\frac{7a^2b^2c^2}{12xyz} \text{ and } \frac{4x^3y^3z^3}{21a^6b^2c^8}\).
5. \(\frac{12m^2n^3}{7xy^2z} \text{ and } \frac{35x^9yz}{96m^8n}\).

Simplify the following:

6. \(\frac{x+1}{x-1} \times \frac{x^2 + x - 2}{x^2 + x}\).
7. \(\frac{a^3 - 9b^3}{a^3 + 3ab} \times \frac{3a^3}{a^2 - 3ab}\).
8. \(\frac{a^3 - b^3}{a^3 + ab} \times \frac{(a + b)^3}{a^2 + ab + b^2}\).
9. \(\frac{a^3 + 8x^8}{a^3 - 2a^2x} \times \frac{a^2 - 4ax + 4x^2}{a^2 - 2ax + 4x^2}\).
10. \(\frac{x^3 + 4x^3 + 3x - 3x + 2}{x^3 - 4x - 2}\).
11. \(\frac{x^3 - 7x + 10}{x^3 - 2x - 15} \times \frac{x^2 - 3x - 18}{x^2 - 8x + 12}\).
12. \(\frac{x^3 - 6x^3 + 5x - 6}{x^3 - 5x - 2x + 2}\).
13. \(\frac{x^3 - 6x - 16}{x^3 - 4x - 21} \times \frac{x^2 - 11x + 28}{x^2 - 12x + 32}\).
14. \(\frac{a^3 - x^3}{a + b} \times \frac{a^2 - b^2}{ax + x^2} \times \left(\frac{a + bx}{a + x}\right)\).
15. \(\left(\frac{a^3 - x^3}{a + b}\right) \left(\frac{a^2 - b^2}{ax + x^2}\right) \left(\frac{a + bx}{a + x}\right)\).
16. \(\frac{4a + \frac{3x}{2b}}{3x + \frac{3b}{4a}}\).
17. \(\left(\frac{a + b}{a}\right) \left(\frac{c + d}{c}\right) - \left(\frac{a - b}{a}\right) \left(\frac{c - d}{c}\right)\).
18. \(\frac{2x^3 - 7x + 3}{2x^3 - 7x + 3} \times \frac{3x^3 + 11x - 4}{3x^3 + 8x - 3} \times \frac{2x^2 + x - 15}{2x^2 - 11x + 15}\).
19. \(\frac{a^3 - c^3}{a^3 - b^3} \times \frac{b^3 + c^3 - a^3 - 2bc}{2ac} \times \frac{c^3 - a^3 - b^3 + 2ab}{a^3 - b^3 + c^3 - 2ac}\).
20. \(\frac{b^3 - c^3}{b^3 - a^3 + 2ab} \times \frac{a^3 - b^3 + c^3 - 2ac}{a^3 + b^3 - c^3 - 2ab}\).
111. Division of Fractions.

Let \( \frac{a}{b} \) and \( \frac{c}{d} \) be any two fractions; to find the value of \( \frac{a}{b} + \frac{c}{d} \).

Let \( x = \frac{a}{b} + \frac{c}{d} \).

Then, we have \( x \times \frac{c}{d} = \frac{a}{b} + \frac{c}{d} \times \frac{c}{d} = \frac{a}{b} \).

\[ \therefore \quad m \div n \times n = m, \text{ whatever } m \text{ and } n \text{ may be.} \]

\[ \therefore \quad x \times \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}; \text{ or, } x = \frac{a}{b} \times \frac{d}{c}. \]

\[ \therefore \quad \frac{c}{d} \times \frac{d}{c} = 1. \]

Thus, to divide one fraction by another we have to multiply the former by the reciprocal of the latter.

Cor. \( \frac{a}{b} + c = \frac{a}{b} + \frac{c}{1} = \frac{a}{b} \times \frac{1}{c} = \frac{a}{bc} \).

Example 1. Simplify \( \frac{a^2 + b^2}{a^2 - b^2} + \frac{a^2 - ab + b^2}{a - b} \).

The required result \( \frac{a^2 + b^2}{a^2 - b^2} \times \frac{a - b}{a^2 - ab + b^2} = \frac{(a^2 + b^2)(a - b)}{(a^2 - b^2)(a^2 - ab + b^2)} \)

\[ = \frac{(a + b)(a^2 - ab + b^2)(a - b)}{(a + b)(a^2 - ab + b^2)} = 1. \]

Example 2. Simplify \( \frac{x^2 + x - 2}{x^2 + 7x + 12} + \frac{x^2 - 3x - 10}{x^2 + x - 12} \times \frac{x^2 - 4x - 5}{x^2 - 4x + 3} \).

The required result \( \frac{x^2 + x - 2}{x^2 + 7x + 12} \times \frac{x^2 + x - 12}{x^2 - 3x - 10} \times \frac{x^2 - 4x - 5}{x^2 - 4x + 3} \)

\[ = \frac{(x - 1)(x + 2)}{(x + 3)(x + 4)} \times \frac{(x + 4)(x - 3)}{(x - 5)(x + 2)} \times \frac{(x - 5)(x + 1)}{(x - 3)(x - 1)} \]

\[ = \frac{(x - 1)(x + 2)(x + 4)(x - 3)(x - 5)(x + 1)}{(x + 3)(x + 4)(x - 5)(x + 2)(x - 3)(x - 1)} = x + 1. \]

Example 3. Simplify \( \frac{a}{b} - \frac{a}{b} + \frac{a}{b} + \frac{a}{b} \times \frac{a^2}{a - b} \frac{a + b}{a - b} \).

\[ \text{We have } \frac{a}{a - b} + \frac{a}{a + b} = \frac{a(a + b) - a(a - b)}{a^2 - b^2} = \frac{2ab}{a^2 - b^2} + \frac{2b^2}{a^2 - b^2} \]

\[ = \frac{2ab}{a^2 - b^2} \times \frac{a^2 - b^2}{2b^2} = \frac{a}{b}; \text{ ... } \]

\[ \text{(1)} \]
\[
\frac{a + b + a - b}{a - b} = \frac{(a + b)^2 + (a - b)^2}{a^2 - b^2} = 2(a^2 + b^2) + \frac{4ab}{a^2 - b^2} = 2(a^2 + b^2) \times \frac{a^2 - b^2}{4ab} = \frac{a^2 + b^2}{2ab}.
\]

Hence, from (1) and (2),

the given expression is

\[
\frac{a^2}{b} + \frac{a^2 + b^2}{2ab} \times \frac{a^2}{a^2 + b^2} = \frac{a}{b} \times \frac{2ab}{a^2 + b^2} \times \frac{a^2}{a^2 + b^2} = \frac{2a^4}{(a^2 + b^2)^2}.
\]

N.B. 'When several fractions are connected by the signs \(\times, \div\), each sign applies only to the fraction which immediately follows it.'

**EXERCISE 58**

1. \(\frac{4a^2bc + 8ab^2c}{15xy^2z + 25x^2yz}\)
2. \(\frac{a^2 + ab}{a - b} + \frac{ab}{a^2 - b^2}\)
3. \(\frac{x^4 - 49}{x^2 - 25} + \frac{x + 7}{x + 5}\)
4. \(\frac{a^4 - b^4}{a^2 + 2ab + b^2} + \frac{a^2 + b^2}{a + b}\)
5. \(\frac{m^2 - 9n^2}{m^2 + 5mn + 6n^2} + \frac{m^2 - 2mn - 3n^2}{m^2 - n^2}\)
6. \(\frac{m^2 - n^2}{m + n} + \frac{m^2 + mn + n^2}{m^2 - n^2}\)
7. \(\left(\frac{2x + y}{x + y} - 1\right) + \left(1 - \frac{y}{x + y}\right)\)
8. \(\left(\frac{a}{a + b} + \frac{b}{a - b}\right) + \left(\frac{a}{a - b} - \frac{b}{a + b}\right)\)
9. \(\left(\frac{x + y}{x - y} + \frac{x - y}{x + y}\right) + \left(\frac{x + y}{x - y} - \frac{x - y}{x + y}\right)\)
10. \(\frac{x^2 - 4}{x^3 + 3x - 18} + \frac{x + 7}{x^3 - 36}\)
11. \(\left(1 - \frac{2pq}{p^2 + q^2}\right) + \left(\frac{p^2 + q^2}{p - q} - 3pq\right)\)
12. \(\frac{a^2 + b^2 + 3ab(a + b)}{(a + b)^2 - 4ab} + \frac{(a - b)^2 + 4ab}{a^2 - b^2 - 3ab(a - b)}\)
13. \(\frac{x^2 + y^2 + (x + y)^2 - 3xy}{(x - y)^2 + 3xy} \times \frac{xy}{x^2 - y^2}\)
14. \(\frac{a(a - b)^2 + 4a^2b + a^2 - b^2}{ab + b^2} \times \frac{b(a + b)^2 - 4ab^2}{a^2 - ab}\)
15. \(\frac{x^2 - x - 30}{x^2 - 36} + \frac{x^2 + 3x - 10}{x^2 + 2x - 8} + \frac{x + 4}{2x^2 + 12x}\)
16. \( \frac{x^2 + 3x - 108}{x^2 - 64} + \frac{x^2 + 6x - 72}{x^2 + x - 56} + \frac{x^2 - 16x + 63}{x^2 - 14x + 48} \)

17. \( \left( \frac{x^2 + y^2}{x^2 - y^2} - \frac{x^2 - y^2}{x^2 + y^2} \right) + \left( \frac{x + y}{x - y} - \frac{x - y}{x + y} \right) \)

18. \( \left( \frac{a + b}{a - b} + \frac{a^2 + b^2}{a^2 - b^2} \right) + \left( \frac{a - b}{a + b} - \frac{a^3 - b^3}{a^3 + b^3} \right) \)

19. \( \frac{a^4 - b^4}{a + b} + \frac{(a + b)^2 - 4ab}{a + b} + \frac{a}{(a + b)^2 - 2ab} \)

20. \( \frac{(a - b)(a + b)^2 - ab}{(a - b)^2 + 2ab} + \frac{(a - b)^2 + 3ab}{(a + b)^2 + ab} \times \frac{(a + b)^2 - 2ab}{(a + b)^2 - 3ab} \)

21. \( \frac{a^2 - b^2}{b^2 - a^2} \times \frac{1}{b} - \frac{1}{a} \times \frac{1}{a^2 + b^2 + 1} \)

[ C. U. 1868 ]

[ C. U. 1871 ]

CHAPTER XVII
SIMPLE EQUATIONS AND PROBLEMS

1. Simple Equations

112. In Chapter V, we have explained the process of solving simple equations. We propose to consider the subject more fully here.

We have stated that the process of solving any equation is primarily based upon certain axioms [Art. 63] from which it has been noticed that an equation is not altered.

(i) if any term be transposed from one side of the equation to the other; and (ii) if both the sides be multiplied or divided by any the same quantity.

Hence, the general rule for solving a simple equation involving one unknown quantity may be put as follows:

(1) **Simplify the two sides separately by clearing of fractions and brackets, if any, and by performing operations indicated by the symbols.**

(2) **Transpose all the terms involving the unknown quantity to the left-hand side of the equation and the remaining terms to the right-hand side.**

(3) **Next, simplify the two sides again.**
(4) Finally, divide both the sides by the coefficient of the unknown quantity.

The value of the unknown quantity, thus obtained, is the required solution.

Note. The student should verify for his own satisfaction that this value does really satisfy the given equation.

Example 1. Solve \((6x + 9)^2 + (3x - 7)^2 = (10x + 3)^2 - 71\). [C. U. 1882]

The left side \(= (36x^2 + 108x + 81) + (64x^2 - 112x + 49)\)
\[= 100x^2 - 4x + 130;\]
and the right side \(= (100x^2 + 60x + 9) - 71\)
\[= 100x^2 + 60x - 62.\]

Hence, the equation stands thus:
\[100x^2 - 4x + 130 = 100x^2 + 60x - 62.\]
Removing \(100x^2\) from both sides, we have
\[-4x + 130 = 60x - 62.\]
Hence, by transposition,
\[-4x - 60x = -130 - 62,\]
or, \[-64x = -192;\]
and therefore, dividing both sides by \(-64\), we have \(x = 3\).
Thus, the required root is 3.

Example 2. Given \(\frac{x - 6}{8} - \frac{2x - 15}{9} + 1 = \frac{x}{15} - \frac{x - 12}{6};\) find \(x\).

Multiplying both sides by \(8 \times 9 \times 5\) or 360, which is the L.C.M. of the denominators, we have
\[\frac{360(x - 6)}{8} - \frac{360(2x - 15)}{9} + 360 = \frac{360x}{15} - \frac{360(x - 12)}{6},\]
or, \[45(x - 6) - 40(2x - 15) + 360 = 24x - 60(x - 12),\]
or, \[45x - 270 - 80x + 600 + 360 = 24x - 60x + 720,\]
or, \[-35x + 690 = -36x + 720.\]
Hence, by transposition, \(-35x + 36x = 720 - 690,\)
or, \(x = 30.\)

Example 3. Solve \(\frac{1}{4}4a(1 + x) - \frac{3}{4}(a - x) = \frac{1}{4}3a(1 - x) - \frac{3}{4}(a + x).\)

The left side \(= \frac{4a}{8} (1 + x) - \frac{3a}{4} (a - x) = \left(\frac{4a}{8} - \frac{3a}{4}\right) + \left(\frac{4a}{8} + \frac{3}{4}\right)x\)
\[= \frac{7a}{12} + \frac{16a + 9}{12}x;\]
and the right side
\[
\frac{3a}{4} (1-x) - \frac{3}{6}(a+x) = \left(\frac{3a}{4} - \frac{4a}{3}\right) - \left(\frac{3a}{4} + \frac{4}{3}\right)x
\]
\[
= -\frac{7a}{12} - \frac{9a+16}{12}x.
\]
Hence, the equation stands thus:
\[
\frac{7a}{12} + \frac{16a+9}{12}x = -\frac{7a}{12} - \frac{9a+16}{12}x.
\]
Multiplying both sides by 12,
\[
7a + (16a+9)x = -7a - (9a+16)x.
\]
Hence, by transposition,
\[
(16a+9) + (9a+16)x = -14a,
\]
or,
\[
25(a+1)x = -14a;
\]
\[
\therefore \text{ dividing both sides by } 25(a+1), \text{ we have }
\]
\[
x = \frac{-14a}{25(a+1)} \text{ which is the required root.}
\]

**Example 4.** Given \[\frac{x}{a+b} + 1 = \frac{x}{a-b} + \frac{a-b}{a+b};\] find \(x\).

Multiplying both sides by \(a^2-b^2\), which is the L.C.M. of the denominators, we have,
\[(a-b)x + (a^2-b^2) = (a+b)x + (a-b)^2.\]
Hence, by transposition,
\[(a-b)x - (a+b)x = (a-b)^2 - (a^2-b^2),\]
or,
\[
(a-b) - (a+b)x = -2ab + 2b^2,
\]
or,
\[
-2bx = -2b(a-b).
\]
Therefore, dividing both sides by \(-2b\), we have \(x = a-b\).

**EXERCISE 59**

Find the value of \(x\), when
1. \(3(x-4)^2 + 5(x-3)^2 = (2x-5)(4x-1) + 24.\)
2. \((12x+9)^2 + (5x+3)^2 = (13x+9)^2 + 33.\)
3. \(5(x+1)^2 + 7(x+3)^2 = 12(x+2)^2.\)
4. \((3x-14)^2 + (4x-19)^2 = (5x-23)^2 = 22.\)
5. \((5x-8)^2 + (12x-7)^2 = (13x-10)^2 + 37.\)
6. \((x-1)^2 + (x+1)^2 = 2x(x^2-1) + 4.\)
7. \((x-2)^2 + 2x^2 + (x+2)^2 = 4x^2(x+2).\)
8. \[(x + 2)(x + 3)(x + 4) + 96 = x^2(x + 9) + 5(3x + 13)\].

9. \[3(x^2 - 14) = (x + 1)^2 + (x - 2)^2 + (x - 5)^2\].

10. \[a(x - a) = b(x - b)\].

Solve the following equations:

12. \[(x + a)(x + b) - (a + b)^2 = (x - a)(x - b)\].

13. \[a^2(x - a) + b^2(x - b) = abx\].

14. \[m^2(x - m) + n^2(x + n) + mnx = 0\].

15. \[b(x - 2a) + a(x - 2b) = (a - b)^2\].

16. \[a(4x - a) + b(4x - b) - 2ab = 0\].

17. \[x(x - a) + x(x - b) - 2(x - a)(x - b) = 0\].

18. \[(x + 3a)(x - 3b) + 3(x - 3a)(x + 3b) = 4(x - 3a)(x - 3b)\].

19. \[(2b + 2c - x)^2 + (2b - 2c + x)^2 = (2b + 2d - x)^2 + (2b - 2d + x)^2\].

20. \[(x - a)^2 + (x - b)^2 + (x - c)^2 = 3(x - a)(x - b)(x - c)\].

21. \[(x + a)^2 + (x + b)^2 + (x + c)^2 = 3(x + a)(x + b)(x + c)\].

22. \[\frac{x}{a} + a = \frac{x}{b} + b\].

23. \[\frac{a}{b}x - \frac{b}{ax} = a^2 - b^2\].

24. \[\frac{1}{3}(x + 1) + \frac{1}{3}(x + 2) + \frac{1}{3}(x + 3) = 16\].

25. \[\frac{x - 6}{5} + \frac{x - 4}{3} = 8 - \frac{x - 2}{7}\].

26. \[\frac{x}{10} + \frac{2x - 13}{9} = 8 - \frac{4x - 36}{15}\].

27. \[\frac{x + 7}{2} + \frac{x + 13}{5} + \frac{x + 17}{7} = \frac{x + 27}{4}\].

28. \[6 \frac{1}{3} - \frac{x - 7}{3} = \frac{4x - 2}{5}\].

29. \[\frac{x - 1}{3} - \frac{x - 9}{2} + \frac{3x - 2(x - 2)}{7} = 4 \frac{1}{2}\].

30. \[\frac{2x - 9}{27} + \frac{x}{18} - \frac{x - 3}{4} = 8 \frac{1}{2} - x\].

31. \[\frac{9x + 7}{2} - \frac{(x - x - 2)}{7} = 36\].

32. \[\frac{7x + 9}{4} - \frac{(x - 2x - 1)}{9} = 7\].

33. \[\frac{x + 7}{3} - 5 \frac{2}{3} = \frac{2x + 5}{7} + \frac{10 - 5x}{8}\].

34. \[x - \left(\frac{3x - 2x - 5}{10}\right) = \frac{1}{6} \left(2x - 57\right) - \frac{5}{3}\].

35. \[\frac{4x - 21}{7} + 7 \frac{1}{2} + \frac{7x - 28}{3} = x + 3 \frac{1}{2} - \frac{9 - 7x}{8} + \frac{1}{12}\].

36. \[\frac{1}{2} \left(x - \frac{a}{3}\right) - \frac{1}{3} \left(x - \frac{a}{4}\right) + \frac{1}{4} \left(x - \frac{a}{5}\right) = 0\].

37. \[\frac{x - 3}{7} - \frac{1}{2}x - \frac{3}{8} = \frac{1}{2}x + 2 - \frac{x - 6}{8} + \frac{x}{8}\].

38. \[\frac{1}{3}(x - 2) - \frac{1}{4}(x - 4) = \frac{1}{3}(2x - 3) - 2 \frac{2}{3}\].

39. \[\frac{a - x}{a} + \frac{2a - x}{2a} = \frac{3a - x}{3a}\].

[ C. U. 1861 ]
40. \( \frac{2x-13}{9} - \frac{x-1}{11} = \frac{x}{8} + \frac{x}{7} - 9. \)  
41. \( \frac{2x-3}{6} + \frac{3x-8}{11} = \frac{4x+15}{33} + \frac{1}{2} \).  
42. \( \frac{4x+3}{9} + \frac{13x}{108} = \frac{8x+19}{18} \).  
43. \( \frac{x^2-2\frac{1}{2}}{4} - \frac{x-3\frac{1}{2}}{5} = \frac{2x^2-3}{8} - \frac{x-5\frac{1}{2}}{3} \).  
44. \( \frac{a-x^2}{bx} - \frac{b-x}{c} = \frac{c-x}{b} - \frac{b-x^2}{cx} \).  
45. \( \frac{x+2\frac{1}{2}}{15} + \frac{x+3\frac{1}{2}}{25} = \frac{x+4\frac{1}{2}}{55} \).  
46. \( \frac{11x-13}{25} + \frac{19x+3}{7} - \frac{5x-25\frac{1}{2}}{4} = \frac{28\frac{1}{2} - 17x + 4}{21} \).  
47. \( \frac{x-1\frac{3}{8}}{2} - \frac{2-6x}{13} = \frac{x - 5x - \frac{1}{2}(10 - 3x)}{39} \).  
48. \( \frac{3x-\frac{3}{4}(1+x)}{4} + \frac{1-\frac{3}{2}x}{5\frac{1}{2}} = \frac{2\frac{1}{2} + \frac{3}{4}(x-1)}{2\frac{1}{4}} \).  
49. \( \frac{1}{4}(x-a) - \frac{1}{4}(2x-3b) - \frac{1}{2}(a-x) = 10a + 11b \).  
50. \( \frac{2x+a}{b} - \frac{x-b}{a} = \frac{3ax + (a-b)x}{ab} \).  
51. \( \frac{2x+1}{29} - \frac{402-3x}{12} = \frac{9 - 471 - 6x}{2} \).  
52. \( \frac{15 - \frac{3}{4}x}{5} - \frac{2x+5}{2\frac{1}{2}} = \frac{17 - \frac{3}{4}x}{3} \).  

113. Equations involving Decimals.

The decimals, if necessary, may be converted into vulgar fractions.

**Example 1.** Solve \( \frac{x-2}{0.5} - \frac{x-4}{0.0625} = 56 \).

Since, \( '0.5 = \frac{5}{10} = \frac{1}{2} \) and \( '0.625 = \frac{625}{10000} = \frac{1}{16} \),

we have \( \frac{x-2}{\frac{1}{2}} - \frac{x-4}{\frac{1}{16}} = 56 \),

or, \( 18(x-2) - 16(x-4) = 56 \),

or, \( 2x + 28 = 56 \); \( \therefore \)

or, \( 2x = 28 \),

or, \( x = 14 \).
Example 2. Solve \( 0.65x + \frac{535x - 975}{6} = \frac{1.56 - 39x - 78}{9} \).

Since, \( \frac{535x - 975}{6} = \frac{5.85x - 9.75}{6} = \frac{1.95x - 3.25}{2} \),

\( \frac{1.56}{2} = 15.6 \div 2 = 7.8 \),

and \( \frac{39x - 78}{9} = \frac{3.9x - 7.8}{9} = \frac{1.3x - 2.6}{3} \),

the equation stands thus:

\[ 0.65x + \frac{1.95x - 3.25}{2} = 7.8 - \frac{1.3x - 2.6}{3} \]

Hence, multiplying both sides by 6, we have

\[ 3.9x + (5.85x - 9.75) = 46.8 - (2.6x - 5.2) \]

By transposition, \((3.9 + 5.85 + 2.6)x = 46.8 + 5.2 + 9.75\),

or, \(12.35x = 61.75\),

\[ x = \frac{61.75}{12.35} = 5 \]

EXERCISE 60

Solve the following equations:

1. \( 0.6x - 0.2x = 3x - 1.5 \)
2. \( 3.75x + 5 = 2.25x + 8 \)
3. \( 1.2x - \frac{1.18x - 0.05}{0.5} = 4x + 8.9 \)
4. \( \frac{x + 0.75}{125} - \frac{x - 0.25}{25} = 15 \)
5. \( \frac{x}{6} - \frac{1}{0.05} + \frac{x}{0.005} - \frac{1}{0.0005} = 0 \) \[ C. U. 1883 \]
6. \( 0.5x + \frac{45x - 75}{6} = \frac{1.2}{2} - \frac{3x - 6}{9} \)
7. \( 7x + 0.4 = 67x + 6 \)
8. \( 1.6x + \frac{135x - 225}{6} = \frac{36}{2} - \frac{0.9x - 18}{9} \)
9. \( 0.6x + \frac{0.02x + 0.07}{0.03} - \frac{x + 2}{9} = 9.5 \)
10. \( 0.011x + \frac{0.001x - 0.125}{0.03} = \frac{5 - x}{0.03} - 145 \) \[ C. U. 1886 \]
114. Solution of equations facilitated by suitable transposition and combination of terms.

Example 1. Solve \( \frac{23x - 29}{12} + \frac{19x + 13}{7} = \frac{97x + 72\frac{1}{2}}{35} + \frac{7x - 8\frac{1}{2}}{4} \).

By transposition, we have
\[
\frac{23x - 29}{12} - \frac{7x - 8\frac{1}{2}}{4} = \frac{97x + 72\frac{1}{2}}{35} - 19x + 13 ;
\]
or,
\[
\frac{(23x - 29) - (21x - 25)}{12} = \frac{97x + 72\frac{1}{2}}{35} - (95x + 65) ;
\]
or,
\[
\frac{x - 2}{6} = \frac{2x + 7\frac{1}{2}}{35}.
\]

Hence, multiplying both sides by \( 6 \times 35 \),
\[35x - 70 = 12x + 45.\]
Hence, \( 23x = 115 \), or, \( x = 5 \).

Example 2. Solve \( \frac{x - a(b + c)}{bc} + \frac{x - b(c + a)}{ca} + \frac{x - c(a + b)}{ab} = 3. \)

The equation may be written as
\[
\frac{x - a(b + c)}{bc} + \frac{x - b(c + a)}{ca} + \frac{x - c(a + b)}{ab} = 1 + 1 + 1.
\]

By transposition, we have
\[
\left\{ \frac{x - a(b + c)}{bc} - 1 \right\} + \left\{ \frac{x - b(c + a)}{ca} - 1 \right\} + \left\{ \frac{x - c(a + b)}{ab} - 1 \right\} = 0 ;
\]
or,
\[
\frac{x - a(b + c)}{bc} - \frac{bc}{bc} + \frac{x - b(c + a)}{ca} - \frac{ca}{ca} + \frac{x - c(a + b)}{ab} - \frac{ab}{ab} = 0 ;
\]
or,
\[
\frac{x - (ab + ac + bc)}{bc} + \frac{x - (bc + ba + ca)}{ca} + \frac{x - (ca + cb + ab)}{ab} = 0 ;
\]
or,
\[
\left\{ x - (ab + bc + ca) \left[ \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} \right] \right\} = 0.
\]

When the product of two quantities equals to 0, at least one of them must be equal to 0. Since the sum of three known quantities, \( \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} \) cannot be zero, the other must be equal to 0.

\[
\therefore \quad x - (ab + bc + ca) = 0.
\]
Hence, \( x = ab + bc + ca \).
EXERCISE 61

Solve the following equations:

1. \[ \frac{5x + 6}{4} + \frac{64x - 35}{15} = \frac{20x + 23}{16} + \frac{13x - 7}{8} \]

2. \[ \frac{17x - 13}{9} + \frac{108x + 75}{32} = \frac{27x + 19}{8} + \frac{50\frac{7}{16}x - 39}{27} \]

3. \[ \frac{29x - 18}{8} + \frac{189x - 93}{49} = \frac{86\frac{1}{4}x - 54}{24} + \frac{27x - 13}{7} \]

4. \[ \frac{16x - 17}{9} - \frac{23x - 15}{16} = \frac{142\frac{1}{4}x - 153}{22} - \frac{92x - 65}{64} \]

5. \[ \frac{18x - 19}{7} + \frac{135x + 62\frac{1}{2}}{65} = \frac{27x + 14}{13} + \frac{106\frac{1}{2}x - 114}{42} \]

6. \[ \frac{33 - 19x}{15} - \frac{41 + 27x}{28} + \frac{164 + 107\frac{1}{4}x - 164\frac{1}{4}}{112} - \frac{95x}{75} = 0 \]

7. \[ \frac{18 - 41x}{9} - \frac{17 - 16x}{8} + \frac{9\frac{1}{4} - 10x}{5} - \frac{14 - 32x}{7} = 0 \]

8. \[ \frac{x - a^2}{b^2 + c^2} + \frac{x - b^2}{c^2 + a^2} + \frac{x - c^2}{a^2 + b^2} = 3 \]

9. \[ \frac{3x - bc}{b + c} + \frac{3x - ca}{c + a} + \frac{3x - ab}{a + b} = a + b + c \]

10. \[ \frac{ax - b^2 + c^2}{a - c} + \frac{bx - c^2 + a^2}{b - a} + \frac{cx - a^2 + b^2}{c - b} = 2(a + b + c) \]

11. \[ \frac{x - (b^2 + c^2)}{a^2 - 3bc} + \frac{x - (c^2 + a^2)}{b^2 - 3ca} + \frac{x - (a^2 + b^2)}{c^2 - 3ab} = a + b + c \]

12. \[ \frac{p^2x + (l^2 + m^2)}{l^2 - lm + m^2} + \frac{q^2x + (m^2 + n^2)}{m^2 - mn + n^2} + \frac{r^2x + (n^2 + l^2)}{n^2 - ln + l^2} = 2(l + m + n) \]

II. Problems

115. We have already explained in Chapter VI how simple algebraic problems can be expressed symbolically and solved. We have now to consider examples of a harder type.

As pointed out before, the chief difficulty in the solution of a problem lies in constructing its symbolical expression. The student should, therefore, become proficient in it by constant and varied practice.

No general rule for solution can be stated. The following advice can, however, be offered:

Read the problem several times and consider its meaning carefully.

See what quantity is required to be found out in the problem. Represent it by \( x \).
Next, express the conditions of the problem in terms of the symbol $x$ and obtain a simple equation in $x$.

Finally, solving this equation, find the value of $x$.

The process is explained by the following examples. For further illustrations, the student is referred to Chapter VI.

**Example 1.** How old is a man now, who, 20 years ago, was five times as old as his son who will be 41 years old 16 years after?

The present age of the man is to be found out. Let it be $x$ years.

\[ \therefore \text{20 years ago, the man's age} = (x - 20) \text{ years}, \]

\[ \text{16 years after, the son's age will be} = 41 \text{ years}; \]

\[ \therefore \text{the son's present age} = 41 - 16 = 25 \text{ years}. \]

Hence, 20 years ago, the son's age $= 25 - 20 = 5$ years.

\[ \therefore \text{from the condition of the problem,} \]

\[ x - 20 = 5 \times 5, \]

or, $x = 20 + 5 \times 5 = 20 + 25 = 45$ years.

Thus, the man's present age $= 45$ years.

**Example 2.** The sum of five consecutive odd numbers is 1185. What are the numbers?

[In solving problems, $2x$ and $2x + 1$ are taken as even and odd number respectively; because for any integral value of $x$, the value of $2x$ is even and that of $2x + 1$ is odd.]

Let $2x + 1$ = the smallest of the consecutive odd numbers. Since consecutive odd numbers differ from each other by 2, the numbers after $2x + 1$ are $2x + 3$, $2x + 5$, $2x + 7$, $2x + 9$, etc. In the present problem, the five consecutive odd numbers are, therefore, $2x + 1$, $2x + 3$, $2x + 5$, $2x + 7$ and $2x + 9$.

By the condition of the problem, their sum $= 1185$;

or, $\ (2x + 1) + (2x + 3) + (2x + 5) + (2x + 7) + (2x + 9) = 1185$,

or, $10x + 25 = 1185$, \quad or, \quad $10x = 1185 - 25 = 1160$;

\[ \therefore \ x = \frac{1160}{10} = 116. \]

Thus, the smallest of the consecutive odd numbers is 233.

Hence, the five required consecutive odd numbers are 233, 235, 237, 239, 241.

**Example 3.** Two persons started at the same time from $A$. One rode on horseback at the rate of $\frac{7}{2}$ kilometres an hour and arrived at $B$, 30 minutes later than the other who travelled the same distance by train at the rate of 30 kilometres an hour. Find the distance between $A$ and $B$. 
Let $x$ be the distance in kilometres between $A$ and $B$. Then the
time taken by the first man to travel the distance $= \frac{x}{\frac{7}{3}}$ hours $= \frac{2x}{15}$ hours
and the time taken by the other $= \frac{x}{30}$ hours.

But the time taken by the former is half an hour more than that
taken by the latter.

Hence, $\frac{2x}{15} = \frac{x}{30} + \frac{1}{2}$; or, $4x = x + 15$;

$\therefore 3x = 15$; $\therefore x = 5$.

Thus, the distance between $A$ and $B = 5$ kilometres.

Example 4. A person being asked his age, replied, "Ten years ago
I was 5 times as old as my son, but 20 years hence I shall be only twice
as old as he." What is his age?

Let the present age of the person be $x$ years.

Then 10 years ago his age was $(x - 10)$ years, and $\therefore$ that of his son
was $\frac{1}{2}(x - 10)$ years.

Hence, the present age of his son $= \frac{1}{2}(x - 10) + 10$ years, and $\therefore$ the son’s age 20 years hence will be $\frac{1}{2}(x - 10) + 30$ years; and the age of the person 20 years hence will evidently be $(x + 20)$ years.

Hence, by the second condition of the problem, we must have

$x + 20 = 2\left(\frac{1}{2}(x - 10) + 30\right)$

$= \frac{1}{2}(x - 10) + 60$;

$\therefore 5x + 100 = 2x - 20 + 300$;

$\therefore 3x = 180$; $\therefore x = 60$.

Note. Fractions might have been avoided by assuming the present age of the person to be 60 years. The student can easily proceed on this assumption.

Example 5. $A$ and $B$ have the same annual income. $A$ lays by
a fifth of his, but $B$, by spending annually Rs. 80 more than $A$, at the
end of 4 years finds himself Rs. 220 in debt. What was their income?

Let Rs. $x$ be the income of each.

Then $A$ spends Rs. $\frac{4}{5}x$ annually. Hence, $B$ spends annually
Rs. $(\frac{4}{5}x + 80)$.

But spending at this rate $B$ contracts a debt of Rs. 220 in 4 years, or
a debt of Rs. 55 per year. His annual income, therefore, falls short of
his annual expenses by Rs. 55.

Hence, we must have $x = (\frac{4}{5}x + 80) - 55$;

$\therefore \frac{1}{5}x = 25$; $\therefore x = 125$.

Thus, $A$ and $B$ had each an income of Rs. 125.
Example 6. A market woman bought a certain number of eggs at 5 a rupee, and as many at 7 a rupee, and sold them at the rate of 12 for two rupees, losing rupee one by her bargain. What number of eggs did she buy?

Let \( x \) = the number of eggs bought.

Then, since one half of them were bought at 5 a rupee, and the other half at 7 a rupee, the whole cost in buying the eggs

\[
= \( \frac{x}{2} \cdot \frac{1}{5} + \frac{x}{2} \cdot \frac{1}{7} \) \text{ rupees} = \( \frac{x}{10} + \frac{x}{14} \) \text{ rupees.}
\]

By selling the eggs at 12 for two rupees,

the amount realised = \( x \times \frac{1}{12} \) rupees.

Hence, by the equation, \( \frac{2x}{12} = \left( \frac{x}{10} + \frac{x}{14} \right) - 1 \);

or, \( 35x = 21x + 15x - 210 \);

Thus, altogether 210 eggs were bought.

Example 7. Divide 28 into two such parts that the difference between their squares is equal to 112.

Suppose, \( x \) is the greater part, so the other is \( 28 - x \).

By the condition of the problem,

\[
x^2 - (28 - x)^2 = 112;
\]

or, \( (x + 28 - x)(x - 28 + x) = 112 \);

or, \( 28(2x - 28) = 112 \);

or, \( (x - 14) = 2 \); \quad [ \text{dividing both sides by 28 \times 2} \]

or, \( x = 16 + 2 \);

\( \therefore \ x = 16 \); \quad \therefore \ the parts are 16 and 12.

Example 8. There is a number consisting of two digits, the digit in the units' place is twice that in the tens' place, and if 2 be subtracted from the sum of the digits, the difference is equal to \( \frac{1}{6} \)th of the number. Find the number.

Let \( x \) = the digit in the tens' place.

Then \( 2x \) = \( \ast \ast \ast \ast \) units' \( \ast \ast \ast \ast \) .

Clearly, therefore, the number = \( 10x + 2x \).

\[ \text{[ See Example 4 worked out in Art. 65.]} \]

Hence, by the second condition of the problem,

\[
(x + 2x) - 2 = \frac{10x + 2x}{6};
\]

whence, \( 18x - 12 = 12x \);

or, \( 6x = 12 \);

\( \therefore \ x = 2 \).

Hence, the required number = 24.
Example 9. Divide 127 into 4 parts, such that if the first be increased by 18, the second diminished by 5, the third multiplied by 6 and the fourth divided by \( \frac{24}{2} \), the results will all be equal. [B. U. 1883]

Let \( x \) be the result in all the cases.

By the condition of the problem,

The 1st part \(+ 18 = x\); \therefore \text{ the 1st part} = x - 18.

The 2nd part \(- 5 = x\); \therefore \text{ the 2nd part} = x + 5.

The 3rd part \times 6 = x\); \therefore \text{ the 3rd part} = \( \frac{x}{6} \).

The 4th part \( + \frac{5}{2} = x\); \therefore \text{ the 4th part} = \( \frac{2}{5} x \).

Therefore, \((x - 18) + (x + 5) + \frac{x}{6} + \frac{5}{2} x = 127\);

or, \(6(x - 18) + 6(x + 5) + 6 \cdot \frac{x}{6} + 6 \cdot \frac{5}{2} x = 6 \times 127\);

or, \(6x - 108 + 6x + 30 + x + 15x = 762\);

or, \(28x = 762 + 108 - 30\);

or, \(28x = 840\);

\therefore \( x = \frac{840}{28} = 30\).

\therefore \text{ the parts are} (30 - 18), (30 + 5), (30 + 6) \text{ and } 30 \times \frac{5}{2};

i.e., 12, 35, 5, 75.

EXERCISE 62

1. The length of a field is twice its breadth; another field which is 50 metres longer and 10 metres broader, contains 6800 square metres more than the former; find the size of each.

2. The length of a room exceeds its breadth by 3 metres; if the length had been increased by 3 metres, and the breadth diminished by 2 metres, the area would not have been altered; find the dimensions.

3. \( A \) and \( B \) began to play with equal sums, and when \( B \) has lost \( \frac{1}{4} \)th of what he had to begin with, \( A \) has gained Rs. 6 more than half of what \( B \) has been left with; what had they at first?

4. The ages of a father and his son together are 80 years; and if the age of the son be doubled, it will exceed the father's age by 10 years. Find the age of each.

5. A person distributed Rs. 100 among 36 persons, men and women, giving rupees three to each man and two rupees and a half to each woman. How many were there of each?
6. The sum of four consecutive odd numbers is 488. What are the numbers?

7. The sum of six consecutive even numbers is 1362. What are the numbers?

8. There are two places, 154 kilometres distant from each other, from which two persons A and B set out at the same instant with a desire to meet on the road. A travelling at the rate of 3 kilometres in 2 hours and B at the rate of 5 kilometres in 4 hours. How long and how far did each travel before they met?

9. A labourer was engaged for 36 days, upon the condition that he should receive two rupees and fifty paise for every day he worked, but should pay rupee one and fifty paise for every day he was idle. At the end of the time he received fifty-eight rupees. How many days did he work?

10. A person bought a picture at a certain price and paid the same price for the frame; if the frame had cost rupees twenty less and the picture rupees fifteen more, the price of the frame would have been only half that of the picture. Find the cost of the picture.

11. A post has a fourth of its length in the mud, a third of its length in the water and 10 metres above the water, what is its length?

12. Divide 20 into two such parts that the difference between their squares is 160.

13. A labourer is engaged for 30 days on condition that he receives two rupees and fifty paise for each day he works, and loses rupee one for each day he is idle; he receives rupees forty-seven in all. How many days does he work, and how many days is he idle?

14. A can do a piece of work in 9 days, B in twice that time; C can do only $\frac{2}{3}$ as much as A, in a day; how long would A, B and C, working together, require to do the same piece of work?

15. Two sums of money are together equal to rupees fifty-seven and twenty paise and there are as many rupees in the one as 10 paise in the other. What are the sums?

16. A certain sum is to be divided among A, B and C. A is to have Rs. 30 less than the half, B is to have Rs. 10 less than the third part, and C is to have Rs. 8 more than the fourth part. What does each receive?

17. A farmer wishing to purchase a number of sheep, found that if they cost him Rs. 42 a head, he would not have money enough by Rs. 28; but if they cost him Rs. 40 a head, he would then have Rs. 40 more than he required. Find the number of sheep, and the money which he had.

18. Two coaches start at the same time from York and London, a distance of 320 kilometres travelling, one at 9 kilometres an hour, the other at 11. Where will they meet and in what time from starting?
19. I bought a certain number of apples at three a rupee, and five-sixths of that number at four a rupee; by selling them at sixteen for six rupees I gained rupees three and a half. How many apples did I buy?

20. A number consists of two digits; the sum of the digits is 5, and if the left digit be increased by 1, it will be equal to the right of the number. Find the number.

21. A number consists of two digits, the digit in the tens' place exceeds that in the units' place by 5, and if 5 times the sum of the digits be subtracted from the number, the digits will be inverted. Find the number.

22. There is a number consisting of two digits, the sum of whose digits is 5, and if 10 times the digit in tens' place be added to 4 times the digit in the units' place, the number will be inverted. What is the number?

23. Divide the number 99 into four parts, such that if the first be increased by 1, the second diminished by 2, the third multiplied by 3, and the fourth divided by 4, the results will all be equal.

24. Divide 60 into 4 parts, such that if the first be diminished by 3, the second increased by 11, the third multiplied by 4, and the fourth divided by 2, the results will all be equal.

25. Divide the number 116 into four such parts that if the first be increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the result in each case shall be the same.

CHAPTER XVIII
SIMPLE SIMULTANEOUS EQUATIONS AND PROBLEMS

I. Simple Simultaneous Equations

116. Introductory remarks. The equation \( x - y = 2 \), in which \( x \) and \( y \) are both unknown, evidently admits of an infinite number of solutions; for any pair of numbers, whose difference is 2 will satisfy it. [For instance, the equation will be satisfied if \( x = 3, y = 1 \); if \( x = 4, y = 2 \); if \( x = 5, y = 3 \); if \( x = 6, y = 4 \); and so on.] If, however, \( x \) and \( y \) be such that they must also satisfy the equation \( x + y = 8 \), then of the different pairs of numbers whose difference is 2, we shall have to reject all excepting that of which the sum is 8. Thus the two equations,

\[
\begin{align*}
x - y &= 2 \\
x + y &= 8
\end{align*}
\]

will both be satisfied by the same values of \( x \) and \( y \), only when \( x = 5 \) and \( y = 3 \).
Again, it may be seen that the three equations,
\[
\begin{align*}
x + y + z &= 6 \\
x - y + z &= 4 \\
x + y - z &= 2
\end{align*}
\]
will be satisfied by the same values of \(x, y, z\) only when \(x = 3, y = 1, z = 2\). The equations may be individually satisfied by innumerable sets of values of the unknown quantities, but there is only one set which will satisfy them all.

Two or more equations (like those just referred to) which are all satisfied by the same values of the unknown quantities involved in them are called simultaneous equations. They are said to be simple or of the first degree when each unknown quantity occurs only in the first power and the product of the unknown quantities does not occur.

We shall consider first of all simultaneous equations involving two unknown quantities, and later on, those that involve more than two. There are three general methods for solving such equations and we shall treat them successively in the next three articles.

117. First Method: Method of Substitution: From either equation find the value of one of the unknown quantities in terms of the other and substitute the value thus found in the other equation.

**Example 1.** Solve \(5x - 24y = 16\) \(\begin{align*}
4x - y &= 31
\end{align*}\)

From the 2nd equation, we have
\[
y = 4x - 31
\]

Substituting this value of \(y\) in the 1st equation, we have
\[
5x - 24(4x - 31) = 16,
\]
or,
\[
5x - 96x + 744 = 16;
\]
\[
\therefore \quad -91x = -728; \quad \therefore \quad x = 8.
\]

Hence, from (1), \(y = 4 \times 8 - 31 = 1\).

Thus, we have \(x = 8\) and \(y = 1\).

Note. The student is recommended to verify for his own satisfaction that these values of \(x\) and \(y\) do really satisfy both of the given equations.

**Example 2.** Solve \(\frac{3x - 5y}{2} + 3 = \frac{2x + y}{5}; 8 - \frac{x - 2y}{4} = \frac{x}{2} + \frac{y}{3}\).

Multiplying both sides of the 1st equation by 10, we have
\[
5(3x - 5y) + 30 = 2(2x + y),
\]
or,
\[
15x - 25y + 30 = 4x + 2y;
\]
\[
\therefore \quad 11x = 27y - 30.
\]...  ...  ... (1)
Multiplying both sides of the 2nd equation by 12, we have
\[ 96 - 3(x - 2y) = 6x + 4y, \]
or, \[ 96 - 3x + 6y = 6x + 4y; \]
\[ \therefore 2y - 9x + 96 = 0, \]
\[ \therefore \quad 2y - 9x + 96 = 0; \quad \ldots \quad \ldots \quad (2) \]

From (1), we have \[ x = \frac{27y - 30}{11}, \]
\[ \ldots \quad \ldots \quad (3) \]

Substituting this value of \( x \) in (2), we have
\[ 2y - \frac{9(27y - 30)}{11} + 96 = 0; \]
\[ \therefore 22y - 9(27y - 30) + 1056 = 0; \]
\[ \therefore 22y - 243y + 270 + 1056 = 0; \]
\[ \therefore 221y = 1326; \quad \therefore y = 6. \]

Hence, from (3), \[ x = \frac{27 \times 6 - 30}{11} = \frac{132}{11} = 12. \]

Thus, we have \( x = 12 \) and \( y = 6 \).

**EXERCISE 63**

Solve the following equations:

1. \[ \begin{align*}
   x + 4y &= 14 \quad \text{[C. U. 1872]} \\
   7x - 3y &= 5
\end{align*} \]

2. \[ \begin{align*}
   5x - 8y &= 9 \quad \text{[C. U. 1872]} \\
   13x + 7y &= 79
\end{align*} \]

3. \[ \begin{align*}
   2x + 3y &= 32 \\
   11y - 9x &= 3
\end{align*} \]

4. \[ \begin{align*}
   9x - 4y &= 8 \\
   13x + 7y &= 101
\end{align*} \]

5. \[ \begin{align*}
   x + ay &= b \\
   ax - by &= c
\end{align*} \]

6. \[ \begin{align*}
   2x - \frac{1}{2}(y - 3) &= 4 \\
   3y + \frac{1}{3}(x - 2) &= 9
\end{align*} \]

7. \[ \begin{align*}
   \frac{1}{4}(x + y) &= \frac{1}{2}(2x + 4) \\
   \frac{1}{2}(x - y) &= \frac{1}{3}(x - 24)
\end{align*} \]

8. \[ \begin{align*}
   \frac{1}{3}(x - y) &= \frac{1}{2}(y - 1) \\
   \frac{1}{4}(4x - 5y) &= x - 7
\end{align*} \]

9. \[ \begin{align*}
   \frac{1}{3}(3x - 2y) - 3 &= \frac{1}{2}(2x - y) \\
   \frac{1}{2}(5x - 4y) - 3 &= \frac{1}{3}(4x - 3y)
\end{align*} \]

10. \[ \begin{align*}
   \frac{1}{2}(2x + 3y) + \frac{1}{3} x &= 8 \\
   \frac{1}{3}(7y - 3x) - y &= 11
\end{align*} \]

118. **Second Method: Method of Comparison:** From each equation find the value of the same unknown quantity in terms of the other and equate the values thus found.

**Example 1.** Solve \[ \begin{align*}
   6x - 5y &= 11 \\
   2x + 3y &= 27
\end{align*} \]

From the 1st equation, we have
\[ 5y = 6x - 11. \]
\[ \therefore y = \frac{6x - 11}{5} \quad \ldots \quad \ldots \quad (1) \]
From the 2nd equation, we have
\[3y = 27 - 2x.\]
\[\therefore \quad y = \frac{27 - 2x}{3}.\]
\[\therefore \quad \frac{6x - 11}{5} = \frac{27 - 2x}{3}.
\]  
\[\therefore \quad 3(6x - 11) = 5(27 - 2x),\]

or, \[18x - 33 = 135 - 10x;\]

\[\therefore \quad 28x = 168; \quad \therefore \quad x = 6.\]  
Hence, from (1), \[y = \frac{6 \times 6 - 11}{5} = \frac{35}{5} = 7.\]  
Thus, we have \[x = 6\] and \[y = 7.\]  

**Example 2.** Solve \[
\begin{align*}
\frac{7 + x}{5} - \frac{2x - y}{4} &= 3y - 5 \\
\frac{5y - 7}{2} + \frac{4x - 3}{6} &= 18 - 5x
\end{align*}
\]  

Multiplying both sides of the 1st equation by 20, we have
\[4(7 + x) - 5(2x - y) = 20(3y - 5),\]

or, \[28 - 6x + 5y = 60y - 100;\]

\[\therefore \quad 55y + 6x = 128.\]

Multiplying both sides of the 2nd equation by 6, we have
\[3(5y - 7) + (4x - 3) = 6(18 - 5x),\]

or, \[15y + 4x - 24 = 108 - 30x;\]

\[\therefore \quad 15y + 34x = 132.\]

From (1), \[y = \frac{128 - 6x}{55}.\]

From (2), \[y = \frac{132 - 34x}{15}.\]

Hence, from (3) and (4), we have
\[
\frac{128 - 6x}{55} = \frac{132 - 34x}{15}; \quad \text{or}, \quad \frac{64 - 3x}{11} = \frac{66 - 17x}{3};
\]

\[\therefore \quad 3(64 - 3x) = 11(66 - 17x),\]

or, \[192 - 9x = 726 - 187x;\]

\[\therefore \quad 178x = 534; \quad \therefore \quad x = 3.\]

Hence, from (3), \[y = \frac{128 - 18}{55} = \frac{110}{55} = 2.\]

Thus, we have \[x = 3\] and \[y = 2.\]
EXERCISE 64

Solve the following equations:

1. \(5x - 3y = 9\)
   \(5y + 2x = 16\)
2. \(3y - 4x = 1\)
   \(3x + 4y = 18\)
3. \(3x - 7y = 7\)
   \(11x + 5y = 57\)
4. \(y(3 + x) = x(7 + y)\)
   \(4x + 9 = 5y - 14\)
5. \(32x - 25y = 28\)
   \(14x + 15y = 116\)
6. \(\frac{1}{2}(3x + y) = \frac{1}{2}(2x + y + 1)\)
   \(8 - \frac{1}{2}(x - y) = 6\)
7. \(\frac{5}{2}(5x - 6y) + 3x = 4y - 2\)
   \(\frac{1}{2}(5x + 6y) - \frac{1}{2}(3x - 2y) = 2y - 2\)
8. \(2x - \frac{1}{2}(y + 3) = 7 + \frac{1}{2}(3y - 2x)\)
   \(4y + \frac{1}{2}(x - 2) = 26\frac{1}{2} - \frac{1}{2}(2y + 1)\)
9. \(2x - \frac{1}{2}(2y - 1) = 3x + \frac{1}{2}(3x - 2y)\)
   \(4y - \frac{1}{2}(5 - 2x) = 6 - \frac{1}{2}(3 - 2y)\) \[C. U. 1873\]
10. \(\frac{x}{3} - \frac{2}{y} = 1, \frac{x}{3} + \frac{3}{y} = 3\) \[A. U. 1923\]

119. Third Method: Method of Elimination: "Multiply the equations by such numbers as will make the coefficient of one of the unknown quantities the same in the two resulting equations; then by addition or subtraction we can form an equation containing only the other unknown quantity."

**Example 1.** Solve \(3x - 4y = 5\)
   \(5x + 2y = 17\)

Multiplying the 2nd equation by 2, we have

\[10x + 4y = 34\]
and the 1st equation is
\[3x - 4y = 5\]
Hence, by addition, \(13x = 39\); \(\therefore x = 3\).
Substituting this value of \(x\) in the 1st equation, we have
\(4y = 9 - 5 = 4\);
\(\therefore y = 1\).
Thus, we have \(x = 3, y = 1\).

**Example 2.** Solve \(5x + 9y = 89\)
   \(2x - 17y = 15\)

Multiplying the 1st equation by 2, and the 2nd by 5, we have
\[10x + 18y = 178\]
and
\[10x - 85y = 75\]
Hence, by subtraction, we have
\[109y = 103; \therefore y = 1\]
Substituting this value of \(y\) in the 2nd equation, we have
\(2x - 15 + 17 = 32\);
\(\therefore x = 16\).
Thus, we have \(x = 16, y = 1\).
Note. We might as well have multiplied the 1st equation by 17 and the 2nd equation by 9 and added the two resulting equations; this would have given us the value of $x$. But we have preferred the other alternative because, the coefficients of $x$ being smaller, the required multiplications have been more easily effected.

**Example 3.** Solve \[
\begin{align*}
23x - 24y &= 21 \\
25x - 16y &= 43
\end{align*}
\]

Multiplying the 1st equation by 2, and the 2nd by 3, we have \[
\begin{align*}
46x - 48y &= 42 \\
75x - 48y &= 129
\end{align*}
\]

Hence, by subtraction, we have \[
29x = 87; \quad \therefore \quad x = 3.
\]

Substituting this value of $x$ in the 2nd equation, we have \[
16y = 75 - 43 = 32; \quad \therefore \quad y = 2.
\]

Thus, we have $x = 3, y = 2$.

**Note.** It may be noticed that the coefficient of $y$ in each of the resulting equations is the least common multiple of 24 and 16 and this is all that is required. The process would have been unnecessarily tedious if the 1st equation were multiplied by 16 and the 2nd by 24.

**Example 4.** Solve \[
\begin{align*}
\frac{x - 2}{2} - \frac{x + y}{14} + \frac{y + 12}{8} \\
\frac{x + 7}{3} + \frac{y - 5}{10} &= 1 - \frac{5(y + 1)}{7}
\end{align*}
\]

From the 1st equation, we have \[
\frac{7(x - 2) - (x + y)}{14} = \frac{(x - y - 1) - 2(y + 12)}{8},
\]

or, \[
\frac{6x - y - 14}{7} = \frac{x - 3y - 25}{4},
\]

or, \[
24x - 4y - 56 = 7x - 21y - 175,
\]

or, \[
x + 17y = -119,
\]

or, \[
x + y = -7. \quad \ldots \quad \ldots \quad (1)
\]

From the 2nd equation, we have \[
\frac{10(x + 7) + 3(y - 5)}{30} = \frac{7(1 - x) - 5(y + 1)}{7},
\]

or, \[
\frac{10x + 3y + 55}{30} = \frac{2 - 7x - 5y}{7},
\]

or, \[
70x + 21y + 385 = 60 - 210x - 150y,
\]

or, \[
230x + 171y = -325. \quad \ldots \quad \ldots \quad (2)
\]
Multiplying (1) by 171, we have
\[
171x + 171y = -1197 \]
also
\[
280x + 171y = -325 \]
Hence, by subtraction,
\[
109x = 872; \quad \therefore \ x = 8.
\]
Substituting this value of \( x \) in (1), we have
\[
y = -7 - 8 = -15.
\]
Thus, we have \( x = 8, y = -15 \).

**Example 5.** Solve
\[
\begin{align*}
\frac{2}{x} + \frac{3}{y} &= 1 \\
\frac{7}{x} + \frac{4}{y} &= 17
\end{align*}
\]
Multiplying the 1st equation by 4, and the 2nd by 3, we have
\[
\frac{8}{x} + \frac{12}{y} = 4 \quad \text{and} \quad \frac{21}{x} + \frac{12}{y} = \frac{45}{8}
\]
Hence, by subtraction,
\[
\frac{13}{x} = \frac{13}{8}; \quad \therefore \ x = 8.
\]
Substituting this value of \( x \) in the 1st equation, we have
\[
\frac{3}{y} = 1 - \frac{1}{4} = \frac{3}{4}; \quad \therefore \ y = 4.
\]
Thus, we have \( x = 8, y = 4 \).

**Alternative Method:**
Supposing \( \frac{1}{x} = u, \frac{1}{y} = v \),
we get
\[
\begin{align*}
2u + 3v &= 1 \quad \ldots \quad (1) \\
7u + 4v &= \frac{17}{8} \quad \ldots \quad (2)
\end{align*}
\]
Multiplying the equation (1) by 4 and (2) by 3, we have
\[
8u + 12v = 4
\]
and \( 21u + 12v = \frac{45}{8} \).
Hence, by subtraction,
\[
13u = \frac{17}{8};
\]
\[
\therefore \quad u = \frac{1}{8}.
\]
\[
\therefore \quad \frac{1}{x} = u = \frac{1}{8}; \quad \therefore \ x = 8.
\]
Substituting the value of $u$ in equation (1),

\[ \frac{3}{4} + 3v = 1; \]

or, \[ 3v = 1 - \frac{3}{4}; \]

or, \[ 3v = \frac{1}{4}; \]

\[ \therefore v = \frac{1}{4}. \]

\[ \therefore \frac{1}{y} = v = \frac{1}{4}; \quad \therefore y = 4. \]

Thus, we have $x = 8$, $y = 4$.

**Example 6.** Solve \[
\begin{align*}
\frac{12}{x+y} + \frac{8}{x-y} &= 8 \\
\frac{27}{x+y} - \frac{12}{x-y} &= 6
\end{align*}
\]

Supposing $u = \frac{1}{x+y}$ and $v = \frac{1}{x-y}$,

we get,

\[ 12u + 8v = 8 \quad \ldots \quad (1) \]
\[ 27u - 12v = 6 \quad \ldots \quad (2) \]

Multiplying the equation (1) by 3 and (2) by 2, we have

\[ 36u + 24v = 24 \]
and \[ 54u - 24v = 12. \]

Hence by addition,

\[ 90u = 36; \]

\[ \therefore u = \frac{36}{90} = \frac{2}{5}; \]

\[ \therefore \frac{1}{x+y} = \frac{2}{5}; \]

\[ \therefore x + y = \frac{5}{2} \quad \ldots \quad (3) \]

Substituting the value of $u$ in the equation (1), we have

\[ 12 \cdot \frac{2}{5} + 8v = 8; \]

or, \[ 8v = 8 - \frac{24}{5} = \frac{4}{5}; \]

or, \[ v = \frac{1}{5}; \]

\[ \therefore \frac{1}{x-y} = \frac{2}{5}; \]

\[ \therefore x - y = \frac{5}{2} \quad \ldots \quad (4) \]

Hence, by adding (3) and (4), we have

\[ 2x = 5; \]

\[ \therefore x = \frac{5}{2}. \]
Substituting the value of \( x \) in the equation (3), we have
\[
\frac{1}{4} + y = \frac{1}{4};
\]
or,
\[
y = 0.
\]
\[
\therefore \quad x = \frac{1}{4}, \quad y = 0.
\]

**EXERCISE 65**

Solve the following equations:

1. \[
\begin{aligned}
7x - 5y &= 11 \\
3x + 2y &= 13
\end{aligned}
\]
2. \[
\begin{aligned}
13x + 6y &= 58 \\
5x - 11y &= 9
\end{aligned}
\]
3. \[
\begin{aligned}
8x - 9y &= 20 \\
7x - 10y &= 9
\end{aligned}
\]
4. \[
\begin{aligned}
25x - 14y &= 8 \\
12x + 7y &= 45
\end{aligned}
\]
5. \[
\begin{aligned}
12x + 11y &= 70 \\
8x - 7y &= 18
\end{aligned}
\]
6. \[
\begin{aligned}
13x - 14y &= 22 \\
17x - 21y &= 18
\end{aligned}
\]
7. \[
\begin{aligned}
28x - 15y &= 41 \\
21x + 13y &= 55
\end{aligned}
\]
8. \[
\begin{aligned}
19x + 24y &= 34 \\
23x + 36y &= 62
\end{aligned}
\]
9. \[
\begin{aligned}
47x - 56y &= 128 \\
25x + 84y &= 298
\end{aligned}
\]
10. \[
\begin{aligned}
51x - 16y &= 3 \\
68x + 28y &= 137
\end{aligned}
\]
11. \[
\begin{aligned}
52x - 9y &= 34 \\
39x + 14y &= 67
\end{aligned}
\]
12. \[
\begin{aligned}
12x + 85y &= -49 \\
19x - 34y &= 91
\end{aligned}
\]
13. \[
\begin{aligned}
65x - 14y &= 9 \\
15x + 46y &= -17
\end{aligned}
\]
14. \[
\begin{aligned}
13x + 69y &= 78 \\
17x + 135y &= 101
\end{aligned}
\]
15. \[
\begin{aligned}
5x + 11y &= 146 \\
11x + 5y &= 110
\end{aligned}
\]
16. \[
\begin{aligned}
a + b y &= c \\
a^2 x + b^2 y &= c^2
\end{aligned}
\]
[O. U. 1881]
17. \[
\begin{aligned}
\frac{x + y}{2} + \frac{3x - 5y}{4} &= 9 \\
\frac{x}{14} + \frac{y}{18} &= -1
\end{aligned}
\]
[C. U. 1876]
18. \[
\begin{aligned}
\frac{4x + 5y}{40} &= x - y \\
\frac{2x - y}{3} + 2y &= -1
\end{aligned}
\]
19. \[
\begin{aligned}
\frac{4x - 3y - 7}{5} &= \frac{3x - 2y - 5}{10} \\
\frac{y}{3} + \frac{x}{2} - \frac{3y}{20} &= \frac{y - x}{15} + \frac{x}{6} + \frac{11}{10}
\end{aligned}
\]
20. \[
\begin{aligned}
\frac{6x - 3y + 7x - 5y}{12} &= \frac{19}{15} - \frac{17x - 10y + 2}{3} \\
\frac{(3\frac{1}{2})x + 2y - 5}{16} + \frac{11x - (4\frac{1}{2})y + 17}{11} &= \frac{19}{22} + \frac{17x - 10y + 2}{3}
\end{aligned}
\]
21. \[
\begin{aligned}
\frac{3x - 5y - 2x - 3y - 33}{3} &= \frac{y}{2} + \frac{x}{3} + \frac{1}{4} \\
3\frac{1}{2} \left( \frac{x}{7} + \frac{y}{4} + 1\frac{1}{3} \right) &= -3\frac{1}{3} \left( 4x - \frac{y}{8} - 24 \right)
\end{aligned}
\]
22. \[
\begin{aligned}
\frac{2\frac{1}{2}x + 3\frac{3}{2}y - \frac{18x - 025}{20}}{19} &= \frac{8x + 5\frac{2}{3} + 01y}{5}
\end{aligned}
\]
23. \[
\begin{aligned}
\frac{2y + 05}{15} &= \frac{49x - 7}{42}
\end{aligned}
\]
24. \[
\frac{4}{x} + \frac{10}{y} = 2
\]
\[
3 + \frac{2}{y} = 19
\]
\[
\frac{5}{x} + \frac{10}{y} = 5\frac{5}{6}
\]
[ C. U. 1879 ]

25. \[
\frac{2}{x} + \frac{3}{y} = 2
\]
\[
\frac{5}{x} + \frac{10}{y} = 5\frac{5}{6}
\]
[ C. U. 1887 ]

27. \[
\frac{1}{3x} + \frac{1}{5y} = 1
\]
\[
\frac{1}{5x} + \frac{1}{3y} = 1\frac{3}{10}
\]
[ C. U. 1870 ]

28. \[
\frac{3}{y} - \frac{1}{x} = 1
\]
\[
\frac{2}{5x} + \frac{5}{2y} = 7
\]

29. \[
\frac{x}{4} + \frac{2}{y} = 2
\]
\[
\frac{2x}{5} + \frac{3}{2y} = 2\frac{7}{10}
\]

30. \[
\frac{1}{5x} + \frac{y}{9} = 5
\]
\[
\frac{1}{3x} + \frac{y}{2} = 14
\]

31. \[
\frac{14}{x+y} + \frac{3}{x-y} = 5
\]
\[
\frac{21}{x+y} - \frac{1}{x-y} = 2
\]

II. Problems leading to simple equations with more than one unknown quantity

120. Easy Problems.

Example 1. The present age of the father is double of that of the son. 16 years ago the father's age was thrice that of the son. Find their present ages.

Let \( x \) = the present age of the father,
and \( y \) = the present age of the son.

By the given condition of the problem,

\[
x = 2y \quad \ldots \quad (1)
\]
and \( (x - 16) = 3(y - 16) \quad \ldots \quad (2)\)

Substituting \( 2y \) for \( x \) in equation (2),

\[
2y - 16 = 3(y - 16)
\]
or, \( 2y - 16 = 3y - 48 \)
or, \( 2y - 3y = 16 - 48 \)
or, \( -y = -32 \)
\[
\therefore \quad y = 32.
\]
Therefore, the son's age is 32 years and that of the father is \((32 \times 2 - 1)\) 64 years.

Example 2. \(A\) and \(B\) each had a number of mangoes. \(A\) said to \(B\), "If you give me 30 of your mangoes, my number will be twice yours." \(B\) replied, "If you give me 10, my number will be thrice yours." How many had each?

Let \(x\) = the number of mangoes \(A\) had,

and \(y\) = the number of mangoes \(B\) had.

Then, in accordance with what \(A\) said, we must have the equation

\[x + 30 = 2(y - 30)\]  \(\ldots\)  (1)

and in accordance with \(B\)'s reply, we must have the equation

\[y + 10 = 3(x - 10)\]  \(\ldots\)  (2)

From (2), \(3x - y = 40\), or, \(6x - 2y = 80\)  \(\ldots\)  (3)

and from (1), \(x - 2y = -90\).  \(\ldots\)  (4)

Hence, by subtraction, \(5x = 170\); \(\therefore x = 34\).

Substituting this value of \(x\) in (4), we have

\[2y = 34 + 90 = 124\]

\(\therefore y = 62\).

Thus, \(A\) had 34 mangoes and \(B\) had 62.

Example 3. A certain fraction becomes 2 when 7 is added to its numerator, and 1 when 2 is subtracted from the denominator. What is the fraction?

Let \(\frac{x}{y}\) represent the fraction.

Then, we have

\[\frac{x + 7}{y} = 2\]  \(\ldots\)  (1)

and \(\frac{x}{y - 2} = 1\).  \(\ldots\)  (2)

From (1), \(x + 7 = 2y\); \(\therefore x = 2y - 7\)

From (2), \(x = y - 2\)

Therefore, \(2y - 7 = y - 2\), whence \(y = 5\).

Hence, \(x = 5 - 2 = 3\).

Thus, the fraction is \(\frac{3}{5}\).

Example 4. 2 men and 7 boys can do in 4 days a piece of work which would be done in 3 days by 4 men and 4 boys. How long would it take one man or one boy to do it?

Let \(x\) = the number of days in which one man would do the work,

and \(y\) = the number of days in which one boy would do it.
Then, in one day a man does $\frac{1}{x}$ th of the work and a boy does $\frac{1}{y}$ th of it.

Hence, since 2 men and 7 boys do $\frac{1}{4}$ th of the work in one day, we must have

$$\frac{2}{x} + \frac{7}{y} = \frac{1}{4}.$$  

Again, since 4 men and 4 boys do $\frac{1}{3}$ rd of the work in one day, we must have

$$\frac{4}{x} + \frac{4}{y} = \frac{1}{3}.$$  

Multiplying (1) by 2, and subtracting (2) from the resulting equation, we must have

$$\frac{10}{y} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6};$$  

Hence, from (2), 

$$\frac{4}{x} = \frac{1}{3} - \frac{1}{15} = \frac{4}{15};$$  

Thus, one man would do the work in 15 days and one boy in 60 days.

**Example 5.** Cost of 3 doors and 5 windows is Rs. 487 and that of 5 doors and 3 windows is Rs. 561. Find the value of a door and a window.

Let $x$ = the cost of a door and $y$ = the cost of a window.

By the given conditions of the problem,

$$3x + 5y = 487$$  

and  

$$5x + 3y = 561.$$  

Multiplying (1) by 3 and (2) by 5, we have

$$9x + 15y = 1461$$  

$$25x + 15y = 2805.$$  

Subtracting (3) from (4),

$$16x = 1344;$$  

$$x = 84.$$  

Substituting the value of $x$ in (1), we have

$$y = 47.$$  

Thus, the value of a door is Rs. 84 and that of a window Rs. 47.

**Example 6.** Two plugs are opened in the bottom of a cistern containing 192 litres of water; after 3 hours one of them becomes stopped, and the cistern is emptied by the other in 11 hours; bad
8 hours elapsed before the stoppage, it would have only required 6 hours more to empty it. How many litres will each plug-hole discharge in one hour, supposing the discharge to be uniform?

Let \( x \), \( y \) be the numbers of litres of water which the plugs can respectively discharge in an hour.

In the first case, the first plug remains opened for 3 hours, and the second for 3 + 11 or 14 hours.

\[
3x + 14y = 192.
\]

In the second case, the first plug remains opened for 6 hours, and the second for 6 + 6 or 12 hours.

\[
6x + 12y = 192.
\]

Multiplying (1) by 2 and subtracting (2) from the resulting equation, we have

\[
16y = 2 \times 192 - 192
\]

\[
= 192 \; ;
\]

\[
\therefore \quad y = 12.
\]

Hence, from (2), \( 6x = 192 - 144 = 48 \); \( \therefore \quad x = 8. \)

Thus, the plug-holes respectively discharge 8 and 12 litres in an hour.

**Example 7.** The dimension of a rectangular court is such that if the length were increased by 3 metres, and the breadth diminished by the same, its area would be diminished by 18 square metres; and if its length were increased by 3 metres, and its breadth increased by the same, its area would be increased by 60 square metres, find the dimensions.

Let \( x \) metres = length of the court,
and \( y \) metres = its breadth.

Then, from the first condition of the problem, we have

\[
(x + 3)(y - 3) = xy - 18; 
\]

\[
\therefore \quad (1)
\]

and from the second condition,

\[
(x + 3)(y + 3) = xy + 60.
\]

\[
\therefore \quad (2)
\]

From (1), \( 3y - 3x = -9 \), or, \( y - x = -3 \). \( \therefore \quad (3) \)

From (2), \( 3y + 3x = 51 \), or, \( y + x = 17 \). \( \therefore \quad (4) \)

From (3) and (4), by addition,

\[
2y = 14; \quad \therefore \quad y = 7;
\]

and by subtraction, \( 2x = 20; \quad \therefore \quad x = 10. \)

Thus, the length of the court is 10 metres, and the breadth is 7 metres.

**Example 8.** There is a certain number consisting of two digits, to the sum of whose digits if you add 7, the result will be three times the left-hand digit; and if from the number itself you subtract 18, the digits will be inverted. Find the number.
Let \( x \) and \( y \) be the left and right-hand digits respectively; then the required number is represented by \( 10x + y \), and the number with inverted digits = \( 10y + x \).

Hence, by the conditions of the problem,
\[
\begin{align*}
  x + y + 7 &= 3x, & \quad \text{... (1)} \\
  (10x + y) - 18 &= 10y + x, & \quad \text{... (2)} \\
\end{align*}
\]

From (1), \( 2x - y = 7 \); \quad \text{... (3)}
and from (2), \( 9x - 9y = 18 \), or, \( x - y = 2 \). \quad \text{... (4)}

Subtracting (4) from (3), we have
\[
x = 7 - 2 = 5.
\]
Hence, from (4), \( y = 5 - 2 = 3 \).

Thus, the required number is 53.

**Example 9.** \( A \) and \( B \) play at bowls, and \( A \) bets \( B \) three shillings to two upon every game; after a certain number of games it appears that \( A \) has won three shillings; but if \( A \) had bet five shillings to two and lost one game more out of the same number, he would have lost thirty shillings. How many games did each win?

Let \( x \) = number of games that \( A \) won,
and \( y \) = number of games that \( B \) won.

Then, the total number of games played is evidently \( x + y \).

Now, since \( A \) receives from \( B \), 2s. for every game that he wins and gives \( B \), 3s. for every game that he loses (i.e., for every game that \( B \) wins), his total gain must be equal to \( (2x - 3y) \) shillings.

Hence,
\[
2x - 3y = 3. \quad \text{... \text{... (1)}}
\]

According to the other condition, \( A \) would have gained \( 2(x - 1) \) shillings and lost \( 5(y + 1) \) shillings; and therefore, his total loss would have been \( [5(y + 1) - 2(x - 1)] \) shillings.

Hence,
\[
5(y + 1) - 2(x - 1) = 30,
\]
or,
\[
5y - 2x = 23. \quad \text{... \text{... (2)}}
\]

From (1) and (2), by addition, \( 2y = 26 \); \quad \therefore \quad y = 13.

Hence, from (1), \( x = \frac{3 + 39}{2} = 21 \).

Thus, \( A \) won 21 games and \( B \) won 13 games.

**EXERCISE 66**

What fraction is that whose numerator being doubled and denominator increased by 7, the value becomes \( \frac{1}{3} \); but the denominator being doubled, and the numerator increased by 2, the value becomes \( \frac{1}{2} \)?
2. Find two numbers such that if the first be added to 5 times the second, the sum is 52; and if the second be added to 8 times the first, the sum is 65.

3. Find two numbers such that five times the greater exceeds four times the less by .22, and three times the greater together with seven times the less is 32.

4. What numbers are those whose difference is 45, and the quotient of the greater by the less is 4?

5. The age of the father exceeds twice that of his son by 10 years. Twenty years ago, the age of the father was five times that of his son. Find their present ages.

6. Ten years ago the age of the father was seven times that of his son. Two years hence twice the age of the father will be equal to five times that of his son. Find their present ages.

7. There are two numbers such that one-fourth of the greater added to one-third of the less is 11; and if one-fifth of the less be taken from one-eighth of the greater, the remainder is nothing; find the numbers.

8. A certain fraction becomes \( \frac{1}{2} \) when 1 is subtracted from its denominator, and 1 when 7 is added to its numerator. What is the fraction?

9. What fraction is that which, if 1 be added to the numerator, becomes 1, and if 1 be added to the denominator, becomes \( \frac{1}{2} \)? [C. U. 1862]

10. A certain fraction becomes \( \frac{1}{2} \) when its numerator is increased by unity, and \( \frac{1}{3} \) when its denominator is increased by unity. What is the fraction?

11. The denominator of a fraction exceeds the numerator by 4 and if 5 be taken from each, the sum of the reciprocal of the new fraction and 4 times the original fraction is 5. Find the original fraction.

[ Solving the problem, we will find the fraction to be \( \frac{9}{4} \). Students should note that the fraction should not be reduced to its lowest terms as is generally done. If reduced to lowest terms, it will not satisfy the given conditions. ]

12. \( A \) and \( B \) have 39 rupees between them, but if \( A \) were to lose two-thirds of his money, and \( B \) three-fourths of his, they would then have only 11 rupees. How much has each?

13. Two numbers are such that if 7 be added to the less, the sum is twice the greater, and if 4 be added to the greater, the sum is 3 times the less. Find the numbers.

14. Two persons, 27 kilometres apart, setting out at the same time, meet together in 9 hours, if they walk in the same direction, but in 3 hours if they walk in opposite directions; find their rates of walking.

15. A banker was asked to pay Rs. 50 in 50 paise and 25 paise so that the number of the latter should be exactly twice that of the former. How must he do it?
16. A man and a boy can do in 15 days a piece of work which would be done in 2 days by 7 men and 9 boys. How long would it take one man to do it?

17. A rectangle is of the same area as another which is 6 metres longer and 4 metres narrower; it is also of the same area as a third, which is 8 metres longer and 5 metres narrower. What is its area?

18. If 15 kgs. of tea and 17 kgs. of coffee together cost Rs. 189 and 25 kgs. of tea and 13 kgs. of coffee together cost Rs. 213, find the price of each per kilogram.

19. A takes 3 hours longer than B to walk 30 kilometres; but if he doubles his pace he takes 2 hours less time than B; find their rates of walking.

20. Says Charles to William, "If you give me 10 of your marbles, I shall then have just twice as many as you"; but says William to Charles, "If you give me 10 of yours, I shall then have three times as many as you." How many had each?

21. If a certain number be divided by the sum of its two digits the quotient is 6 and the remainder is 3. If the digits be inverted and the resulting number be divided by the sum of the digits, the quotient is 4 and the remainder 9. Find the number.

22. Find that number of 2 figures to which, if the number formed by changing the places of the digits, be added, the sum is 121; and if the smaller number be subtracted from the larger, the remainder is 9.

23. A bill of 25 guineas was paid with crowns and half-guineas; and twice the number of half-guineas exceeded 3 times that of the crowns by 17. How many were there of each?

24. A person sells to one person 9 horses and 7 cows for Rs. 3000; and to another, at the same prices, 6 horses and 13 cows for the same sum. What was the price of each?

25. A and B received £5. 17s. for their wages, A having been employed for 15 and B for 14 days; and A received, for working 4 days, 11s. more than B did for three days. What were their daily wages?

26. A and B can do a piece of work in 16 days; they work together for 4 days, when A leaves, and B finishes it in 36 days more. In what time would each do the work separately?

27. If the numerator of a fraction is increased by 2 and the denominator by 1, it becomes equal to $\frac{1}{3}$; and if the numerator and denominator are each diminished by 1, it becomes equal to $\frac{2}{3}$. Find the fraction.

28. A traveller walks a certain distance; had he gone half a kilometre an hour faster, he would have walked it in four-fifths of the time; had he gone half a kilometre an hour slower, he would have been $\frac{3}{4}$ hours longer on the road. Find the distance.
29. A certain number between 10 and 100 is eight times the sum of its digits, and if 45 be subtracted from it, the digits will be reversed; find the number.

30. A and B lay a wager of 10s. If A loses, he will have twenty-five shilling less than twice as much as B will then have; but if B loses, he will have five-seventeenths of what A will then have; find how much money each of them has.

31. A farmer wishing to purchase a number of sheep found that if they cost him Rs. 42 a head, he would not have money enough by Rs. 28; but if they cost him Rs. 40 a head, he would then have Rs. 40 more than he required; find the number of sheep, and the money which he had.

32. There is a number consisting of two digits; the number is equal to three times the sum of its digits, and if it be multiplied by 3, the result will be equal to the square of the sum of its digits. Find the number.
CHAPTER XIX
GRAPHS OF SIMPLE EQUATIONS

121. In Chapter VII, we have discussed representations of numbers by geometric points. We now propose to show how simple equations are represented graphically. The following examples will make the subject clear.

Example 1. If a point moves in such a manner that its abscissa is always equal to 5 units of length, find the path along which the point will move.

Let five times the side of a small square represent the unit of length.

On $OX$ take the point $M$ such that $OM = 5$ units of length; through $M$ draw the straight line $PMP'$ parallel to $YOY'$.

Now, if any point be taken on the straight line $PMP'$, its $x$ will evidently be equal to 5 units of length; but this will not be so if the point be taken on either side of the line $PMP'$.

Hence, the moving point will always be on the line $PMP'$. 
We see, therefore, that if a point moves in such a manner that its $x$ is always equal to 5 units of length, the path along which the point will move is the straight line $PMP'$. This fact is briefly expressed by saying that the straight line $PMP'$ is the graph of the equation $x = 5$.

Note 1. From the above it is clear that the graph of the equation $y = 5$ is a straight line parallel to $XOX'$.

Note 2. Generally speaking, the graph of the equation $x = a$ is a straight line parallel to the axis of $y$, and passing through a point on the axis of $x$ which is at a distance of $a$ units of length from the origin; and the graph of the equation $y = b$ is a straight line parallel to the axis of $x$, and passing through a point on the axis of $y$, which is at a distance of $b$ units of length from the origin.

Note 3. Evidently, therefore, the graph of the equation $x = 0$ is the axis of $y$ itself, and the graph of the equation $y = 0$ is the axis of $x$ itself.

Example 2. If a point moves in such a manner that its $x$ and $y$ are always connected by the relation $y = 3x$, find the path along which the point will move, i.e., draw the graph of the equation $y = 3x$. 
Giving different values of \( x \) in the given equation, we get different values of \( y \). They may be tabulated as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>9</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

Take three times the length of a side of a small square as the unit of length and plot the points tabulated above.

Join the points \((0, 0), (3, 9), (4, 12)\) and \((5, 15)\) and produce the straight line both ways. Then this straight line will be the required path.

Take any point \( P \) on this straight line. The co-ordinates of \( P \) are found to be 6 and 18, which evidently satisfy the given relation. Similarly, the co-ordinates of any other point on this straight line may be shown to satisfy the given relation. But the co-ordinates of a point which is outside the line \( OP \) will not satisfy the given relation, as can be easily verified.

Hence, the moving point will always be on the line \( OP \) and never stray out of it.

Thus, it is found that if a point moves in such a way that its \( x \) and \( y \) are invariably connected by the relation \( y = 3x \), the path along which the point will move is the straight line \( OP \). In other words, the line \( OP \) is the graph of the equation \( y = 3x \).

Note 1. Generally speaking, the graph of the equation \( y = mx \), where \( m \) is any given number, is a straight line passing through the origin.

Note 2. It should be observed that the greater the number of points plotted and closer their positions to each other, the more accurately the graph will be drawn. No graph should be drawn without plotting at least three points.

Example 3. If a point moves in such a way that its \( x \) and \( y \) are invariably connected by the relation \( y = -4x + 5 \), find the path along which the point will move, i.e., draw the graph of the equation \( y = -4x + 5 \).

The corresponding values of \( x \) and \( y \) in the equation \( y = -4x + 5 \) may be tabulated as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>5</td>
<td>1</td>
<td>-1</td>
<td>-3</td>
</tr>
<tr>
<td></td>
<td>-7</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The points indicated by the tabulated values of $x$ and $y$ are some of the different positions of the moving point.

Let four times the side of a small square represent the unit of length. Plot the points and join them. Produce the straight line both ways. Then this straight line will be the required path.

Take a point $P$ on this straight line. The co-ordinates of $P$, which are found to be $-1$ and $9$, satisfy the given relation. Take another point $Q$ on the straight line; its co-ordinates which are found to be $5$ and $-5$, also satisfy the given relation. Similarly, the co-ordinates of any other point on this straight line may be shown to satisfy the given relation. But if a point be taken outside the line $PQ$, its co-ordinates will not satisfy the given relation, as can be easily seen. Hence, the moving point will always be on the line $PQ$ and never stray out of it.

Thus, it is found that if a point moves in such a manner that its co-ordinates always satisfy the equation $y = -4x + 5$, the path along
which the point will move is the straight line \( PQ \). In other words, the straight line \( PQ \) is the graph of the equation \( y = -4x + 5 \).

Note 1. Generally speaking, the graph of the equation \( y = mx + c \), where \( m \) and \( c \) are any given numbers, is a straight line passing through the point \((0, c)\).

Note 2. As every equation of the first degree in \( x \) and \( y \) can be reduced to the form \( y = mx + c \), it is clear that graphs of all simple equations are straight lines. Suppose, \( ax + by + c = 0 \) is a simple equation with two unknown quantities. By transposition, \( by = -ax - c \). Dividing both sides by \( b \), we get \( y = \left(-\frac{a}{b}\right)x + \left(-\frac{c}{b}\right) \).

Note 3. The graph of the equation \( y = mx + c \) is also said to be the graph of the expression \( mx + c \).

Note 4. The graph of any given equation may be defined to be the path described by a point which moves in such a manner that in every position of the point its co-ordinates satisfy the given equation.

Example 4. Draw the graph of the equation \( 7x + 3y = 11 \).
The corresponding values of \( x \) and \( y \) in the equation \( 7x + 3y = 11 \), may be tabulated as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>3 ( \frac{1}{2} )</td>
<td>1 ( \frac{1}{2} )</td>
<td>-1</td>
</tr>
</tbody>
</table>

Evidently, therefore, \((0, 3 \frac{1}{2})\), \((1, 1 \frac{1}{2})\) and \((2, -1)\) are points on the graph.

Let 6 times the side of a small square represent the unit of length. Join the points \((0, 3 \frac{1}{2})\), \((1, 1 \frac{1}{2})\) and \((2, -1)\) and produce the straight line both ways. Then this straight line will be the required graph. [See the diagram of page 216.]

Take any point \( P \) on the line; its co-ordinates, which are found to be 3 and \(-3 \frac{1}{2}\), satisfy the given relation. Take any other point \( Q \) on the line; its co-ordinates, which are found to be \(-1 \) and 6, also satisfy the given relation. Similarly, it may be shown that the co-ordinates of any point that may be taken on the line \( PQ \) will satisfy the given relation; but the co-ordinates of any point which is outside \( PQ \) will not. Hence, the line \( PQ \) is the required graph.

Note 1. The equation \( 7x + 3y = 11 \) may be written as \( y = \frac{11 - 7x}{3} \) after transposition and division of both sides by the coefficient of \( y \). The graph of the equation \( x + 3y = 11 \) is also said to be the graph of the expression \( \frac{11 - 7x}{3} \).

Note 2. The straight line \( PQ \) being the graph of the equation \( 7x + 3y = 11 \), his equation is said to be the equation of the straight line \( PQ \).

Note 3. The equation of a given straight line means the equation which is satisfied by the co-ordinates of every point on that straight line.

Example 5. Draw the graph of \( \frac{6x + 7}{5} \).

From Note 1 of Example 4, the given quantity is equal to another variable \( y \). Find some corresponding values of \( x \) and \( y \) in the equation \( y = \frac{6x + 7}{5} \) and plot them. The straight line formed by joining them will be the graph of the given quantity.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-7</th>
<th>8</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-7</td>
<td>11</td>
<td>17</td>
</tr>
</tbody>
</table>
Let 3 times the side of a small square represent the unit of length. Join the points $(-7, -7)$, $(3, 5)$, $(8, 11)$ and $(13, 17)$, and produce the straight line both ways. This straight line is the graph of the given quantity.

Example 6. Find the equation of the straight line which passes through the points $(1, 1)$ and $(3, -\frac{1}{2})$.

Let $y = mx + c$ be the required equation.

This equation being satisfied by $(1, 1)$ and also by $(3, -\frac{1}{2})$, we must have

\[
\begin{align*}
1 &= m + c \\
1 &= m + c \quad \text{(from (1, 1))} \\
-\frac{1}{2} &= 3m + c \quad \text{from (3, -\frac{1}{2})}
\end{align*}
\]

Hence, $2m = -\frac{1}{2} \Rightarrow m = -\frac{1}{4}$, and \( \therefore c = 1 + \frac{1}{4} = \frac{5}{4} \).

Thus, the required equation is $y = -\frac{1}{4}x + \frac{5}{4}$; or, $3x + 4y = 7.$
Example 7. Draw the graph of the equation \( \frac{x}{3} + \frac{y}{4} = 1 \) and find the length of its portion intercepted between the two axes.

or. \( 4x + 3y = 12 \). [Multiplying both sides by 12]

From the equation \( 4x + 3y = 12 \), we get

<table>
<thead>
<tr>
<th>x</th>
<th>3</th>
<th>0</th>
<th>6</th>
<th>-8</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>4</td>
<td>-4</td>
<td>8</td>
</tr>
</tbody>
</table>

Taking 5 times the side of a small square as unit of length the graph \( PQ \) is drawn

\( PQ \) cuts the axes at \( A \) and \( B \).

(1) By measurement with a scale \( AB \) is found to be equal to 5 units of length.
(ii) \( AOB \) is a right-angled triangle, of which \( AO = 4 \), \( OB = 3 \) and \( \angle AOB \) is a right angle.

\[ \therefore AB = \sqrt{AO^2 + OB^2} = \sqrt{4^2 + 3^2} = 5. \]

**Example 8.** Draw the graph of \( \frac{x + 3}{2} \). Find the value of the function from the graph when \( x = 3 \). \[ \text{[D. B. 1934]} \]

Find the value of \( x \) when the value of the given function is 0.

The graph of \( \frac{x + 3}{2} \) is that of the equation \( y = \frac{x + 3}{2} \). From the equation \( y = \frac{x + 3}{2} \), we get

\[
\begin{array}{ccccc}
\text{a} & 1 & 5 & -1 \\
\text{y} & 2 & 4 & 1 \\
\end{array}
\]

Taking 5 times the side of a small square as unit of length the straight line \( AB \) is drawn. \[ \text{[See the figure above.]} \]

From the figure \( y = 3 \), when \( x = 3 \).

The straight line \( AB \) cuts the \( XOX' \) axis at \( P \). The ordinate of \( P \) is 0. Therefore, we are to find out that abscissa of \( P \) for which the value of the function = 0. Counting from \( O \), \( OP = -3 \). \[ \therefore x = -3. \]
EXERCISE 67

1. Draw the graphs of the following equations:
   (1) \( x = -8 \).
   (2) \( x = 13 \).
   (3) \( x + 11 = 0 \).
   (4) \( y = -7 \).
   (5) \( y - 9 = 0 \).
   (6) \( y + 10 = 0 \).

2. Draw the graphs of the following equations:
   (1) \( y = x \).
   (2) \( y = -x \).
   (3) \( y = 2x \).
   (4) \( y + 2x = 0 \).
   (5) \( y = -3x \).
   (6) \( 3y = 5x \).
   (7) \( 7y + 8x = 0 \).
   (8) \( 6y + 13x = 0 \).

3. Draw the graphs of the following equations:
   (1) \( y = 3x + 4 \).
   (2) \( y = 7x - 8 \).
   (3) \( y = -5x + 9 \).
   (4) \( y = -8x - 11 \).
   (5) \( 3y = 7x + 4 \).
   (6) \( -6y = 7x - 10 \).

4. Draw the graphs of the following equations:
   (1) \( 2x + 7y = 10 \).
   (2) \( 4x - 5y - 7 = 0 \).
   (3) \( 5x + 6y + 8 = 0 \).
   (4) \( -3x + 7y + 8 = 0 \).
   (5) \( 10y - 9x = 13 \).
   (6) \( 8x - 11y + 13 = 0 \).

5. Draw the graphs of the following equations:
   (1) \( \frac{x}{5} + \frac{y}{4} = 1 \).
   (2) \( \frac{x}{7} + \frac{y}{9} = 1 \).
   (3) \( \frac{x}{-8} + \frac{y}{13} = 1 \).
   (4) \( \frac{y - 5 - 7x}{6} \).
   (5) \( \frac{9x - 13}{4} \).
   (6) \( \frac{3x}{4} - \frac{4y}{3} = 1 \).

6. Draw the graphs of the following expressions:
   (1) \( x - 3 \).
   (2) \( 3x + 4 \).
   (3) \(-7x + 8\).
   (4) \( \frac{7 - 4x}{3} \).
   (5) \( \frac{5x - 9}{4} \).
   (6) \( \frac{8x + 11}{5} \).

7. Find the equation of the straight line which passes through each of the following pairs of points:
   (1) \( (0, 0), (5, 6) \).
   (2) \( (0, 5), (7, 0) \).
   (3) \( (6, -8), (-7, 5) \).
   (4) \( (-4, 8), (-9, -13) \).
   (5) \( (-11, 0), (7, -10) \).

8. Draw the graph of the equation \( 3x - 2y - 4 = 0 \). Find, from the graph, the value of \( y \) when \( x = 2 \).

9. Draw the graph of the equation \( 3x + 4y - 12 = 0 \) and find the length of the graph intercepted by the axes.


CHAPTER XX
EASY QUADRATIC EQUATIONS AND PROBLEMS

122. Definition. Any equation which contains the square of the unknown quantity, but no higher power, is called a quadratic equation or an equation of the second degree.

If an equation contains only the second power of the unknown quantity (and not the first), it is called a pure quadratic; if it contains the second as well as the first power, it is called an affected quadratic.

Thus, \(3x^2 - 75\) is a pure quadratic;
and \(3x^2 - 7x = 6\) is an affected quadratic.

123. Solution of a Pure Quadratic. In solving a Pure Quadratic we have to find the square of the unknown quantity just in the same way as simple equations are solved and then to extract the square root of the value so found.

Example 1. Solve \(5(x^2 + 1) - 2 = 3(x^2 + 7)\).
We have \(5x^2 + 5 = 3x^2 + 21\);
hence, \(2x^2 = 18\); [ by transposition ]
\[x^2 = 9;\]

now, since the unknown quantity is one of which the square is 9, it must be either +3 or -3. (Thus there are two values of \(x\) satisfying the given equation, as the student can easily verify.)

Note. The student should carefully observe that the last step of the above solution amounts to answering the following question: "What quantity is that of which the square is 9?"

Example 2. Solve \(\frac{1}{2}(x - 2)(x - 3) - \frac{1}{3}(x - 21)(x - 14) = 2\).
Multiplying both sides by 21, we have
\[7(x - 2)(x - 3) - (x - 21)(x - 14) = 42,\]
The left side = \((7x^2 - 35x + 42) - (x^2 - 35x + 294)\)
\[= 7x^2 - 35x + 42 - x^2 + 35x - 294\]
\[= 6x^2 - 252.\]
Hence, the equation reduces to
\[6x^2 - 252 = 42,\]
or, \(6x^2 = 252 + 42\) [ by transposition ]
i.e., \(6x^2 = 294.\)
Dividing both sides by 6, we have
\[ x^2 = 49. \]

Now, the unknown quantity is such that its square is 49;
\[ \therefore \text{it must be either } +7 \text{ or } -7. \]

Hence, \( x = \text{either } +7 \text{ or } -7. \)

**Example 3.** Find the side of a square whose area is equal to that of a rectangle of length 9 metres and breadth 4 metres.

Let the side of the square = \( x \) metres.
\[ \therefore \text{the area of the square} = x \times x \text{ sq. metres} \]
\[ = x^2 \text{ sq. metres}. \]

Again, the area of the rectangle = \( 4 \times 9 \) sq. metres
\[ = 36 \text{ sq. metres}. \]

Hence, by the condition of the problem,
\[ x^2 \text{ sq. metres} = 36 \text{ sq. metres}, \]
or, \( x^2 = 36; \therefore x = 6, \text{ or, } -6. \)

Since, the actual length of the side of a square is a positive quantity, the solution \( x = -6 \) is inadmissible.
\[ \therefore \text{the required side} = 6 \text{ metres}. \]

**N. B.** In problems leading to quadratic equations, the solutions which are found inadmissible by the condition of the problem should be rejected.

**EXERCISE 68**

Find the values of \( x \) in each of the following equations:

1. \( 3x^2 = 27. \)
2. \( a^2x^2 = a^4. \)
3. \( \frac{1}{4}x^2 = 28. \)
4. \( 8x + \frac{7}{x} = \frac{65}{7} x. \)
5. \( 2(x^2 - 5) + x(3 - x) = 3(x + 5). \)
6. \( (x - 7)(x - 10) + (x - 3)(x - 2) = (x - 17)(x - 5). \)
7. \( \frac{2x^2 + 10}{15} = 7 - \frac{50 + x^2}{25}. \)
8. \( (x + a)^2 - 2a(x + a) = 3a^2. \)
9. \( x^2 + 2bx - b^2 = a^2 - b(b - 2x). \)
10. \( 2x(3x + 5) - 5x(x + 2) = 36. \)
11. \( \frac{3x^2 + 15}{7} + \frac{2x^2 + 9}{3} = \frac{2x^2 + 87}{21} + 2. \)
12. Find the number four times which is equal to sixteen times its reciprocal.
13. Find the side of a square three times the area of which is equal to four times the area of a rectangle whose length and breadth are respectively 9 metres and 3 metres.
14. A has got a square plot of land which he exchanges with a rectangular garden of area 91 sq. metres, belonging to $B$ and gains by the transaction an area of 10 sq. metres. Find a side of the square plot.

15. Divide a straight line of length 10 cm. into two portions such that five times the square on one exceeds the square on the other by twenty times the former portion.

124. Solution of a Quadratic by the method of resolution into factors. Reducing a Quadratic to the form $ax^2 + bx + c = 0$, if we know the factors of which the left-hand side is the product, then by equating to zero either of these factors, we get a solution of the quadratic.

**Example 1.** Solve $x^2 - 5x + 6 = 0$.

Evidently the left-hand side $= (x - 2)(x - 3)$.

Hence, we have $(x - 2)(x - 3) = 0$.

:. either $x - 2 = 0$ \(\text{or} \), $x - 3 = 0$ \(\text{or} \), $x = 2$ \(\text{or} \), $x = 3$

Thus, 2 and 3 are the roots of the equation, as the student can easily verify.

**Example 2.** Solve $2x^2 - 10x = 3x - 15$.

We have $2x(x - 5) = 3(x - 5)$.

If $x - 5 \neq 0$ (\(\neq \) stands for 'is not equal to'), we may remove the factor from each side of the equation.

Thus $2x = 3$; \(\therefore \) $x = \frac{3}{2}$.

But if $x - 5 = 0$, each side of the equation reduces to zero and the equation is satisfied.

Hence from $x - 5 = 0$ we get another root viz., $x = 5$.

Thus the roots are $\frac{3}{2}$, 5.

If in course of simplification any factor which contains the unknown is found to be common to both sides of the equation, it must not be rejected, since every such linear factor equated to zero will give one root of the equation.

**Example 3.** Solve $10(2x + 3)(x - 3) + (7x + 3)^2 = 20(x + 3)(x - 1)$.

We have, $10(2x^2 - 8x - 9) + (49x^2 + 42x + 9) = 20(x^2 + 2x - 3)$.

\[ \therefore \quad 49x^2 - 28x - 21 = 0; \]

\[ \therefore \quad 7x^2 - 4x - 3 = 0, \text{ or,} \quad (7x^2 - 7x) + (3x - 3) = 0, \]

or \[ (7x + 3)(x - 1) = 0. \]

Hence, either $7x + 3 = 0$ \(\text{or} \), $x - 1 = 0$ \(\text{or} \), $x = -\frac{3}{7}$ \(\text{or} \), $x = 1$.

Thus, $-\frac{3}{7}$ and 1 are the roots of the equation.
Example 4. Find the number which exceeds sixty-five times its reciprocal by 64.

Let \( x \) be the required number.

Then, by the condition of the problem,

\[
x - \frac{65}{x} = 64.
\]

Multiplying both sides by \( x \), we have

\[
x^2 - 65 = 64x,
\]

or, \( x^2 - 64x - 65 = 0 \), \hspace{1cm} \text{[by transposition]}

or, \( (x - 65)(x + 1) = 0 \), \hspace{1cm} \text{[by factorisation]}

\[=\]

either \( x - 65 = 0 \) \hspace{1cm} \text{i.e.,} \hspace{1cm} x = 65

or, \( x + 1 = 0 \) \hspace{1cm} \text{i.e.,} \hspace{1cm} x = -1

Hence, the required number is either 65, or, -1.

EXERCISE 69

Solve the following equations:

1. \( 3x^2 - 12x + 1 = 6x - 23. \)

2. \( 4x^2 - 4x = 30. \)

3. \( x + 2 - \frac{6}{x + 2} = 1. \)

4. \( x^2 + 3x - 52 = 0. \)

5. \( x^2 - \frac{5}{4}x - 4 = 0. \)

6. \( 6x^2 + 5x - 4 = 0. \)

7. \( 3(x - 2)^2 = 18 + (3x + 1). \)

8. \( x - \frac{x^2 - 8}{x^2 + 5} = 2. \)

9. \( \frac{21x^2 - 16}{3x^2 - 4} = 7x - 5. \)

10. \( x^2 - (a + b)x + ab = 0. \)

11. Find two numbers whose product is equal to 399 and sum is equal to 40.

12. The sum of a number and its square is eight times the next higher number; find the number.

13. Find the number whose square exceeds ten times itself by 96.

14. Find the number which exceeds 12 by as much as thirty-nine times its reciprocal falls short of 4.

15. The difference between the ages of a man and his son is 25 years now. If the product of the numbers denoting their ages, ten years back, be 150, find the present age of the father.

16. The length of a rectangular garden of area 100 sq. metres exceeds its breadth by 15 metres. Find the cost of fencing it by wire-net the price of which is Rs. 1 50 P. per metre.
MISCELLANEOUS EXERCISES IV

I

1. Define **Highest Common Factor** and **Lowest Common Multiple** of two or more algebraical expressions. Find the H.C.F. and L.C.M. of $36x^2a^4c^6$, $24xy^2a^3b^4$ and $240y^2a^8b^5c$.

2. Factorise the following expressions and find their H.C.F.:
   \[ x^2 - 6x + 9 \text{ and } 4x^2 - 11x - 3. \]

3. Find the L.C.M. of
   \[ \frac{ab - ac - b^2 + bc}{a^2 - 12ac - 4a^3 - 9c^2}. \]

4. Solve the equation:
   \[
   \frac{2(x-1)}{5} + \frac{15}{2}(1 - \frac{x}{3}) + \frac{19}{10} = \frac{9}{5} \left( \frac{x}{6} - \frac{1}{3} \right).
   \]

5. If $2s = a + b + c$, show that
   \[
   \frac{2bc + (b^2 + c^2 - a^2)}{2bc - (b^2 + c^2 - a^2)} = \frac{s(s-a)}{(s-b)(s-c)}.
   \]

6. Reduce the following to its simplest form:
   \[
   \frac{x^4}{x^2 - 1} - \frac{x^4}{x^2 + 1} - \frac{1}{x^2 - 1} + \frac{1}{x^2 + 1}.
   \]

7. Solve $ax + 1 = by + 1 = ay + bx$.

8. One pipe can fill a cistern in $a$ hours; another can do it in $b$ hours; in what time could the two running together fill it? And if a third pipe could empty the cistern in $c$ hours, how long would it take to do this if the first two were running at the same time?

II

1. Find the H.C.F. of $7x^3 - 26x + 15$ and $5x(x-1) + 3(3x - 11) - 24$.

2. Find the L.C.M. of $x^3 + bx^2 + ax + ab$ and $x^3 - (a - b)x - ab$.

3. Reduce the following to their simplest forms:
   (i) \( \frac{(3x^2y^2 - 3x^2y^4)^3}{(2x^2y - 2xy^2)^2} \);  (ii) \( \frac{3(x^2 - x - 30)(x^2 - 9x + 14)}{(x^2 - 13x + 42)(x^2 + 3x - 10)} \).

4. Find the value of
   \[
   \frac{x+y}{x-y} + \frac{x-y}{x+y}, \text{ when } x = a^2 + b^2 \text{ and } y = a^2 - b^2.
   \]

5. Simplify \( \frac{(2x - 9)^2 - (x - 6)^2}{9(x^2 - 10x + 25)} + \frac{2(x - 3)^2}{9(x^2 - 8x + 15)} \).
6. What value of \( x \) will make the product of \( 2x+1 \) and \( x+1 \) less than the product of \( 2x+3 \) and \( x+3 \) by 20?

7. Find the value of \( x \), when
\[
\frac{5}{7}(2x-11)-\frac{3}{4}(x-5)=\frac{x}{3}-(10-x).
\]

8. Solve \( ax+by=a^2 \) and \( \frac{a+x}{b}-b+y=0 \).

III

1. Find the H.C.F. of \( a^2x^3+a^2-2abx^2+b^2x+ab^3-2a^2b \) and \( 2a^2x^3-5a^2x^2+3a^2-2b^2x^4+5a^2b^2x^2-3a^2b^2 \).

2. Find the L.C.M. of \( x^5+x^4+x^3+x^2+x+1 \) and \( x^5-x^4+x^3-x^2+x-1 \).

3. Find the H.C.F. of \( x^3-9, (x+3)^2 \) and \( x^3+x-6 \). [O. U. 1910]

4. State and prove the rule for finding the Lowest Common Multiple of two algebraical expressions. [B. U. 1902]

Find the L.C.M. of \( x^2+(a+b)x+ab \), \( x^2-b^2 \) and \( x^2+(a-b)x-ab \).

5. Simplify \( \frac{1}{4} \left( \frac{x^3}{x^2+x-6} - \frac{x-5}{x^2-3x-10} \right) - \frac{1}{x^2+4} \).

6. Solve \( ax+y=x+by=\frac{1}{3}(x+y)+1 \).

7. An income of Rs. 196 is derived from two sums invested, one at 4 per cent., the other at 7 per cent. per annum; if the interest on the former had been 5 per cent., and on the latter 6 per cent., the income derived would have been Rs. 212. Find the sums invested.

8. Find the value of \( x \), when \( 3(x^2-4)=15 \).

IV

1. Define H.C.F. and L.C.M. of two or more algebraical expressions.

2. Find the H.C.F. of \( x^2-y^2, x^2-2xy+y^2 \) and \( x^2-y^2 \); and show that when their L.C.M. is divided by \( x^2+xy+y^2 \), the quotient is \( (x-y)(x^2-y^2) \).

3. Find the defect of \( \frac{x+6}{x^2+5x-6} \) from \( \frac{x+5}{x^2+3x-10} \).
4. Simplify \( \frac{1}{m^2 + m + 1} + \frac{2m}{m^2 + m^2 + 1} \).

5. Show that \((x+y)^3 - (y+x)^3 = 3(x-y)(x+y)(y+x) + \frac{3}{2}(x-y)^3\).

6. A number of three digits has 5 in the units’ place and the middle figure is half the sum of the other two; if 108 be added to the number, the hundreds’ figure will take the units’ place, and the units’ the tens’. Find the number.

7. If 3 be added to the numerator and denominator of a certain fraction, the fraction becomes \(\frac{3}{7}\); if 5 be subtracted from the numerator and denominator, it becomes \(\frac{1}{3}\). Find the fraction.

8. Solve \(5(x^2 - 3x + 11) + 3(x^2 + 2x + 4) = 3(3x^2 - 3x + 1)\).

V

1. Find the H.C.F. of \(x^4 - (a^2 + b^2)x^2 + a^2b^2\) and \(x^6 - (a + b)^2x^2 + 2ab(a + b)x - a^2b^2\).

2. Find the L.C.M. of \(35x^2 - 11x - 6\) and \(40x^2 - 29x + 3\).

3. Reduce to simplest form:

\[
\left(\frac{2x}{x+y} - \frac{x^2}{x^2 - y^2} + \frac{2y}{x-y}\right) \times \left(\frac{1}{x} + \frac{1}{y}\right) + \left(\frac{3}{x-y} - \frac{2}{x} + \frac{1}{y}\right).
\]

4. Simplify \(\frac{a^2 + bc + ca + ab}{a^3 + 2bc + 2ca + ab} \times \frac{a^3 + 8b^3}{a^2 + a^2c^2 + 6ac^2 + 4c^4}\).

5. Show that \(\frac{x+2}{1+x+x^2} - \frac{x-2}{1-x+x^2} - \frac{2x^2-4}{1-x^2+x^4} = \frac{4x^4+8}{x^2+x^4+1}\).

6. A and B travel together 120 kilometres by rail. A takes a return ticket for which he has to pay one fare and a half. Coming back they find that A has travelled cheaper than B by 50 paise for every 100 kilometres. Show that the fare per kilometre is 2 paise.

7. The expression \(ax + b\) is equal to 13 when \(x\) is 5, and to 29 when \(x\) is 13. Show that the value of the expression is 4 when \(x\) is 5.

8. The defect of 4 from twice the square of a number is 28. Find the number.

VI

1. Find the H.C.F. of \(2x^2 + x - 10, x^2 - 5x + 6\) and \(x^2 - 3x + 2\).

2. Find the L.C.M. of \(ax^2 - (a^2 + ab)x + a^2b\), \(bx^2 - (b^2 + bc)x + b^2c\) and \(cx^2 - (c^2 + ac)x + c^2a\).
3. There are two quantities $a$ and $b$ of which the L.C.M. is $a$, and the G.C.M. is $y$; if $x + y = ma + \frac{b}{m}$, show that $x^2 + y^2 = ma^2 + \frac{b^2}{m^2}$.

4. Simplify $\frac{x(x^2 - y^2)}{x^2 + xy + y^2} + \frac{y(y^2 - x^2)}{y^2 + yz + z^2} + \frac{z(x^2 - y^2)}{z^2 + zx + x^2}$.

5. If $x = \frac{a}{a + b}$ and $y = \frac{b}{a + b}$, show that
   
   (i) $\frac{x^2 + y^2}{x^2 - y^2} = \frac{a^2 + b^2}{a^2 - b^2}$;
   (ii) $\frac{x^2 - y^2}{x^2 + y^2} = \frac{a^2 - b^2}{a^2 + b^2}$.

6. Solve $\frac{1}{4}(7x - 5) + \frac{1}{4}(34x + 10) - \frac{1}{4}(3x - 2)(5x - 3) = \frac{4}{4} - x(3 + 15x) - 18$.

7. A market-woman bought apples at three for a rupee and as many more at four for a rupee; and thinking to make her money again, she sold them at seven for two rupees. She lost, however, three rupees by the business. How much did she sell them for?

8. Solve $(2x + 3)(x - 5) + (x + 5)(3x + 1) = 34 + (x + 4)(x + 5)$.

**VII**

1. Find the H.C.F. of $a(a + 1)x^2 + x - a(a - 1)$ and $a(a + 2)x^3 + 2x^2 - a^2 + 1$.

2. Find the L.C.M. of $ab - ac + bc - b^2$, $bc - ab + ac - a^2$ and $ac - bc + ab - a^2$.

3. The H.C.F. and L.C.M. of two numbers $x$ and $y$ are respectively 3 and 105; if $x + y = 36$, prove that
   
   \[ \frac{1}{x} + \frac{1}{y} = \frac{1}{35}. \]

4. Simplify $\frac{1}{(x-1)(x-2)} + \frac{1}{(x-2)(x-3)} + \frac{1}{(x-3)(x-1)}$.

5. Find the value of $\frac{x + y}{x - y}$ when $x = \frac{a + b}{a - b}$ and $y = \frac{a - b}{a + b}$.

6. Show that if a number formed by two digits is four times the sum of its digits, the number formed by interchanging the digits is seven times their sum.

7. Solve $\begin{cases} 3x + 20 = 4y - 10 \\ 4(x - 1) - 3(y - 3) = 0 \end{cases}$ \quad [C. U. 1895]

8. Find the number, the square of which exceeds 7 by as much as the square of half the number falls short of 13.