TABLE 10.2

<table>
<thead>
<tr>
<th>Member</th>
<th>Length in cm.</th>
<th>Force $F$</th>
<th>$k$</th>
<th>$k^2$</th>
<th>$Fk\ell$</th>
<th>$k\ell^2$</th>
<th>Final force $F + kX$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>300</td>
<td>0</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{9}{16}$</td>
<td>0</td>
<td>$\frac{2700}{16}$</td>
<td>$-256.8$</td>
</tr>
<tr>
<td>$CD$</td>
<td>300</td>
<td>750</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{9}{16}$</td>
<td>$\frac{6750000}{4}$</td>
<td>$\frac{2700}{16}$</td>
<td>443.2</td>
</tr>
<tr>
<td>$DA$</td>
<td>400</td>
<td>0</td>
<td>1</td>
<td></td>
<td>0</td>
<td>$\frac{400}{16}$</td>
<td>$-342.5$</td>
</tr>
<tr>
<td>$AC$</td>
<td>500</td>
<td>-1250</td>
<td>$\frac{5}{4}$</td>
<td>$\frac{25}{16}$</td>
<td>$\frac{3125000}{4}$</td>
<td>$\frac{12500}{10}$</td>
<td>-822</td>
</tr>
<tr>
<td>$BD$</td>
<td>500</td>
<td>0</td>
<td>$\frac{5}{4}$</td>
<td>$\frac{25}{16}$</td>
<td>0</td>
<td>$\frac{12500}{16}$</td>
<td>+428</td>
</tr>
</tbody>
</table>

$\Sigma \frac{Fk\ell}{AE} \quad + \frac{\Sigma k^2\ell + \ell_0}{AE \cdot E}$

Area and $E$ are same for all members

$X = \frac{\Sigma Fk\ell}{\Sigma k^2\ell + \ell_0}$

$X = \frac{3,800,000}{3,800,000 \cdot 16} + 400$

$X = -342.5$ kg.

Final force in member is $F + kX$. Forces in all the members are given in Table 10.2.

Ex. 10.3. Find the forces in all the members of the frame shown in Fig. 10.6. Area of each top chord member and vertical is 8 cm$^2$. and area of each bottom chord member and diagonal is 6 cm$^2$.

Solution. Member $LU_4$ is taken as redundant member. Determinate frame is shown in Fig. 10.6 (b). Forces in various members due to loading are shown in Table 10.3. Unit forces are
applied at $L_2$ and $U_4$ in directions shown in Fig. 10.6. The forces in various members due to this loading are calculated.

5 Panels at 4m Each

DETERMINATE FRAME

Fig. 10.6

The values of $l$, $A$, $F$, $k$, $\frac{Fkl}{A}$, $\frac{k^4l}{A}$ and final forces in various members are given in Table 10.3.

<table>
<thead>
<tr>
<th>Member</th>
<th>Length</th>
<th>$A$</th>
<th>$F$</th>
<th>$k$</th>
<th>$\frac{Fkl}{A}$</th>
<th>$\frac{k^4l}{A}$</th>
<th>$F + kX$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>6</td>
<td>-3000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-3000</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>6</td>
<td>-6000</td>
<td>$-1/\sqrt{2}$</td>
<td>+2000$\sqrt{2}$</td>
<td>+1/3</td>
<td>-5214.5</td>
</tr>
</tbody>
</table>

(Table Continued)
<table>
<thead>
<tr>
<th>Member</th>
<th>Length</th>
<th>$A$</th>
<th>$F$</th>
<th>$k$</th>
<th>$Fk/A$</th>
<th>$k^2/A$</th>
<th>$F + kX$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>6</td>
<td>-2000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2000</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>8</td>
<td>+3000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+3000</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>8</td>
<td>+8000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+8000</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>8</td>
<td>+4000</td>
<td>-1/\sqrt{2}</td>
<td>-1000\sqrt{2}</td>
<td>1/4</td>
<td>+4785.5</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>8</td>
<td>+4000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+4000</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>8</td>
<td>+2000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+2000</td>
</tr>
<tr>
<td>11</td>
<td>4</td>
<td>8</td>
<td>+1000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+3000</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>8</td>
<td>+3000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+3000</td>
</tr>
<tr>
<td>13</td>
<td>4</td>
<td>8</td>
<td>+3000</td>
<td>-1/\sqrt{2}</td>
<td>-750\sqrt{2}</td>
<td>1/4</td>
<td>+3785.5</td>
</tr>
<tr>
<td>14</td>
<td>4</td>
<td>8</td>
<td>0</td>
<td>-1/\sqrt{2}</td>
<td>0</td>
<td>0</td>
<td>+785.5</td>
</tr>
<tr>
<td>15</td>
<td>4</td>
<td>8</td>
<td>+2000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+2000</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>8</td>
<td>+2000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+2000</td>
</tr>
<tr>
<td>17</td>
<td>$4\sqrt{2}$</td>
<td>6</td>
<td>-3000\sqrt{2}</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-4242</td>
</tr>
<tr>
<td>18</td>
<td>$4\sqrt{2}$</td>
<td>6</td>
<td>-3000\sqrt{2}</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-4242</td>
</tr>
<tr>
<td>19</td>
<td>$4\sqrt{2}$</td>
<td>6</td>
<td>$1+2000\sqrt{2}$</td>
<td>+1</td>
<td>+8000/3</td>
<td>$2\sqrt{2}/3$</td>
<td>+1717</td>
</tr>
<tr>
<td>20</td>
<td>$4\sqrt{2}$</td>
<td>6</td>
<td>$-2000\sqrt{2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2828</td>
</tr>
<tr>
<td>21</td>
<td>$4\sqrt{2}$</td>
<td>6</td>
<td>$-2000\sqrt{2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2828</td>
</tr>
</tbody>
</table>

\[
\Sigma \begin{align*}
&= +8000/3 \\
&= +2\sqrt{2}/3 \\
&= +300/\sqrt{2} \\
&= +192/\sqrt{17} \\
&= +1.776
\end{align*}
\]
Analysis of Structures

\[ X = - \frac{F_k l}{AE} - \frac{k l}{AE + A_0 E} \]

\[ \frac{3020.17}{1.776 + 0.942} = -1111 \text{ kg.} \]

Ex. 10.4. Find the forces in all the members of the continuous truss shown in Fig. 10.7 (a). All the members have same cross-sectional area and are of the same material.

Solution. The frame is made determinate by removing support at B as shown in Fig. 10.7(b). The forces in various members of the determinate frame due to external loading are found. As the frame is symmetrical and symmetrically loaded, forces are found for half of the frame only. The forces in various members are given in Table 10.4 and are denoted by \( F \).

Next, unit load is applied at B to the determinate frame and forces in various members are determined. The force in a member due to unit load is denoted by \( k \).
Table 10.4 gives values of $F$, $k$, $Fkl$, $k^2l$ for various members.

<table>
<thead>
<tr>
<th>Member</th>
<th>$l \times m$</th>
<th>$F$</th>
<th>$k$</th>
<th>$Fkl$</th>
<th>$k^2l$</th>
<th>Final forces $F + kX$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>-2000</td>
<td>-1</td>
<td>+3000</td>
<td>3/4</td>
<td>-448</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>-4000</td>
<td>-1</td>
<td>+12000</td>
<td>3</td>
<td>-896</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>+2000</td>
<td>+1</td>
<td>+3000</td>
<td>3/4</td>
<td>+448</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>+4000</td>
<td>+1</td>
<td>+12000</td>
<td>3</td>
<td>+556</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>+4000</td>
<td>+3/2</td>
<td>+18000</td>
<td>27/4</td>
<td>-656</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>+2000</td>
<td>+1</td>
<td>+3000</td>
<td>3/4</td>
<td>+448</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>+2000</td>
<td>+1</td>
<td>+3000</td>
<td>3/4</td>
<td>+448</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>3/4</td>
<td>-1552</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>$3\sqrt{2}$</td>
<td>-2000</td>
<td>-2/2</td>
<td>-6000</td>
<td>3/2</td>
<td>-448√2</td>
</tr>
<tr>
<td>12</td>
<td>$3\sqrt{2}$</td>
<td>-2000</td>
<td>-2/2</td>
<td>-6000</td>
<td>3/2</td>
<td>-448√2</td>
</tr>
<tr>
<td>13</td>
<td>$3\sqrt{2}$</td>
<td>0</td>
<td>-2/2</td>
<td>0</td>
<td>3/2</td>
<td>+1552√2</td>
</tr>
</tbody>
</table>

$$X = -\frac{\Sigma Fkl}{\Sigma \frac{AE}{k^4l}}$$

$$X = \frac{\Sigma Fkl}{\Sigma \frac{AE}{k^4l}}$$

As $AE$ for all members is same

$$X = \frac{\Sigma Fkl}{\Sigma k^4l}$$
\[ X = -3104 \text{ kg.} \]

The minus sign indicates that the reaction at B is upwards and not downwards. Final forces in all members are given in Table 10.4.

**Ex. 10.5.** Find the forces in all the members of the frame shown in Fig. 10.8 (a) due to vertical settlement of 1 cm at support B. All the members have same cross-sectional area of 20 cm². \( E \) for all members is \( 2 \times 10^6 \text{ kg/cm}^2 \).

**Solution.** Settlement of B will cause downward reaction at B. Let \( R_B \) be reaction at B.

\[ R_B = \Sigma \frac{k^2l}{AE} \]

The frame is made determinate by removing support at B and unit vertical load is applied at B. Let the force in a member due to unit load at B be \( k \).

The vertical displacement of joint B due to unit load will be

\[ \Sigma \frac{k^2l}{AE} = \Sigma \frac{k^2l}{AE} \]

If the reaction at B is \( R_B \). Vertical displacement of B will be

\[ R_B \Sigma \frac{k^2l}{AE} = 8 \text{ cm.} \]

\[ R_B = \frac{8}{\Sigma \frac{k^2l}{AE}} \]
The above result can also be obtained by using equation 10.3, of article 10.2.

\[ \chi = R_E = - \frac{Fk_l}{AE} - \frac{\Sigma k^2l}{AE} \]

As there is no loading on the frame \( F = 0 \)

\[ \chi = R_E = \frac{+\delta}{\Sigma k^2l} \frac{1}{AE} \]

The values of \( k, l, k^2l \) for various members are given in Table 10.5.

<table>
<thead>
<tr>
<th>Member</th>
<th>( l ) in m</th>
<th>( k^2l )</th>
<th>( kR_E ) Force in member</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>-2 3</td>
<td>4/3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>-4/3</td>
<td>16/3</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>-2/3</td>
<td>4/3</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>-1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>+2/3</td>
<td>4/3</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>+4/3</td>
<td>16/3</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>+4/3</td>
<td>16/3</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>+1</td>
<td>3</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>+2/3</td>
<td>4/3</td>
</tr>
</tbody>
</table>

(Table Continued)
<table>
<thead>
<tr>
<th>Member</th>
<th>$l$ in $m$</th>
<th>$k$</th>
<th>$k^2l$</th>
<th>$kR_B$ Force in member</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>3</td>
<td>$+1/3$</td>
<td>$1/3$</td>
<td>$+3307$</td>
</tr>
<tr>
<td>13</td>
<td>3</td>
<td>$+2/3$</td>
<td>$4/3$</td>
<td>$+6614$</td>
</tr>
<tr>
<td>14</td>
<td>$3\sqrt{2}$</td>
<td>$-2\sqrt{2}/3$</td>
<td>$8\sqrt{2}/3$</td>
<td>$-6614\sqrt{2}$</td>
</tr>
<tr>
<td>15</td>
<td>3</td>
<td>$+2/3$</td>
<td>$4/3$</td>
<td>$+6614$</td>
</tr>
<tr>
<td>16</td>
<td>$3\sqrt{2}$</td>
<td>$-2\sqrt{2}/3$</td>
<td>$8\sqrt{2}/3$</td>
<td>$-6614\sqrt{2}$</td>
</tr>
<tr>
<td>17</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>18</td>
<td>$3\sqrt{2}$</td>
<td>$-\sqrt{2}/3$</td>
<td>$2\sqrt{2}/3$</td>
<td>$-3307\sqrt{2}$</td>
</tr>
<tr>
<td>19</td>
<td>3</td>
<td>$+1/3$</td>
<td>$1/3$</td>
<td>$+3307$</td>
</tr>
<tr>
<td>20</td>
<td>$3\sqrt{2}$</td>
<td>$-\sqrt{2}/3$</td>
<td>$2\sqrt{2}/3$</td>
<td>$-3307\sqrt{2}$</td>
</tr>
<tr>
<td>21</td>
<td>3</td>
<td>$+1/3$</td>
<td>$1/3$</td>
<td>$+3307$</td>
</tr>
<tr>
<td>22</td>
<td>$3\sqrt{2}$</td>
<td>$-\sqrt{2}/3$</td>
<td>$2\sqrt{2}/3$</td>
<td>$-3307\sqrt{2}$</td>
</tr>
<tr>
<td>23</td>
<td>3</td>
<td>$+1/3$</td>
<td>$1/3$</td>
<td>$+3307$</td>
</tr>
<tr>
<td>24</td>
<td>$3\sqrt{2}$</td>
<td>$-\sqrt{2}/3$</td>
<td>$2\sqrt{2}/3$</td>
<td>$-3307\sqrt{2}$</td>
</tr>
<tr>
<td>25</td>
<td>3</td>
<td>$+1/3$</td>
<td>$1/3$</td>
<td>$+3307$</td>
</tr>
</tbody>
</table>

$$\Sigma \frac{k^2l}{AE} = 1 \text{ cm.}$$

$$R_B \times \frac{40.312 \times 10^5}{20 \times 2 \times 10^6} = 1$$

$$R_B = \frac{4 \times 10^5}{40.312} = 9922 \text{ kg.}$$

Final forces in all members are given in Table 10.5.
**Ex. 10.6.** The truss shown in Fig. 10.9 is hinged at A, supported on rollers at D and C moves between vertical guides. A load of 8000 kg. is applied at C as shown in the figure. Determine the horizontal force at guide C and vertical displacement of C. The figures in parenthesis show the cross-sectional areas of members in cm$^2$. $E=2 \times 10^6$ kg/cm$^2$.

**Solution.** The frame shown is externally redundant to single degree. There are four unknown reaction components, two at A, one at D, and one at C. Horizontal movement of C is zero. The frame is made determine by making end C free as shown in Fig. 10.9 (b). The force in a member of determinate frame is denoted by $F$. Forces in various members of determinate frame due to loading are given in Table 10.6.

Next unit horizontal load is applied at C and forces in various members are calculated. The force in a member due to unit horizontal load is denoted by $k$.

Horizontal movement of C as shown in Fig. 10.9 (b) is

$$\sum \frac{F_{kl}}{AE}$$

Horizontal movement of C due to unit load is

$$\sum \frac{k_{2l}}{AE}$$

Let $H$ be actual horizontal reaction at C.

$$\sum \frac{F_{kl}}{AE} + HE \frac{k_{2l}}{AE} = 0$$

as horizontal movement of C is zero.
\[ H = - \frac{\sum \frac{F_{kl}}{AE}}{\sum \frac{k^2l}{AE}} \]

The values of \( F, k, \frac{F_{kl}}{A} \) and \( \frac{k^2l}{A} \) for various members are given in Table 10.6.

### Table 10.6

<table>
<thead>
<tr>
<th>Member</th>
<th>( l ) in m</th>
<th>( A )</th>
<th>( F )</th>
<th>( k )</th>
<th>( \frac{F_{kl}}{A} )</th>
<th>( \frac{k^2l}{A} )</th>
<th>Force ( P = F + kH )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>12</td>
<td>+8000</td>
<td>-1/\sqrt{3}</td>
<td>-2000/\sqrt{3}</td>
<td>1/12</td>
<td>+11,428</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>6</td>
<td>-8000</td>
<td>-1/\sqrt{3}</td>
<td>+4000/\sqrt{3}</td>
<td>1/6</td>
<td>-4572</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
<td>-8000</td>
<td>-1/\sqrt{3}</td>
<td>+4000/\sqrt{3}</td>
<td>1/6</td>
<td>-4572</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>12</td>
<td>+8000</td>
<td>-1/\sqrt{3}</td>
<td>-2000/\sqrt{3}</td>
<td>1/12</td>
<td>+11,428</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>12</td>
<td>-8000</td>
<td>+1/\sqrt{3}</td>
<td>+2000/\sqrt{3}</td>
<td>1/12</td>
<td>+4572</td>
</tr>
</tbody>
</table>

\[ \Sigma \frac{6000}{\sqrt{3}} = 7.12 \]

\[ II = - \frac{\sum \frac{F_{kl}}{AE}}{\sum \frac{k^2l}{AE}} = - \frac{\sum \frac{F_{kl}}{A}}{\sum \frac{k^2l}{A}} \]

\[ 6000/\sqrt{3} \]

\[ 7/12 \]

\[ H = - 5938.3 \text{ kg.} \]

Forces for all members are given in column 8 of Table 10.6.

To find vertical displacement of \( C \), unit vertical load is applied at \( C \) and forces in various members are calculated. When load of 8000 kg. is applied at \( C \), force in a member is \( P \) and is given in Table 10.6. When unit load is applied at \( C \), force in a member will be \( k' = \frac{P}{8000} \). These forces are given in column 9 of Table 10.6.

Values of \( \frac{P_{kl}}{A} \) for various members are calculated and are given in column 10 of Table 10.6.
Vertical deflection of $C$,

$$\Delta = \sum \frac{P_k l}{AE}$$

$$= \frac{11428.54}{2 \times 10^4} \times 100 \text{ cm.}$$

$$= 0.8714 \text{ cm.}$$

10.3. Frames with more than one degree of redundancy.

The analysis of frames with more than one degree of redundancy is carried out in a manner similar to frames with one degree.
of redundancy. The frame is made determinate by removing redundant members and reactions and forces in various members of determinate frame due to external loading are found. Let the force in member be $F$.

Next unit forces are applied at the joints of the redundant member and along the redundant reactions separately as shown in Fig. 10-10 (c) to (e).

Let the force in a member due to unit load applied separately be $k$, $k'$, $k''$ and so on.

Total force in a member will be

$$F + kX + k'Y + \ldots$$

where $X$, $Y$ \ldots are forces in redundant members.

Total strain energy of the frame, including strain energy in redundant member is

$$U = \sum \frac{(F + kX + k'Y + k''Z)lk}{2AE} + \frac{X^2l_x}{2AE} + \frac{Y^2l_y}{2AE} + \frac{Z^2l_z}{2AE}$$

For strain energy to be minimum

$$\frac{\partial U}{\partial X} = \frac{\partial U}{\partial Y} = \frac{\partial U}{\partial Z} = 0$$

in case there is no yielding of supports.

$$\frac{\partial U}{\partial X} = \sum \frac{(F + kX + k'Y + k''Z)lk}{AE} + \frac{Xl_x}{A_E} = 0 \quad \ldots (1)$$

$$\frac{\partial U}{\partial Y} = \sum \frac{(F + kX + k'Y + k''Z)lk'}{AE} + \frac{Yl_y}{A_E} = 0 \quad \ldots (2)$$

$$\frac{\partial U}{\partial Z} = \sum \frac{(F + kX + k'Y + k''Z)lk''}{AE} + \frac{Zl_z}{A_E} = 0 \quad \ldots (3)$$

By solving equations (1), (2) and (3) values of $X$, $Y$ and $Z$ can be evaluated and forces in all members can be evaluated.

In case of more than three redundant members, there will be as many equations as the redundant forces and by solving these simultaneous equations, the forces in all members can be found.

**Ex. 10-7.** Find the forces in all the members of the frame shown in Fig. 10-11. The figures in parenthesis show the cross sectional areas of the members in cm$^2$.

**Solution.** The frame is indeterminate to second degree as there are 8 joints and 15 members. Determinate frame should have $2j - 3 = 8 \times 2 - 3 = 13$ members. The frame is made determinate by removing members $FD$ and $DH$. The determinate frame is shown in Fig. 10-11 (b). The forces in determinate frame due to external loading are found and are denoted by $F$, given in Table 10-7.

Next unit loads are applied at $F$ and $D$ in direction $FD$ as shown in Fig. 10-11 (c) and forces in all members computed. These forces are denoted by $k$ and are given in Table 10-7.
Next unit loads are applied at $D$ and $H$ in direction $DH$ as shown in Fig. 10.11 (d) and forces in all members computed. These forces are denoted by $k'$ and are given in Table 10.7, which also gives the values of $\frac{Fkl}{A}$, $\frac{Fk'l}{A}$, $k'l$, $k'^2l$, $kk'l$.
<table>
<thead>
<tr>
<th>Members</th>
<th>m in m</th>
<th>A</th>
<th>F</th>
<th>k</th>
<th>k'</th>
<th>$\frac{Fkl}{A}$</th>
<th>$\frac{Fk\ell}{A}$</th>
<th>$\frac{kl}{A}$</th>
<th>$\frac{kll}{A}$</th>
<th>$F + kX + k'Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>6</td>
<td>-3000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-3000</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>6</td>
<td>-3000</td>
<td>-1/2</td>
<td>0</td>
<td>0</td>
<td>+1000/2</td>
<td>0</td>
<td>-1/2</td>
<td>-4031</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>6</td>
<td>-5000</td>
<td>0</td>
<td>-1/2</td>
<td>0</td>
<td>0</td>
<td>+500/3</td>
<td>1/3</td>
<td>-4340</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>6</td>
<td>-5000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-5000</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>8</td>
<td>6000</td>
<td>-1/2</td>
<td>0</td>
<td>-1500/2</td>
<td>0</td>
<td>1/3</td>
<td>0</td>
<td>4969</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>8</td>
<td>6000</td>
<td>0</td>
<td>-1/2</td>
<td>0</td>
<td>-1500/2</td>
<td>1/3</td>
<td>0</td>
<td>6600</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>-1/√2</td>
<td>-1/√2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>-------</td>
<td>-------</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>4</td>
<td>-4000</td>
<td>0</td>
<td>-1/√2</td>
<td>0</td>
<td>+2000√2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>-------</td>
<td>---</td>
<td>-------</td>
<td>---</td>
<td>--------</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>10</td>
<td>4√2</td>
<td>8</td>
<td>-3000√2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>-------</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>11</td>
<td>4√2</td>
<td>6</td>
<td>-3000√2</td>
<td>+1</td>
<td>0</td>
<td>-4000</td>
<td>0</td>
<td>2√2/3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>-------</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>12</td>
<td>4√2</td>
<td>6</td>
<td>-1000√2</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>-1000√3</td>
<td>0</td>
<td>2√2/3</td>
<td>0</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>-------</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>13</td>
<td>4√2</td>
<td>8</td>
<td>+5000√2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
\begin{array}{c|c|c|c|cc|c|c|c|c|c}
\hline
-4000 & -4000 & -3 & 19/12 & 19/12 & 1
\hline
-600√2 & -600√2 & +2√2/3 & +2√2 & +3
\hline
\Sigma = -470 & +173 & 2.526 & 2.526
\hline
\end{array}
\]
Strain energy due to axial force in the beam

\[ U_a = \frac{F^2l}{2AE} = \frac{(2P)^2 \times 12}{2AE} \]

\[ \frac{\partial U_a}{\partial P} = \frac{2P \times 2 \times 2 \times 12}{2AE} = \frac{48P}{AE} = \frac{48P}{240 \times 10^5 \times 1 \times 10^6 \times (100)^3} = \frac{48P}{240 \times 10^5} \]

Strain energy \( U_a \) in two struts will be

\[ U_a = 2 \times \frac{P^2l}{2AE} \]

\[ \frac{\partial U_a}{\partial P} = \frac{2P}{AE} \]

\[ = \frac{2P \times 2}{100 \times 1 \times 10^6} = \frac{4P}{100 \times 10^5} \]

Strain energy \( U_4 \) in the rods \( AE, EF \) and \( FD \) is

\[ U_4 = 2 \times \frac{(\sqrt{5}P)^2l_1}{2AE} + \frac{(2P)^2 \times l_3}{2AE} \]

\[ \frac{\partial U_4}{\partial P} = \frac{2 \times 2 \times \sqrt{5}P \times \sqrt{5} \times 2 \times \sqrt{5}}{2 \times \pi \times 2 \times 10^6} + \frac{2 \times 2 \times 2 \times 2 \times 4}{2 \times \pi \times 2 \times 10} \]

\[ = \frac{10 \sqrt{5}P}{\pi \times 10^6} + \frac{8P}{\pi \times 10^6} \]

Total strain energy of the structure

\[ U = U_1 + U_2 + U_3 + U_4 \]

\[ \frac{\partial U}{\partial P} = \frac{\partial U_1}{\partial P} + \frac{\partial U_2}{\partial P} + \frac{\partial U_3}{\partial P} + \frac{\partial U_4}{\partial P} \]

As total strain energy is to be minimum \( \frac{\partial U}{\partial P} = 0 \)

\[ \frac{1}{E_I} \left[ \int_0^4 (1600 - P)x(-x)dx + \int_0^4 (800 - P)x(-x)dx \right] \]

\[ + \frac{1}{E_I} \left[ \int_4^8 (800x - 4P)(-4)dx + \frac{48P}{240 \times 10^5} + \frac{4P}{100 \times 10^5} \right] \]

\[ = \frac{10 \sqrt{5}P}{\pi \times 10^6} + \frac{8P}{\pi \times 10^6} = 0 \]

\[ \frac{(P-1600)}{E_I} \left[ \frac{x^3}{3} \right]_0^4 + \frac{(P-800)}{E_I} \left[ \frac{x^3}{3} \right]_0^4 + 4 \frac{4Px - 800x^2}{2} \right]_4 \]

\[ + \frac{2P}{10^6} + \frac{0.4P}{10^6} + \frac{10 \sqrt{5}P}{\pi \times 10^6} + \frac{8P}{\pi \times 10^6} = 0 \]
\[
\frac{(P-1600)}{8000 \times 1 \times 10^8} \times \frac{64}{3} + \frac{(P-800)}{8000 \times 1 \times 10^8} \times \frac{64}{3} (100)^2
\]

\[+ \frac{4}{8000 \times 1 \times 10^8} \frac{[4P(8-4)-400(64-16)]}{(100)^2}\]

\[+ \frac{2.4P}{10^8} + \frac{10 \sqrt{P}}{\pi \times 10^8} + \frac{8P}{\pi \times 10^8} = 0\]

\[\therefore \frac{800}{3} (P-1600) + \frac{800}{3} (P-800) + 50 [16P-19,200]\]

\[+ 24P + 7.12P + 2.542P = 0\]

\[\therefore \frac{800}{3} P - 12,80,000 + \frac{800}{3} P - \frac{640,000}{3} + 800P - 960,000 + 12.062P\]

\[1345.395P = 16,00,000\]

\[P = \frac{16,00,000}{1345.395} = 11.90 \text{ kg.}\]

Force in tie rod \(EF = 2 \times 1190 = 2380 \text{ kg.}\)

Force in tie rods \(AE\) and \(FD = \sqrt{5}P\)

\[= \sqrt{5} \times 1190\]

\[= 2661 \text{ kg.}\]

Ex. 10.11. A continuous mast \(ABC\) is pinned at \(A\) and stayed at \(B\) and \(C\) by guys \(BD\) and \(CD\) as shown in Fig. 10.17 (a). The flexural stiffness of the mast is \(6 \times 10^8\) kg. cm.\(^2\) and the extensional stiffness of each guy is \(4 \times 10^8\) kg. Find the force in each guy when a horizontal load of \(W\) is applied at \(C\).

Solution. Consider free body of mast \(ABC\). Let the horizontal component of the force in guy \(CD\) be \(T_1\) and horizontal component of force in guy \(BD\) be \(T_2\). The axial force in guy \(CD\) will be \(\frac{T_1}{\sin \theta_1} = \sqrt{5}T_1\) and the axial force in the guy \(BD\) will be \(\frac{T_2}{\sin \theta_2} = \sqrt{2}T_2\).

The forces acting on the mast \(ABC\) are shown in the free body diagram in Fig. 10.17 (c).

Taking moments about \(A\)

![Fig. 10.17](a)
\[ W \times 10^{-T_1} = 0 \]
\[ \therefore T_1 = W - \frac{P}{2} \]

Total strain energy of the structure will be the sum of strain energy due to moment and axial force in the mast ABC and strain energy due to axial force in guys BD and CD. Strain energy due to axial force in the mast must be neglected as cross-sectional area of mast is not given. For the mast strain energy due to bending,

\[ U_1 = \int_0^1 \frac{M^2}{2EI} \, dx \]
\[ \therefore \frac{\partial U_1}{\partial T_2} = \frac{1}{EI} \int_0^1 M \frac{\partial M}{\partial T_2} \, dx. \]

The value of \( M, \frac{\partial M}{\partial T_2} \) and limits of integration for two portions of the mast are given in Table 10.11.

<table>
<thead>
<tr>
<th>Origin at</th>
<th>Portion</th>
<th>B.M.</th>
<th>( \frac{\partial M}{\partial T_2} )</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>BC</td>
<td>( (W - T_1)x = \frac{T_2x}{2} )</td>
<td>( x/2 )</td>
<td>0–5</td>
</tr>
<tr>
<td>A</td>
<td>BA</td>
<td>( (W - T_1 - T_2)x = -\frac{T_2x}{2} )</td>
<td>(-x/2)</td>
<td>0–5</td>
</tr>
</tbody>
</table>

\[ \frac{\partial U_1}{\partial T_2} = \frac{1}{EI} \int_0^5 \frac{T_2}{2} \times x \times \frac{x}{2} \, dx \]
\[ + \frac{1}{EI} \int_0^5 \left( -\frac{T_2}{2} \right) \frac{-x}{2} \, dx \]
\[ = \frac{T_2}{2EI} \int_0^5 x^2 \, dx \]

Strain energy due to axial force in CD is

\[ U_a = \left[ \sqrt{5} \left( W - \frac{T_2}{2} \right) \right] \frac{5\sqrt{5}}{2AE} \]
\[ \frac{\partial U_2}{\partial T_2} = \sqrt{5} \left( W - \frac{T_2}{2} \right) \left( -\frac{\sqrt{5}}{2} \right) 5\sqrt{5} \]

\[ \frac{\partial U_2}{\partial T_2} = \left( W - \frac{T_2}{2} \right) 25\sqrt{5} \]

\[ \frac{\partial U_2}{\partial T_2} = \frac{10\sqrt{2} T_2}{AE} \]

Strain energy due to axial force in BD is

\[ U_2 = \frac{(\sqrt{2} T_2)^2}{2AE} 5\sqrt{2} \]

\[ \frac{\partial U_2}{\partial T_2} = \frac{10\sqrt{2} T_2}{AE} \]

Total strain energy \( U = U_1 + U_2 + U_3 \)

Strain energy is to be minimum.

\[ \frac{\partial U}{\partial T_2} = 0 \]

\[ \frac{T_2}{2EI} \int_0^5 x^2 \, dx - \frac{\left( W - \frac{T_2}{2} \right) 25\sqrt{5}}{2AE} + \frac{10\sqrt{2} T_2}{AE} = 0 \]

\[ \frac{T_3}{2 \times 6 \times 10^{10}} \left[ \frac{x^3}{3} \right]_0^5 - \frac{\left( W - \frac{T_2}{2} \right) 25\sqrt{5}}{2 \times 2 \times 10^6} + \frac{10\sqrt{2} T_2}{4 \times 10^8} = 0 \]

(100)

\[ 125 \]

\[ \frac{T_3}{30} = \frac{25\sqrt{5}}{8} W + \frac{25\sqrt{5}}{16} T_3 + \frac{5\sqrt{2}}{2} T_2 \]

\[ \frac{125}{9} T_3 + \frac{25\sqrt{5}}{4} T_3 + 10\sqrt{2} T_2 = \frac{25\sqrt{5}}{2} W \]

\[ 42009 \]

\[ T_2 = 27.96 W \]

\[ T_3 = 0.6654 W \]

\[ T_1 = W - \frac{T_2}{2} = W - 0.3327 W \]

\[ = 0.6673 W \]

Force in guy CD = \( \sqrt{5} T_1 = \sqrt{5} \times 0.6673 W \)

\[ = 1.492 W \]

Force in guy BD = \( \sqrt{2} T_2 = \sqrt{2} \times 0.6654 W \)

\[ = 0.9408 W \]

Ex 10.12. The structure shown in Fig. 10.18 consists of a bent cantilever ABC of flexural rigidity EI and a rigid cross member DE each end of which is pinned to the cantilever. Calculate the load in DE and vertical deflection of C.

Solution. Let the force in the member DE be \( \sqrt{2} T \), so that horizontal and vertical components of the force will be T each. The free body diagram of the bent ABC is shown in Fig. 10.18 (6) and free body of ADB is shown in Fig. 10.18 (c).
As the member $DE$ is rigid, strain energy due to direct forces in $DE$ is zero. A strain energy is to be minimum, partial derivative of strain energy with respect to $T$ will be zero.

Partial derivative of strain energy with respect to $W$ will give deflection of point $C$.

![Diagram](image)

Fig. 10.18

Table 10.12 gives values of $M$, $\frac{\partial M}{\partial T}$, and $\frac{\partial M}{\partial W}$ and limits of integration for portions $CE$, $EB$, $BD$ and $DA$.

<table>
<thead>
<tr>
<th>Portion</th>
<th>Origin at</th>
<th>Moment</th>
<th>Moment rewritten</th>
<th>$\frac{\partial M}{\partial T}$</th>
<th>$\frac{\partial M}{\partial W}$</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CE$</td>
<td>$E$</td>
<td>$W \times x$</td>
<td>$W \times x$</td>
<td>$-$</td>
<td>$x$</td>
<td>$0-2$</td>
</tr>
<tr>
<td>$EB$</td>
<td>$E$</td>
<td>$W \times x - T(x-2)$</td>
<td>$W - T(x-2)$</td>
<td>$-(x-2)$</td>
<td>$x$</td>
<td>$2-4$</td>
</tr>
<tr>
<td>$BD$</td>
<td>$B$</td>
<td>$W \times 4 - T \times 2 + T \times x$</td>
<td>$4W + T(x-2)$</td>
<td>$(x-2)$</td>
<td>$4$</td>
<td>$0-2$</td>
</tr>
<tr>
<td>$DA$</td>
<td>$B$</td>
<td>$W \times 4 - T \times 2 + T \times x - T(x-2)$</td>
<td>$4W$</td>
<td>$-$</td>
<td>$4$</td>
<td>$2-6$</td>
</tr>
</tbody>
</table>

$$\frac{\partial U}{\partial T} = \frac{1}{EI} \left[ \int_0^4 [W - T(x-2)][- (x-2)] dx \right]$$

$$+ \frac{1}{EI} \left[ \int_0^4 [4W + T(x-2)][(x-2)] dx \right]$$

$$- \frac{1}{EI} \left[ \int_0^4 [T(x^2 - 4x + 4) - W(x^3 - 2x)] dx \right]$$

$$+ \frac{1}{EI} \left[ \int_0^2 T(x^3 - 4x + 4) dx + \frac{1}{EI} \int_0^2 4W(x - 2) dx \right]$$
\[
T = \frac{44}{3} W
\]

\[
T = 11 W.
\]

Force in DE = \(\sqrt{2}T = \frac{11\sqrt{2}}{4}W\).

\[
\delta_c = \frac{352}{3EI} W
\]

Ex. 10'13. The beam shown in Fig. 10'19 is simply supported at \(A\), hinged at \(A\) and supported by pin-jointed members PQ, PR and PC. I for the beam is 80,000 cm\(^4\) and areas of members PQ, PR and PC are 15 cm\(^2\), 25 cm\(^2\) and 15 cm\(^2\) respectively. \(E = 2 \times 10^6\) kg/cm\(^2\) for members. If the beam carries uniformly distributed load of \(a/m\) throughout the span, determine the deflection of point C.
Solution. Let \( F \) be force in member \( CP \).
Consider free body of beam \( AB \).

Downward deflection of \( C \) due to uniformly distributed load = \( \frac{5}{384} \frac{wl^4}{EI} \).

- Upward deflection of \( C \) due to force \( F \) is \( \frac{Fl^3}{48EI} \).

Net downward deflection of \( C \) is
\[
\delta = \frac{5}{384} \frac{wl^4}{EI} - \frac{Fl^3}{48EI}
\]

Downward deflection of point \( C \) will be same for beam \( ACB \) and frame \( QRPC \).

Downward deflection of \( C \) is given by
\[
\sum \frac{Pkl}{AE}
\]

Force in \( PQ \) is \(-\frac{\sqrt{17}}{2} F\)
and in \( PR \) is \(+\frac{\sqrt{17}}{2} F\)

\[
\delta = \frac{F \times 1 \times 2}{15 \times 2 \times 10^6} + \left( -\frac{\sqrt{17}}{2} \frac{F}{2} \right) \left( -\frac{\sqrt{17}}{2} \right) \times \sqrt{17}
\]

\[
+ \left( \frac{\sqrt{17}}{2} \right) \left( \frac{\sqrt{17}}{2} \right) \times \sqrt{17}
\]

Equating two values of \( \delta \)
\[
\frac{2F}{30 \times 10^6} + \frac{17\sqrt{17} F}{120 \times 10^6} + \frac{17\sqrt{17} F}{200 \times 10^6}
\]

\[
= \frac{5}{384} \frac{wl^4}{EI} - \frac{Fl^3}{48EI}
\]

\[
\frac{2F}{30 \times 10^6} + \frac{17\sqrt{17} F}{120 \times 10^6} + \frac{17\sqrt{17} F}{200 \times 10^6}
\]

\[
\frac{5}{384} \times \frac{4000 \times 8}{80,000 \times 2 \times 10^6} - \frac{F \times 8^3}{48 \times 80,000 \times 2 \times 10^6 (100)^3}
\]
\[
\frac{2F}{15} + \frac{17\sqrt{17}}{60} F + \frac{17\sqrt{17}}{100} F + \frac{4}{3} F = \frac{5}{384} \times 4000 \times 8^8
\]
\[
3.3333F = \frac{80,000}{3}
\]
\[
F = \frac{8000}{3}
\]

Deflection of \(C\) \[
\frac{5 \times \frac{wL^4}{EI}}{384 \times \frac{EI}{48EI}} = \frac{5 \times 4000 \times 8^4}{384 \times 48EI} [8000 \times 8^2]
\]
\[
= \frac{8^8}{48EI} \left[ \frac{5 \times 32,000}{8} - 8000 \right]
\]
\[
= 32 \times \frac{80,000 \times 2 \times 10^6}{(100)^3} [20,000 - 8,000]
\]
\[
= 8 \times 2 \times 10^5 \times 12,000
\]
\[
= \frac{8}{1000} \text{ m.}
\]
\[
= 0.8 \text{ cm.}
\]

**Ex. 10.14.** A spring consists of a segment of a thin circular ring of radius 10 cm. and uniform flexural stiffness \(1 \times 10^8\) kg. cm.² held rigidly at \(A\) while the end \(B\), which is free to rotate, is constrained by frictionless slides to move along the vertical diameter \(BC\). Estimate the vertical displacement of \(B\) when a weight of 1000 kg. is placed there.

**Solution.** When the spring is loaded by a horizontal force \(H\) is brought into play at \(B\). This reaction is indeterminate. To find the value of \(H\), partial derivative of strain energy with respect to \(H\) is equated to zero as there is no deflection of the segment in the direction of \(H\).

The B.M. at any angular position \(\theta\) of the ring is
\[
M_r = WR \sin \theta - HR(1 - \cos \theta)
\]
\[
\frac{\partial M}{\partial H} = -R(1 - \cos \theta)
\]
\[
\frac{\partial U}{\partial H} = \frac{1}{EI} \int_0^L M \frac{\partial M}{\partial H} \, ds
\]
\[
\frac{1}{MY} \int_0^L [WR \sin \theta - HR(1 - \cos \theta)][-B(1 - \cos \theta)] R d\theta = 0
\]
\[= \frac{R^3}{EI} \int_0^{\frac{3\pi}{2}} [H(1 - \cos \theta)^2 - W(1 - \cos \theta)\sin \theta] = 0\]

\[\therefore \int_0^{\frac{3\pi}{2}} [H(1 - 2\cos \theta + \cos^2 \theta) - W(\sin \theta - \sin \theta \cos \theta)] = 0\]

\[\int_0^{\frac{3\pi}{2}} \left[ (1 - 2\cos \theta + \frac{1 + \cos 2\theta}{2}) - W \left( \sin \theta - \frac{\sin 2\theta}{2} \right) \right]^2 = 0\]

\[\therefore H \left[ \frac{3}{2} \theta - 2\sin \theta + \frac{\sin 2\theta}{4} \right]_0^{\frac{3\pi}{2}} - W \left[ -\cos \theta + \frac{\cos 2\theta}{4} \right]_0^{\frac{3\pi}{2}} = 0\]

\[\therefore H \left[ \frac{3}{2} \times \frac{3\pi}{2} + 2 \right] - W \left[ -\frac{1}{4} + 1 - \frac{1}{4} \right] = 0\]

\[\therefore H \left[ \theta \pi + 8 \right] = W \times \frac{1}{2}\]

\[\therefore H = \frac{2W}{8 + 9\pi}\]

Vertical displacement of \(B\) is given by \(\frac{\partial U}{\partial W}\).

\[\frac{\partial M}{\partial W} = R \sin \theta\]

\[\therefore \frac{\partial U}{\partial W} = \frac{1}{EI} \int_0^{\frac{3\pi}{2}} [WR \sin \theta - HR(1 - \cos \theta)]R \sin \theta \, Rd\theta\]

\[= \frac{R^3}{EI} \int_0^{\frac{3\pi}{2}} \left[ W \sin^2 \theta - H(\sin \theta - \sin \theta \cos \theta) \right] d\theta\]

\[= \frac{R^3}{EI} \int_0^{\frac{3\pi}{2}} \left[ \frac{W(1 - \cos 2\theta)}{2} - H \left( \sin \theta - \frac{\sin 2\theta}{2} \right) \right] d\theta\]

\[= \frac{R^3}{EI} \left[ W \left( \frac{\theta}{2} - \frac{\sin 2\theta}{2} \right) - H \left( -\cos \theta + \frac{\cos 2\theta}{2} \right) \right]_0^{\frac{3\pi}{2}}\]

\[= \frac{R^3}{EI} \left[ W \times \frac{3\pi}{4} - H \left( -\frac{1}{4} + 1 - \frac{1}{4} \right) \right]\]

\[\therefore \delta_B = \frac{R^3}{EI} \left[ W \times \frac{3\pi}{4} - \frac{H}{2} \right]\]

Substituting the value of \(H\)

\[\delta_B = \frac{R^3}{EI} \left[ \frac{3\pi}{4} W - \frac{W}{8 + 9\pi} \right]\]

\(R = 10\) cm. \quad and \quad \(W = 100\) kg.
\[ \delta_B = \frac{1000}{1 \times 10^3} \times 100 \left( \frac{3\pi}{4} \frac{1}{8+9\pi} \right) \]
\[ = \frac{1}{10} \times 2.33 \]
\[ = 0.233 \text{ cm.} \]

Problems

1. Three members BA, CA and DA are connected as shown in Fig. 10.21. The members AB and AD have area of 4 cm\(^2\) and AC has area of 6 cm\(^2\). Determine the forces in the members.

![Diagram of a triangular structure with forces and dimensions labeled.](image)

\[ \sqrt{3000} \text{ kg} \]

Fig. 10.21

[Ans. \( AB = AC = -931 \text{ kg.} \), \( AD = -1683.6 \text{ kg.} \)]

2. In the continuous truss shown in Fig. 10.22 all members have same cross-sectional area of 10 cm\(^2\). Find the reaction at central support B.

![Diagram of a continuous truss structure with loads and dimensions labeled.](image)

Fig. 10.22

[Ans. \( R_B = 776 \text{ kg.} \)]

3. The queen post truss of Ex. 10.10 is loaded with two equal loads of 2400 kg at B and C. Find the force in the struts \( E \) and \( CF \).

[Ans. \( P = 2378 \text{ kg.} \)]

4. Vertical pile of length 8 m. is hinged at the base A and d by pin connections BD and CE as shown in Fig. 10.23. Determine the force in each of the members BD and CE (neglect the effect of thrust in the pile). \( EA \) for strut = \( 30 \times 10^4 \text{ kg.} \)

\( EI = 5 \times 10^{10} \text{ kg. cm}^2 \).

[Ans. \( P_{BD} = 8585 \text{ kg.} \), \( P_{CE} = 19,369 \text{ kg.} \)]
5. If the right-angled connection at $A$ in the structure shown in Fig. 10·24 were converted into a rigid joint by welding, determine the percentage reduction in the load carried by tie bar. Both angle members are of the same section having $I = 400$ cm$^4$. Area of tie bar = 1 cm$^2$.

6. A portal frame shown in Fig. 10·25 is subjected to the action of two horizontal forces each of 2000 kg at joints $D$ and $E$. The members $ABC$ and $DEF$ are continuous and all joints are pinned. Considering bending effects only show that the horizontal reactions at $A$ and $F$ are equal. Obtain the force in the member $HK$.

[Ans. $P_{HK} = 2000$ kg.]

7. A rectangular frame shown in Fig. 10·26 consists of five bars pinned together at $A$, $B$, $C$ and $D$ and suspended in vertical plane from a rigid support. All the members are of steel and 12 cm$^3$ in cross-sectional area. Find force in the member $AD$ due to 2 loads at $A$ and $D$. If the temperature is raised to 20°C, the
distance $BO$ remaining unchanged, find force in each of the bars. 
$E = 2 \times 10^6$ kg/cm$^2$, $\alpha = 11 \times 10^{-6}$/°C.

8. Find the force in the members of redundant frame shown in Fig. 10.27. All the members have same cross-sectional area.

![Fig 10.27](image)

9. Find the forces in all the members of the frame shown in Fig. 10.28. All the members have the same cross-sectional area.

![Fig 10.28](image)

[Ans. $P_{BC} = 708$ kg.  
$P_{CB} = 206$ kg.  
$P_{BE} = 292$ kg.]

10. Find the forces in all the members of the frame shown in Fig. 10.29. The figures in parenthesis show the cross-sectional areas of the members.

![Fig 10.29](image)
11. The member $BC$ of redundant frame $ABCD$ shown in Fig. 10.30 is too long by $0.15$ cm. If the frame is acted upon by a horizontal force at $B$ as shown, determine the forces in all the members of the frame. The figures in brackets show the cross-sectional areas of the members in cm$^2$. $E = 2 \times 10^8$ kg/cm$^2$.

![Diagram of Frame ABCD]

Fig. 10.30

12. A vertical steel frame, rectangular in shape, consists of two vertical members, 6 m high and 4.5 m apart, a horizontal member at their top and diagonal bracings. The sectional area of each vertical member is 25 sq cm, that of horizontal member is 18.75 sq cm, and that of each diagonal member is 31 sq cm. The frame is hinged to the grand at its bottom joints. It is subjected to a horizontal force of 50 t at the level of the top horizontal member. Calculate the stresses in various members of the frame.

(AMIE Nov 1969)

13. A truss $ABCD$ has both its ends $A$ and $D$ provided with hinged supports and carries two load 5 t and 10 t at $B$ and $C$ respectively as shown in fig 10.31. Treating member $BC$ as redundant, calculate the forces in all the members. All members have the same cross-sectional area and are made of the same material.

(AMIE May 1971)

![Diagram of Truss ABCD]

Fig. 10.31
TWO HINGED ARCHES

11.1. Types of Arches
Single span arches are of the following type:

(i) Three-hinged arch.

(ii) Two-hinged arch.

(iii) Fixed arch.

(iv) Fixed arch with hinge at crown.

Fig. 11.1 (a) shows three hinged arch. This type of arch is statically determinate and has been dealt with in Chapter 13 of Analysis of Structures Vol. I.

Figs. 11.1 (b), (c) and (d) show a two-hinged arch, a fixed arch and a fixed arch with a hinge at the crown.

A two-hinged arch has four unknown reaction components, two at each support. There are only three equations of statics and thus two-hinged arch is statically indeterminate to first degree.

11.2. Analysis of Two-hinged Arches
Fig. 11.2 shows two hinged arch having supports at different levels. At A there will be reaction components $V_A$ and $H_A$. At B also there will be two reaction components $V_B$ and $H_B$. Using three equations of statics

\[ \Sigma V = 0 \text{ gives } V_A + V_B = P \quad \ldots(i) \]

\[ \Sigma H = 0 \text{ gives } H_A = H_B \quad \ldots(ii) \]

Taking moments about A,

\[ \Sigma M = 0 \text{ gives, } V_B \times l + H_B \times H_1 = P \times a \]

\[ \frac{P \times a - H_B \times H_1}{l} \quad \ldots(iii) \]

439
Fourth equation can be obtained from deflection consideration. Partial derivation of strain energy with respect to \( H_A \) or \( H_B \) will give deflection in the direction of \( H_A \) or \( H_B \). If the supports do not yield, the partial derivative of strain energy with respect to horizontal thrust will be zero. If the support yields by \( \delta \) in direction of horizontal thrust, the partial derivative of strain energy with respect to horizontal thrust will be \( \delta \).

In case the supports of a two-hinged arch are at the same level, the vertical reactions are statically determinate and can be found by taking moments about one support. Thus in the two-hinged arch shown in Fig. 11·3, the vertical reactions at \( A \) and \( B \) will be \( \frac{P(l-a)}{l} \) and \( \frac{P_a}{l} \) respectively. These reactions are same as for a simply supported beam of span \( l \). The B.M. at a section will be due to vertical loads and due to horizontal reactions. The B.M. at any section due to vertical loads will be same as for a simply supported beam and due to horizontal thrust B.M. will be \( -H \times y \) where \( y \) is the vertical height of the section from the springings. If \( M_s \) is the bending moment at a section as a simply supported beam, the total B.M. at the section will be

\[
M_s = M_s - H \times y
\]

Neglecting strain energy due to direct thrust, total strain energy due to bending moment will be

\[
U = \int_0^l \frac{(M_s - H y)^2}{2EI} \, ds
\]

\[
\frac{\partial U}{\partial H} = 0 \text{ if the supports do not yield}
\]
\[
\frac{\partial U}{\partial H} = \int_0^l \frac{(M_z - H y) \times (-y)}{E I} \, ds
\]
\[
\therefore \quad -\delta = \int_0^l \frac{M_z \, y \, ds}{E I} + \int_0^l \frac{H y^2 \, ds}{E I}
\]
\[
H = \int_0^l \frac{M_z \, y \, ds}{E I} - \int_0^l \frac{y^2 \, ds}{E I} \quad \ldots (11.1)
\]

In case any support yields by \( \delta \)

\[
\frac{\partial U}{\partial H} = \int_0^l \frac{(M_z - H y) \times (-y)}{E I} \, ds
\]
\[
\therefore \quad -\delta = \int_0^l \frac{M_z \, y \, ds}{E I} + \int_0^l \frac{H y^2 \, ds}{E I}
\]
\[
H = \int_0^l \frac{M_z \, y \, ds}{E I} - \int_0^l \frac{y^2 \, ds}{E I} \quad \ldots (11.2)
\]

In case moment of inertia at any section of the arch varies as secant of the angle of slope \( \theta \) at the section, \( I = I_o \sec \theta \) where \( I_o \) is moment of inertia at the crown.

\[
dx \sec \theta
\]

Equation 11.1 becomes \( H = \int_0^l \frac{M_z \, y \, dx}{E I_o} - \int_0^l \frac{y^2 \, dx}{E I_o} \quad \ldots (11.3) \)

If one of the supports yields by \( \delta \)

\[
H = \int_0^l \frac{M_z \, y \, dx}{E I_o} - \delta \]
\[
H = \int_0^l \frac{M_z \, y \, dx - E I_o \delta}{y^2 \, dx} \quad \ldots (11.4)
\]

A two-hinged arch can also be analysed by unit load method.

The arch is made determinate by putting one end on rollers as shown in Fig. 11.4 (a). Let \( M_z \) be the B.M. at any section of determinate arch. Applying unit load at roller end, B.M. at any section \( m = -y \).

The horizontal movement of roller end due to load is,

\[
\delta_1 = \int_0^l \frac{M_z \, y \, ds}{E I}
\]
Horizontal movement $\delta_2$ of the arch due to unit load at roller end will be

$$\delta_2 = \int_0^l \frac{y^2 ds}{EI}$$

If $H$ is the horizontal thrust at hinge, the deflection of roller end due to force $H$ will be

$$H \int_0^l \frac{y^2 ds}{EI}$$

In case there is no change in span i.e., supports do not yield

$$H \int_0^l \frac{y^2 ds}{EI} = \int_0^l \frac{M_s y ds}{EI}$$

$$H = \frac{\int_0^l \frac{M_s y ds}{EI}}{\int_0^l \frac{y^2 ds}{EI}}$$

If

$$I = I_c \sec \theta$$

$$H = \frac{\int_0^l \frac{M_s y dx}{EI}}{\int_0^l \frac{y^2 dx}{EI_c}} = \frac{\int_0^l M_s y dx}{\int_0^l y^2 dx}$$

11.3. Shear Force and Normal Thrust in Two-hinged Arch.

Consider a portion of two-hinged arch. Let $\theta$ be the slope at the section where B.M. and S.F. are required.

Let $V$ be the vertical force and $H$ the horizontal thrust resisted at the section as shown in Fig. 11.5.

Shear force at the section

$$F = V \cos \theta - H \sin \theta \quad \ldots (11.5)$$

Normal thrust at the section

$$N = V \sin \theta + H \cos \theta \quad \ldots (11.6)$$

11.4. Effect of Rib-shortening in Two-hinged Arch.

In the analysis of two-hinged arch in article 11.2, the effect of rib-shortening due to direct thrust in rib was not considered. The effect of normal thrust in the arch is to shorten the rib of the arch and thus release part of the horizontal thrust. The normal thrust at any section of the arch is given by

$$N = V \sin \theta + H \cos \theta$$
Strain energy due to normal thrust

\[ U_1 = \int_0^l \frac{N' ds}{2EA} \]

Total strain energy due to bending and normal thrust will be

\[ U = \int_0^l \frac{(M_x - H y)^2}{2EI} ds + \int_0^l \frac{(V \sin \theta + H \cos \theta)^2}{2EA} ds \]

\[ \frac{\partial U}{\partial H} = \text{Displacement in the direction of horizontal thrust. In case supports do not yield } \frac{\partial U}{\partial H} = 0. \text{ If one of the supports yields by } \delta, \]

\[ \frac{\partial U}{\partial H} = -\delta \]

\[ \frac{\partial U}{\partial H} = \int_0^l \frac{(M_x - H y)(-y) ds}{EI} \]

\[ + \int_0^l \frac{V \sin \theta + H \cos \theta}{EA} \cos \theta ds \]

\[ - \delta \text{ (if the supports yield)} \]

\[ - \int_0^l \frac{M_y ds}{EI} - H \int_0^l \frac{y ds}{EI} + \int_0^l \frac{V \sin \theta \cos \theta ds}{EA} \]

\[ + H \int_0^l \frac{\cos^2 \theta ds}{EA} = -\delta \]

\[ H = \int_0^l \frac{M_y ds}{EI} - \delta \]

Neglecting terms containing \( V \) as they are small

\[ H = \int_0^l \frac{M_y}{EI} ds - \delta \]

If the cross-section of the arch varies such that moment of inertia at any section \( I \) is \( I_x \sec \theta \), where \( I_x \) is moment of inertia at the crown and \( \theta \) is the slope at the section,

\[ H = \int_0^l \frac{M_y dx}{EI_x} - \delta \]

\[ + \int_0^l \frac{y dx}{EI_x} + \int_0^l \frac{\cos^2 \theta dx}{EA} \]

In case of arches with small rise to span ratio, the slope at any section is small and \( \cos \theta \) is approximately equal to unity. Putting area at any section approximately equal to area at crown \( A_c \),

\[ H = \int_0^l \frac{M_y dx}{EI_c} \]

\[ \frac{y dx}{EI_c} + \frac{l}{EA_c} \]

\[ \ldots (11.7) \]
If the supports yield by an amount \( k \) for unit thrust, \( \delta = kH \).

Eqn. (11.7) becomes

\[
H = \frac{\int_0^l M_s y dx}{\frac{EI_o}{E I_e} + \frac{I_e}{EA_e}} - kH
\]

\[
= \frac{\int_0^l M_s y dx}{\frac{EI_o}{E I_e} + \frac{I_e}{EA_e}} + k
\]

\[
\therefore \quad H = \frac{\int_0^l M_s y dx}{\frac{EI_o}{E I_e} + \frac{I_e}{EA_e}} + k \quad \ldots (11.8)
\]

11.5. Parabolic Arch Subjected to Concentrated Load

Consider a parabolic arch of span \( l \) and central rise \( h \). Let a single concentrated load \( W \) act at distance \( a \) from one support.

It is assumed that moment of inertia at any section \( I = I_e \sec \theta \) where \( I_e \) is moment of inertia at the crown and \( \theta \) is slope at the section.

The equation of centre line of arch with one support as origin is given by

\[
y = Ax(l - x), \text{ where } A \text{ is a constant.}
\]

At \( x = l/2 \), \( y = h \).

\[
\therefore \quad A = \frac{4h}{l^2}
\]

The equation of centre line of arch becomes

\[
y = \frac{4hx(l - x)}{l^2}.
\]

If the supports of arch do not yield and effect of rib shortening is neglected, horizontal thrust is given by

\[
H = \frac{\int_0^l M_s y dx}{EI} \quad \frac{\int_0^l y^2 dx}{EI} - \frac{\int_0^l M_s y dx}{EI} \quad \frac{\int_0^l y^2 dx}{EI}
\]

The vertical reactions at the abutments will be

\[
R_s = \frac{W(l - a)}{l}
\]

and

\[
R_1 = \frac{Wa}{l}
\]
B.M. at a section at distance $x$ from $A$, considering as simply supported beam is

$$M_x = W(l-a) \times x$$

B.M. at a section at distance $x$ from $B$, considering as simply supported beam, is $M_x = \frac{Wa}{l} \times x$.

$$\int_0^l M_y \, dx = \int_0^a \frac{W(l-a)}{l} \times x \times y \, dx + \int_0^{l-a} \frac{Wa}{l} \times x \times y \, dx$$

$$= \int_0^a \frac{W(l-a)}{l} \times x \times \frac{4hx(l-x)}{l^2} \, dx$$

$$+ \int_0^{l-a} \frac{Wa}{l} \times x \times \frac{4hx(l-x)}{l^2} \, dx$$

$$= \frac{4hW(l-a)}{l^2} \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^a$$

$$+ \frac{4hWa}{l^2} \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^{l-a}$$

$$= \frac{4hW(l-a)}{l^2} \left[ \frac{l}{3} - \frac{a}{4} \right]$$

$$+ \frac{4hWa(l-a)^3}{l^2} \left[ \frac{l}{3} - \frac{(l-a)}{4} \right]$$

$$= \frac{hW(l-a) \cdot a^3 (4l-3a)}{3l^2} + \frac{hWa(l-a)^3(l+3a)}{3l^2}$$

$$= \frac{hW(l-a) \cdot a^3 (4l-3a) + (l-a)^3(l+3a)}{3l^3}$$

$$= \frac{hW(l-a) \cdot a^3 (4l-3a^3 + l^3 + 3a^3 - 5a^3)}{3l^3}$$

$$= \frac{hW(l-a) \cdot a^3 (l^3 + al^3 - a^3)}{3l^3}$$

$$= \frac{hW(l-a) \cdot a^3 (l^3 + al^3 - a^3)}{3l^2}$$

$$\int_0^l y^2 \, dx = \int_0^l \left[ \frac{4hx(l-x)}{l^2} \right]^2 \, dx$$

$$= \frac{16h^2}{l^4} \int_0^l x^2(l^2 - 2lx + x^2) \, dx$$

$$= \frac{16h^2}{l^4} \left[ \frac{x^3}{2} - \frac{2lx^4}{4} + \frac{x^5}{5} \right]_0^l$$

$$= \frac{16h^2}{l} \cdot x^5 \left[ \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right]$$
\[ = 16h^2 \times l \left[ \frac{10 - 15 + 16}{30} \right] \]
\[ = \frac{8}{15} h^2 l \]
\[ H = \frac{\int_0^l M_y dx}{\int_0^l y^2 dx} = \frac{\frac{hW}{3l^2} (l^3 + al - a^2)}{\frac{8}{15} h^2 l} = \frac{5W}{8h} (l - a)(l^2 + al - a^2) \]

Putting \( a = kl \)

\[ H = \frac{5Wl^2(1-k)(1+k-k^2)}{8h} = \frac{5Wl}{8h} (k^4 - 2k^3 + k) \]

(11.10)

The B.M. diagram for the arch is obtained by superposing the B.M. diagram due to \( H \times y \), B.M. diagram due to \( H \times y \), B.M. diagram due to \( H \times y \), B.M. diagram due to \( H \times y \) equal to \( \frac{Wa(l-a)}{l} \). B.M. diagram due to \( H \) will be a parabola.

As the value of \( H \) for a particular position of the load is constant, B.M. diagram due to \( H \) is same as shape of the arch with central ordinate at centre of span as \( H \times n \).

11.1. Parabolic Arch Loaded with Uniformly Distributed Load.

Consider a parabolic arch loaded with uniformly distributed load throughout its span.

The thrust for the parabolic arch loaded with uniformly distributed load coincides with the profile of the arch and thus there is no bending moment and shear force at any section of the arch. The arch will be subjected to only normal thrust.

Vertical reaction at each support = \( \frac{1}{2} \)

\[ \int_0^l M_y dx = \int_0^l \left( \frac{wl}{2} x - \frac{wx^2}{2} \right) dx \]
\[ = \int_0^l \left( \frac{wl}{2} x - \frac{wx^2}{2} \right) \frac{4h(l-x)}{l^2} \]
\[ = \frac{2hw}{l} \int_0^l (lx - x^2)(lx - x^2) dx \]
\[ \int_0^l y^2 \, dx = \frac{8}{15} h^2 l \] (See Art. 11.5)

\[ H = \frac{\int_0^l M \, y \, dx}{\int_0^l y^2 \, dx} = \frac{\frac{hw l^3}{15}}{\frac{8}{15} h^2 l} = \frac{w l^3}{8A} \]
B.M. at any section of the arch = \( M_\alpha = M_\alpha - H x y \)

\[
\begin{align*}
&= \frac{wl}{2} x - \frac{wx^2}{2} - \frac{wl^3}{8h} \times \frac{4hl(l-x)}{l^2} \\
&= \frac{wl}{2} x - \frac{wx^2}{2} - \frac{w}{2} x(l-x) \\
&= 0
\end{align*}
\]

\[\therefore \text{ B.M. at every section of the arch is zero.}\]

S.F. at any section is \( F = V \cos \theta - H \sin \theta \).

Now \( \frac{dy}{dx} = \frac{4h}{l^2} (l-2x) = \tan \theta \)

\[\cos \theta = \frac{\sqrt{16h^2 (l-2x)^2 + 1}}{k} \]

\[\sin \theta = \frac{4h}{l^2} (l-2x)/k \]

\[\begin{align*}
&= \left( \frac{wl}{2} - wx \right) \cos \theta - \frac{wl^2}{8h} \sin \theta \\
&= \left( \frac{wl}{2} - wx \right) \frac{1}{k} - \frac{wl^2}{8h} \times \frac{4h}{l^2} (l-2x) \\
&= \frac{w}{k} \left[ \left( \frac{l}{2} - x \right) - \frac{1}{2} (l-2x) \right] \\
&= 0.
\end{align*}\]

\[\therefore \text{ S.F. is zero at every section.}\]

Normal thrust at any section is

\[N = V \sin \theta + H \cos \theta\]

\[\begin{align*}
&= \left( \frac{wl}{2} - wx \right) \frac{4h}{l^2} (l-2x) + \frac{wl^2}{8h} \times \frac{1}{k} \\
&= \frac{w}{k} \left[ \frac{2h}{l^2} (l-2x)^2 + \frac{l^2}{8h} \right] \\
&= \frac{w}{\sqrt{16h^2 (l-2x)^2 + 1}} \left[ \frac{2h}{l^2} (l-2x)^2 + \frac{l^2}{8h} \right]
\end{align*}\]

Thus the parabolic arch loaded with uniformly distributed load throughout the span is subjected to only normal thrust.

**Ex. 11.1.** Find the horizontal thrust for the two-hinged parabolic arch shown in Fig. 11.8 (a). The moment of inertia at any section is \( I_\alpha \), \( sec \theta \) where \( \theta \) is the slope at the section and \( I_\alpha \) is moment of inertia at the crown. Neglect effect of rib shortening.
Solution.

\[ H = \int M_y \, dx \int y^2 \, dx \]

\[ y = \frac{4hx(l-x)}{l^2} = \frac{4 \times 4x(24-x)}{24 \times 24} = \frac{x(24-x)}{36} \]

Vertical reactions are

\[ R_A = \frac{15(24-6)}{24} = 11.25 \, t \]

\[ B.M \text{ diagram} \]

\[ B = \frac{15 \times 6}{24} = 3.75 \, t \]

\[ \int_0^l M_y \, dx = \int_0^6 11.25x \times y \, dx + \int_0^{18} 3.75x \times y \, dx \]

\[ = \int_0^6 11.25x \times \frac{x(24-x)}{36} \, dx + \int_0^{18} 3.75 \times \frac{x(24-x)}{36} \, dx \]

\[ = 11.25 \times \frac{24x^4}{36} - \frac{x^4}{4} \bigg|_0^6 + 3.75 \times \frac{24x^3}{36} - \frac{x^4}{4} \bigg|_0^{18} \]

\[ = 11.25 \times 6 \left[ 8 - \frac{6}{4} \right] + 3.75 \times 18 \left[ 8 - \frac{18}{4} \right] \]
\[ \int y^2 \, dx = \int_0^{24} \left[ \frac{x}{36} \left( 24 - x \right) \right]^2 \, dx \]
\[ = \frac{1}{36 \times 36} \left[ \frac{576}{3} \times \frac{48x^4}{4} + \frac{x^5}{5} \right]_0^{24} \]
\[ = \frac{24^3}{36 \times 36} \times [192 - 288 + 115.2] \]
\[ = \frac{32}{3} \times 192 \]
\[ = 2048 \]
\[ H = 2565 \]
\[ = 12.524 \text{ t.} \]

Rise of arch at 6 m from A
\[ y_e = \frac{6(24-6)}{36} = 3 \text{ m.} \]

B.M. under the load
\[ = 11.25 \times 6 - 12.524 \times 3 \]
\[ = 67.5 - 375.72 = 29.978 \text{ t.m.} \]

B.M. at crown
\[ = 3.75 \times 12 - 12.524 \times 4 \]
\[ = 46 - 50.096 = -5.096 \text{ t.m.} \]

B.M. diagram is shown in Fig. 11.3 (b).

Ex. 11.2. A uniformly distributed load of 4000 kg/m. covers left hand half of the span of a parabolic arch, span 36 m. and central rise 8 m. Determine the position and magnitude of maximum bending moment. Also find shear force and normal thrust at the section. Assume that moment of inertia at a section varies as secant of slope at the section. Neglect effect of rib shortening.

Solution.
\[ y = \frac{4h x(l-x)}{l^3} \]
\[ \frac{4 \times 8x(36-x)}{36 \times 36} = \frac{2x(36+x)}{81} \]
For a parabolic arch
\[ \int y^2 \, dx = \frac{8}{15} \cdot \frac{h^2 l}{2} \]
\[ = \frac{8}{15} \times 8 \times 8 \times 36 \]
\[ = \frac{64 \times 96}{5} \]

B.M. as simply supported beam in portion AC of the arch at distance x from A is given by
\[ M_x = 54,000x - 4000x^2 \]

B.M. as simply supported beam in portion BC of the arch at distance x from B is given by
\[ M_x = 18,000x \]

\[ \int M_x y \, dx = \int_0^{18} 18,000 \times x \times y \, dx + \int_0^{18} [54,000x - 2000x^2] \times y \, dx \]
\[ = \int_0^{18} 18,000 \times x \times \frac{2x(36-x)}{81} \, dx \]
\[ + \int_0^{18} [54,000x - 2000x^2] \times \frac{2x(36-x)}{81} \, dx \]
\[ = \frac{4000}{9} \int_0^{18} x^2(36-x) \, dx \]
\[ + \frac{2}{81} \int_0^{18} [54,000x^2(36-x) - 2000x^2(36-x)] \, dx \]
\[
\begin{align*}
&= \frac{4000}{9} \left[ \frac{36x^6}{3} - \frac{x^4}{4} \right]_{0}^{18} + \frac{2}{81} \left[ \frac{54,000}{3} \left( \frac{36x^6}{3} - \frac{x^4}{4} \right) \right]_{0}^{18} - 2000 \left( \frac{36x^6}{4} - \frac{x^5}{5} \right)_{0}^{18} \\
&= \frac{4000}{9} \left[ 12 \times 18^3 - \frac{18^4}{4} \right]_{0} + \frac{2}{81} \left[ 54,000 \left( 12 \times 18^3 - \frac{18^4}{4} \right) \right]_{0} - 2000 \left( 0 \times 18^4 - \frac{18^5}{5} \right)_{0} \\
&= \frac{4000}{9} \times 18^3 \left( 12 - \frac{9}{2} \right) + \frac{2}{81} \times 1 \cdot 3 \left[ 54,000 \left( 12 - \frac{9}{2} \right) \right] - 2000 \times 18 \left( 9 - \frac{18}{5} \right) \\
&= \frac{4000}{9} \times 18^3 \times \frac{15}{2} + \frac{2}{81} \times 18^3 \times 18,000 \left[ 3 \times \frac{15}{2} + 117 \right] \\
&= 4000 \times 18^3 \left[ \frac{15}{2} + \frac{117}{10} \right] \\
&= 4000 \times 18 \times 36 \times 19.2 \\
\int_{0}^{l} M_y \, y \, dx &= \frac{4000 \times 18 \times 36 \times 19.2}{64 \times 96} \\
&= 40,500 \text{ kg.}
\end{align*}
\]

**Maximum -ve B.M. occurs in portion BC of the arch. B.M. at distance \( x \) from \( B \) is given by**

\[
M_x = 18,000 \times 40,500 \times \frac{2x(36 - x)}{91} \\
= 18,000x - 1000x (36 - x)
\]

For maximum B.M. put \( \frac{dM_x}{dx} = 0 \),

\[
\frac{dM_x}{dx} = 18,000 - 1000 (36 - 2x) = 0.
\]

\[
\therefore 36 - 2x = 18 \\
x = 9 \text{ m}
\]

\[
\therefore \text{ Maximum -ve B.M. occurs at left hand quarter span.}
\]

\[
M_x = 18,000 \times 9 - 1000 \times 9 \times (36 - 9) \\
= -81,000 \text{ kg. m.}
\]

**Maximum +ve B.M. occurs in portion AC of the arch. B.M. at a section distance \( x \) from \( A \) is given by**

\[
M_x = 54,000 \times \frac{x - \frac{4000 \times x^2}{2}}{81} \\
= 54,000x - 2000x^3 - 1000x (36 - x)
\]
For maximum B.M., put \( \frac{dM_x}{dx} = 0 \),
\[
\frac{dM_x}{dx} = 54,000 - 4000x - 1000(36 - 2x) = 0
\]
\[
= 54,000 - 4000x - 36,000 + 2000x = 0
\]
\[
\therefore 2000x = 18,000
\]
\[
x = 9 \text{ m}
\]
\[
\therefore \text{Maximum +ve B.M. occurs at right hand quarter of the span.}
\]
Maximum +ve B.M. = 54,000 \times 9 - 2000 \times 9^2 - 1000 \times 9 \times (36 - 9)
\[
= 486,000 - 162,000 - 243,000
\]
\[
= 81,000 \text{ kg. m.}
\]

**S.F. and Normal Thrust at Quarter Span**

\[
y = \frac{2x(36-x)}{81}, \quad \frac{dy}{dx} = \frac{2}{81}(36 - 2x)
\]

At \( x = 9 \), \( \frac{dy}{dx} = \frac{2}{81} \times 18 = \frac{4}{9} \)

\[
\tan \theta = \frac{x}{y} ; \cos \theta = 0.914 ; \sin \theta = 0.406
\]

S.F. at any section = \( H \sin \theta - V \cos \theta \)

Normal thrust = \( H \cos \theta + V \sin \theta \)

It is evident from Fig. 11.9 (c) and (d) that shear force and normal thrust at left hand quarter point and right hand quarter point will be same.

\[
\text{S.F.} = 40,500 \times 0.4062 - 0.914 \times 18,000
\]
\[
= 0
\]

Normal thrust = 40,500 \times 0.914 + 18,000 \times 0.4062
\[
= 37,017 + 7,308
\]
\[
= 44,325 \text{ kg.}
\]

**Ex. 11.3.** Two hinged arch of the form shown in Fig. 11.10 has constant section throughout. Determine the horizontal thrust, neglecting the effect of rib shortening. Draw B.M. diagram.

**Solution.** Horizontal thrust in the arch is given by
\[
H = \int_0^l \frac{M_y}{EI} ds
\]
\[
= \int_0^l \frac{y^2}{EI} ds
\]

As the arch is symmetrical and symmetrically loaded, values of \( \int_0^l \frac{M_y ds}{EI} \) and \( \int_0^l \frac{y^2 ds}{EI} \) will be 2 \( \int_0^{l/2} \frac{M_y ds}{EI} \) and 2 \( \int_0^{l/2} \frac{y^2 ds}{EI} \) respectively.
In the portion $AB$ of the arch, the rise of the arch at horizontal distance $x$ from $A$ will be $y = \frac{4}{3} x$ and $ds = \frac{5}{3} dx$. B.M. as simply supported beam in this portion of the arch is given by

$$M_s = 18,000 x - \frac{2000 x^2}{2}.$$ 

In the portion $BC$ of the arch, the rise of the arch is constant equal to 4 m. and $ds = dx$.

$$\int_0^l \frac{M_s y}{EI} ds = 2 \int_0^3 \frac{(18,000 - 1000x^2)}{EI} \times \frac{4}{3} x \times \frac{5}{3} dx$$

$$= \frac{40}{9EI} \left[ \int_0^3 (18,000 x^3 - 1000 x^4) dx \right]$$

$$+ \frac{8}{EI} \left[ (18,000 x - 1000 x^2) dx \right]$$

$$= \frac{40}{9EI} \left[ \int_0^3 6000 x^3 - \frac{1000 x^4}{4} \right]$$

$$+ \frac{8}{EI} \left[ 8000 x^2 - \frac{1000 x^3}{3} \right]$$

$$= \frac{40}{9EI} \left[ 6000 \times 3^3 - 1000 \times \frac{3^4}{4} \right]$$

$$+ \frac{8}{EI} \left[ 9000 (9^2 - 3^2) - \frac{1000}{3} (9^3 - 3^3) \right]$$

$$= \frac{40}{9EI} \left[ 18,000 - 2,250 \right] + \frac{8}{EI} \left[ 9000 \times 72 - \frac{1000}{3} \times 702 \right]$$
\[
\int_0^1 \frac{y^2}{EI} \, ds = 2 \int_0^3 \frac{(\frac{1}{3}x)^3}{EI} \times \frac{5}{3} \, dx + 2 \int_3^6 \frac{(4)^3}{EI} \, dx
\]
\[
= \frac{160}{27EI} \left[ \frac{x^3}{3} \right]_0^3 + \frac{32}{EI} \left[ x \right]_0^3
\]
\[
= \frac{160 \times 192}{3EI} + \frac{736}{EI} = \frac{3,942,000}{EI}
\]
\[
H = \frac{3,942,000}{736} \times \frac{3}{3EI} = 16067.9 \text{ kg.}
\]

The B.M. diagram will be sum of \( M_s \) diagram and \( H \times y \) diagram. B.M. diagram is shown in Fig. 11.10 (b).


The effect of temperature change in a two hinged arch is to cause shortening or extension of the curved length of the arch.

Due to temperature rise of \( 't' \) the increase in length of the curved portion, \( \delta s \), will be at \( at \delta s \).

Increase in the horizontal direction will be at \( \delta s \cos \theta \).

Total increase in horizontal span if free to increase
\[
= \int_0^t \frac{at \delta s \cos \theta}{EI} \, ds
\]
\[
= \frac{atl}{EI}
\]

If \( H' \) is the thrust due to temperature rise
\[
\frac{\partial U}{\partial H'} = \text{shortening in span}
\]
\[
= \int_0^t \frac{H' y^3 \, ds}{EI} = \frac{atl}{EI}
\]
\[
H' = \frac{atl}{\frac{1}{2} y'ds}
\]
\[
\text{if } I = I_o \cos \theta, H' = \frac{atl}{\int_0^t EI}
\]

(11.11)
In case there is fall of temperature by \( t^\circ \),

\[ H' = \frac{- \alpha t l}{E I} \]

In general, if the arch is subjected to some loading, temperature rise of \( t^\circ \) and yielding of supports by \( \delta \), horizontal thrust at abutments is given by,

\[ H = \frac{\int_0^l M y ds}{E I} + \alpha t l - \delta \frac{\int_0^l y^2 ds}{E I} \]  

...(11.12)

**Ex. 11.4.** A two hinged parabolic arch of span 60 m. and central rise 6 m. is subjected to a crown load of 4000 kg. Allowing for rib-shortening, temperature rise of 20\(^\circ\)C and yield of each support of 0.006 cm/1000 kg., determine \( H. I_c = 600,000 \text{ cm}^4, A_o = 1000 \text{ cm}^2, E = 1 \times 10^5 \text{ kg/cm}^2, \alpha = 11 \times 10^{-6} \text{C}, I = I_c \text{ sec } \theta. \)**

**Solution.** Horizontal thrust in the arch is given by

\[ H = \frac{\int_0^l M y dx}{E I_c} + \alpha t l + \frac{\int_0^l y^2 dx}{E A_o} + k \]

\[ y = \frac{4hx(l-x)}{l^2} \]

\[ 4 \times 6x(60-x) \]

\[ 60 \times 60 \]

\[ \frac{x}{150(60-x)} \]

**Fig. 11.12**

\[ \int_0^l \frac{M y dx}{E I_c} = \frac{1}{E I_c} \times 2 \int_0^{30} 2000x \times \frac{x}{150} (60-x) dx \]

\[ = \frac{80}{3EI_c} \left[ 30 \int_0^{30} (60x^2 - x^3) dx \right] \]

\[ = \frac{80}{EI_c} \left[ 60 \frac{x^3}{3} - \frac{x^4}{4} \right]_0^{30} \]

\[ = \frac{9,000,000}{E I_c} \]

\[ \int_0^l \frac{y^2 ds}{E I_c} = \frac{8}{15} \frac{h^3 l}{E I_c} \]

\[ = \frac{8}{15} \times 6 \times 6 \times 60 \]

\[ = \frac{32 \times 36}{EI_c} \]
Two Hinged Arches

\[ EI_c = \frac{1 \times 10^5 \times 6,000,000}{(100)^2} = 60 \times 10^5 \text{ kg} \cdot \text{m}^2. \]

\[ E \cdot A_c = 1 \times 10^5 \times 1000 = 1000 \times 10^5 \text{ kg} \cdot \text{m}^2. \]

\[ k = 2 \times 0.008 \times \frac{1}{100} = 0.12 \times 10^{-8} \]

\[ H = \frac{32 \times 36}{60 \times 10^5 + 1000 \times 10^5} + 0.12 \times 10^{-8} \]

\[ = \frac{9,000,000 + 79,200}{1000} = 9,079.00 \]

\[ = 1162.8 \]

\[ = 7805 \text{ kg}. \]

**Ex. 11.5.** A two-hinged parabolic arch has a span of 60 metres and central rise of 10 metres. The moment of inertia at any section is equal to 6,000,000 sec. \( \theta \) cm\(^4\), where 6,000,000 cm\(^4\), is the moment of inertia at the crown and \( \theta \) is the slope of the section. Determine the horizontal movement between the constraints so that under a central point load of 4000 kg, the bending moments at the quarter points and crown are numerically equal. \( E = 1 \times 10^8 \text{ kg/cm}^2 \).

**Solution.** Let \( \Delta \) be the movement of constraints.

Horizontal thrust in the arch is given by

\[ \int_0^l \frac{M_\text{yds}}{EI_c} \cdot \Delta \]

\[ y = \frac{4hx(l-x)}{I} \]

\[ = \frac{4 \times 10 \times x(60-x)}{60 	imes 60} \]

\[ = \frac{x(60-x)}{90} \]

\[ \int_0^l \frac{y^2dx}{EI_c} = \frac{8h^3l}{15 EI} \]

\[ 8 \times \frac{10 \times 10 \times 60}{15 \times \frac{3200}{EI_c}} \]

\[ = \frac{g,572}{\text{kg} \cdot \text{m}}. \]

![Diagram of the arch with forces and moments labeled](image-url)
\[
\int_0^t \frac{M_y dx}{EI_c} = \frac{1}{EI_c} \times 2 \int_0^{30} 2000x \times y dx \\
= \frac{2}{EI_c} \int_0^{30} 2000x \times \frac{x(60-x)dx}{90} \\
= \frac{400}{9EI_c} \left[ \frac{60x^3}{3} - \frac{x^4}{4} \right]_0^{30} \\
= \frac{400}{9EI_c} \left\{ 20 \times 30^3 - \frac{30^4}{4} \right\} \\
= \frac{400}{9EI_c} \times 30^3 [20 - 7.5] \\
= 15,000,000 \frac{EI_c}{3200} - \Delta \\
\\
\therefore \quad H = \frac{15,000,000}{EI_c} \frac{90}{3200} - \Delta \\
\\
B.M. at centre of span \\
= 2000 \times 30 - H \times 10, \text{ this B.M. is } +ve. \\
\\
Rise of arch at quarter span \\
= \frac{15 \times (60-15)}{90} = 7.5 \text{ m.} \\
\\
B.M. at quarter span = 2000 \times 15 - H \times 7.5, \text{ this B.M. is } -ve. \\
\\
B.M. at mid span is to be numerically equal to B.M. at quarter span. \\
2000 \times 30 - 10H = - (2000 \times 15 - H \times 7.5) \\
60,000 + 30,000 = 17.5H \\
H = 90,000 \\
\frac{15,000,000}{EI_c} - \Delta \\
\therefore \quad \frac{90,000}{17.5} = \frac{3200}{EI_c} \\
\\
\therefore \quad EI_c = \frac{6,00,000 \times 110^5}{100 \times 100} = 6 \times 10^7 \text{ kg. m}^2. \\
\\
\therefore \quad \frac{15,000,000}{6 \times 10^7} - \Delta = \frac{90,000}{17.5} \times \frac{3200}{6 \times 10^7} \\
0.25 - \Delta = \frac{28.8}{105} \\
\Delta = 0.25 - 0.27333 \\
\Delta = -0.02333 \text{ m.} \\
\Delta = -2.333 \text{ cm.}
Minus sign indicates that hinges \( A \) and \( B \) should move inwards.

\[ H = 5142.8 \text{ kg}. \]

B.M. diagram is shown in Fig. 11.13.

11.8. Tied Arches.

In a two hinged arch sometimes one end is hinged and other is provided on rollers, with two supports connected by a tie rod as shown in Fig. 11.14.

![Fig. 11.14](image)

When such an arch is loaded the roller end has tendency to move but its movement is restricted due to the restraint provided by the tie rod. The movement of roller end \( B \) will be equal to extension of the tie rod. The tie rod will be subjected to tensile force equal to horizontal thrust in the arch. As seen in article 11.3, the horizontal thrust in an arch with supports yielding and \( I \) varying as \( \sec \theta \), was given by

\[
H = \int_0^l \frac{M_y \, dy}{Et} - \Delta
\]

\[
= \int_0^l \frac{y^2 \, dy}{Et} + \frac{l}{EA_t} + \frac{l}{A_t E_t}
\]

Extension of the rod \( \Delta = \frac{H}{A_t E_t} \times l \) where \( A_t \) is the area of tie rod and \( E_t \) is Young's modulus of the material of the tie rod.

\[
H - \int_0^l \frac{M_y \, dy}{Et} - \Delta \frac{l}{A_t E_t}
\]

\[\therefore\]

\[
H = \int_0^l \frac{y^2 \, dy}{Et} + \frac{l}{EA_t} + \frac{l}{A_t E_t}
\]

Ex. 11.6. A two-hinged parabolic arch of span 30 m. ha., central rise 3 m. The horizontal thrust is taken by a tie rod. Calculate the bending moment at the quarter span due to a crown load of 400 kg. Allow for the extension of the tie rod and rib shortening. Area of rib at crown = 240 cm². \( E \) for arch = 1.5 × 10⁵ kg/cm². \( E \) for tie rod = 2 × 10⁵ kg/cm². \( I = 18,000 \sec^2 \theta \) cm⁴. Area of tie rod = 10 cm².

**Solution.** Horizontal thrust is given by

![Fig. 11.15](image)

\[
H = \int_0^l \frac{M_y \, dy}{Et} - \frac{l}{EA_t} - \frac{l}{A_t E_t}
\]
For parabolic arch,

\[ \int_0^l \frac{y^2 dx}{EI_o} = \frac{8}{15} \frac{h^3 l}{EI_o} \]

\[ = \frac{8}{15} \times \frac{3 \times 3 \times 30}{EI_o} = \frac{144}{EI_o} \]

\[ y = \frac{4h}{l^2} \times \frac{x(30-x)}{30 \times 30} \]

\[ = \frac{x(30-x)}{75} \]

\[ \int_0^l \frac{M_s ydx}{EI_o} = \frac{2}{EI_o} \int_0^2 M_s ydx \]

\[ = \frac{2}{EI_o} \left[ \int_0^{15} 2000x \times \frac{x(30-x)}{75} dx \right] \]

\[ = \frac{160}{3EI_o} \left[ 30 \times \frac{x^3}{3} - \frac{2^4}{4} \right]_0^{15} \]

\[ = \frac{160}{3EI_o} \left[ 19 \times 15^3 - \frac{15^4}{4} \right] \]

\[ = \frac{160}{3EI_o} \times 15^3 \left[ 10 - \frac{15}{4} \right] \]

\[ = \frac{1,125,000}{EI_o} \]

\[ EI_o = \frac{18,000 \times 1.5 \times 10}{(100)^2} = 27 \times 10^4 \]

\[ EA_o = 240 \times 1.5 \times 10^3 = 3600 \times 10^4 \]

\[ E_t A_t = 2 \times 10^6 \times 10 = 2000 \times 10^4 \]

\[ H = \frac{1,125,000}{27 \times 10^4} \]

\[ = \frac{1,125,000}{14 + 0.225 + 0.405} = \frac{1,125,000}{144.63} \]

\[ = 7778 \text{ kg.} \]

Rise at quarter span = \[ \frac{7.5(30-7.5)}{75} \] = 2.25 m.

B.M. at quarter point = \[ 2000 \times \frac{15}{2} = 7778 \times 2.25 \]

\[ = 15,000 - 17,500.5 = -2500.5 \text{ kg. m.} \]
11.9. Symmetrical Circular Arch

Consider a segmental arch radius \( R \) and subtending angle \( 2\alpha \) at the centre.

Central rise of the arch is
\[
h = R - R \cos \alpha = R[1 - \cos \alpha]
\]

Consider an elementary width \( ds \) of the arch at an angle \( \theta \) from the centre line.

Rise of the arch at this section is
\[
y = h - (R - R \cos \theta) = h - R(1 - \cos \theta) = R(1 - \cos \alpha) - R(1 - \cos \theta) = R(\cos \alpha - \cos \theta)
\]

Horizontal distance \( x \) from the support is given by
\[
x = \frac{l}{2} - R \sin \theta = R \sin \alpha - R \sin \theta = R(\sin \alpha - \sin \theta)
\]

B.V. at any section as a simply supported beam will be
\[
M_s = \frac{wI}{2} x - \frac{wx^2}{2} = \frac{wR \sin \alpha}{2} R(\sin \alpha - \sin \theta) - \frac{w}{2} R^2(\sin \alpha - \sin \theta)^2
\]
\[
wR^2 \sin \alpha (\sin \alpha - \sin \theta)
\]
\[
wR^2 (\sin \alpha - \sin \theta)^2
\]

Horizontal thrust in the arch is given by
\[
\tau = \int \frac{M_s y \, ds}{EI}
\]
\[
\tau = \int \frac{y^2 ds}{EI}
\]
\[
\frac{ds}{R \, d\theta}
\]
If arch is of constant moment of inertia

\[ H = \frac{\int_{0}^{l} M_s y \, ds}{\int_{0}^{l} y^2 \, ds} \]

\[ \int_{0}^{2\alpha} \left[ \frac{wR^2}{2} \sin \alpha (\sin \alpha - \sin \theta) - \frac{wR^2}{2} (\sin \alpha - \sin \theta)^2 \right] \]

\[ \therefore H = \frac{R(\cos \theta - \cos \alpha)}{\int_{0}^{2\alpha} R^2(\cos \theta - \cos \alpha)^2 \, Rd\theta} \int_{0}^{2\alpha} \sin \alpha (\sin \alpha - \sin \theta) - (\sin \alpha - \sin \theta)'(\cos \theta - \cos \alpha) \, d\theta \]

\[ \int_{0}^{2\alpha} (\cos \theta - \cos \alpha)^2 \, d\theta \]

The value of horizontal thrust can be found by integration.

The analysis of two-hinged circular arches involve tedious integration. These arches can be easily analysed by graphical integration given in article 11.10.

11.10. Graphical Integration Method of Calculating Horizontal Thrust.

Graphical integration method of calculating horizontal thrust in an arch can be easily used when the arch section has varying moment of inertia or the profile of the arch axis is such that it is not possible or very cumbersome to do actual integration.

![Diagram showing the arch and hinges](image-url)

**M_x Diagram**

Fig. 11.17

The central line of the arch is divided into a number of parts. These parts need not be equal. The length \( \delta s \) of each part along the centre line of the axis is measured. The moment of inertia at mid-section of each part is calculated. Let it be \( I \). The rise of the arch at mid-section of each part \( y \) is also calculated.

B.M. diagram as a simply supported beam is plotted for the arch and B.M. at mid-section of each part \( M_s \) is calculated. The
values of \( \frac{y^2 S_s}{EI} \) and \( \frac{M_s y S_s}{EI} \) for each part are calculated and tabulated as shown in Table 11.1.

**TABLE 11.1.**

<table>
<thead>
<tr>
<th>Parts</th>
<th>8s</th>
<th>I</th>
<th>y</th>
<th>( \frac{y^2 S_s}{EI} )</th>
<th>M_s</th>
<th>( \frac{M_s y S_s}{EI} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\Sigma \frac{y^2 S_s}{EI} \quad \Sigma \frac{M_s y S_s}{EI}
\]

\[
\mu = \frac{\Sigma \frac{M_s y S_s}{EI}}{\Sigma \frac{y^2 S_s}{EI}}
\]

**Ex. 11.7.** A segmental arch of uniform moment of inertia has a span of 60 m and subtends 90° at the centre. Left hand half of the arch is loaded with uniform distributed load of 2 tonnes/metre. Draw the B.M. diagram for the arch.

**Solution.** Radius of arch axis

\[
= \frac{l/2}{\sin 45} = \frac{30}{\sin 45} = 30 \sqrt{2}.
\]

Rise of the arch at any section \( X \), at angle \( \theta \) from \( OC \) is,

\[
y = R \cos \theta - 30
\]

Horizontal distance of this section from \( OC \) is,

\[
x = R \sin \theta.
\]

B.M. at section \( X \) in \( AC \) is

\[
M_s = 45 (30 - x) - \frac{(30-x)^2}{2} \times 2
\]

Fig. 11.18 (a), and (b)
B.M. at any section \( X \) in \( BC \) is given by

\[
M_\theta = 15(30-x) = 450 - 15R\sin\theta
\]

\[
\int_0^1 M_{\theta}yds = \int_0^{\pi/4} (450 + 15R\sin\theta - R^2\sin^2\theta)(R\cos\theta - 30) Rd\theta
\]

\[
= R \left[ 450R\sin\theta - \frac{15R^2}{2}\cos\frac{2\theta}{2} - R^2\sin^2\theta \right]_{0}^{\pi/4}
\]

\[
-13,500\theta + 450R\cos\theta + 15R^2 \left( \theta - \frac{\sin\frac{2\theta}{2}}{2} \right) \right]_{0}^{\pi/4}
\]

\[
+ R \left[ 450R\sin\theta + \frac{15R^2}{2}\cos\frac{2\theta}{2} - 13,500\theta \right]_{0}^{\pi/4}
\]

\[
-450R\cos\theta \right]_{0}^{\pi/4}
\]

\[
R \left[ \frac{450R}{\sqrt{2}} - \frac{R^2}{6\sqrt{2}} - 13,500\times\frac{\pi}{4} + 450R\times\frac{1}{\sqrt{2}} \right]
\]

\[
+ 15R^2\times\frac{\pi}{4} - \frac{15}{2}R^2 + \frac{15R^3}{4} - 450R \right]
\]

\[
+ R \left[ \frac{450R}{\sqrt{2}} - 13,500\times\frac{\pi}{4} - 450R\times\frac{1}{\sqrt{2}} - \frac{15R^3}{4} + 450R \right]
\]

\[
= R \left[ 450\sqrt{2} - \frac{R^2}{6\sqrt{2}} - 13,500\times\frac{\pi}{4} + \frac{15}{4}R^2\pi - \frac{15}{2}R^3 \right]
\]

Substituting \( R = 30\sqrt{2} \)

\[
\int_0^1 y^2 ds = 2\int_0^{\pi/4} (R\cos\theta - 30)^2 Rd\theta
\]

\[
= 2R \left[ \frac{15}{4}R^2 \cos^2\theta - 60R\cos\theta + 900 \right]_{0}^{\pi/4}
\]
\[ = 2R \left[ \frac{R^2}{2} \left( \theta + \sin \frac{2\theta}{2} \right) - 60R \sin \theta + 900 \theta \right]_0^\pi/4 \]
\[ = 2R \left[ \frac{R^2}{2} \times \frac{\pi}{4} + \frac{R^2}{4} - 60R \times \frac{1}{\sqrt{2}} + 900 \times \frac{\pi}{4} \right] \]

Substituting \( R = 30\sqrt{2} \)

\[ \int y^2 \, ds = 2 \times 30 \sqrt{2} \left[ \frac{900\pi}{4} \frac{900}{2} - 1800 + \frac{900\pi}{4} \right] \]
\[ = 60 \sqrt{2} \left[ \frac{900\pi}{2} - \frac{2700}{2} \right] \]
\[ = 27,000 \sqrt{2}(\pi - 3) = 5423 \]
\[ H = \frac{190,917}{5423} = 35.22 \text{ t} \]

B.M. at crown \( = 45 \times 30 - 30 \times 2 \times 15 - 35.22 \times 12.42 \]
\[ = 1350 - 900 - 436.7 \]
\[ = 133 \text{ t.m.} \]

B.M. diagram is shown in Fig. 11 18 (b).

**Ex. 11 8.** Analyse the two-hinged arch of Ex. 11 7 graphical integration.

**Solution.** The span of the arch is divided in 10 equal parts. The length of segments 1, 2, 3, 4 and 5 will be different. Origin is taken at \( D \) centre of span. The rise of the arch is calculated at centre of each segment.
\[
R^2 = x^2 + (y + (R \cdot h))^2
\]
\[
y + (R \cdot h) = \sqrt{R^2 - x^2}
\]
\[
\therefore y = \sqrt{R^2 - x^2} - (R \cdot h)
\]
Substituting \( R = 30\sqrt{2} \)
\[
y = \sqrt{1800 - x^2} - 30
\]
Substituting different values of \( x \), values of \( y \) can be calculated at centre of different segments.

B.M. at distance \( \pi \) from centre of span in right hand half of the arch is given by \( M_x = 15(30-x) \).

B.M. at \( x \) distance from centre of span in left hand half of the arch is given by \( M_x = 45(30-x) - \frac{2(30-x)^2}{2} \)
\[
= 45(30-x) - (30-x)^2
\]
Substituting various values of \( x \), B.M. at centre of each segment is calculated.

\[
\sin \theta_1 = \frac{x}{R} = \frac{6}{30\sqrt{2}} = \frac{1}{5\sqrt{2}} = 0.1414 \therefore \theta_1 = 8^\circ 8'
\]
\[
\sin \theta_2 = \frac{12}{30\sqrt{2}} = 0.2828 \therefore \theta_2 = 16^\circ 26'
\]
\[
\sin \theta_3 = \frac{18}{30\sqrt{2}} = 0.4242 \therefore \theta_3 = 25^\circ 6'
\]
\[
\sin \theta_4 = \frac{24}{30\sqrt{2}} = 0.5656 \therefore \theta_4 = 34^\circ 26'
\]
\[
\sin \theta_5 = \frac{30}{30\sqrt{2}} = \frac{1}{\sqrt{2}} \therefore \theta_5 = 45^\circ
\]
\[
\delta s_1 = R \theta = 30\sqrt{2}(\theta_3 - \theta_4) = 30\sqrt{2}(45^\circ - 34^\circ 26') \times \frac{\pi}{180}
\]
\[
= 7.82 \text{ m.}
\]
\[
\delta s_2 = 30\sqrt{2}(34^\circ 26' - 25^\circ 6') \frac{\pi}{180} = 6.91 \text{ m.}
\]
\[
\delta s_3 = 30\sqrt{2}(25^\circ 6' - 16^\circ 19') \frac{\pi}{180} = 6.42 \text{ m.}
\]
\[
\delta s_4 = 30\sqrt{2}(16^\circ 19' - 8^\circ 8') \frac{\pi}{180} = 6.15 \text{ m.}
\]
\[
\delta s_5 = 30\sqrt{2}(8^\circ 8') \frac{\pi}{180} = 6.03 \text{ m.}
\]

The values of \( y, M_x, y \delta s \) and \( y^2 \delta s \) for various segments are given in Table 11.
### Table 11.2

<table>
<thead>
<tr>
<th>Section</th>
<th>$x$</th>
<th>$y$</th>
<th>$M_y$</th>
<th>$5s$</th>
<th>$y^2s$</th>
<th>$M_ys$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>27</td>
<td>2.73</td>
<td>126</td>
<td>7.62</td>
<td>58.29</td>
<td>2690</td>
</tr>
<tr>
<td>2</td>
<td>21</td>
<td>6.86</td>
<td>324</td>
<td>6.91</td>
<td>325.2</td>
<td>15,360</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>9.69</td>
<td>450</td>
<td>6.42</td>
<td>602.7</td>
<td>27,990</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>11.46</td>
<td>504</td>
<td>6.15</td>
<td>807.6</td>
<td>35,520</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>12.33</td>
<td>486</td>
<td>6.03</td>
<td>916.8</td>
<td>36,130</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>12.33</td>
<td>405</td>
<td>6.03</td>
<td>916.8</td>
<td>30,130</td>
</tr>
<tr>
<td>7</td>
<td>9</td>
<td>11.46</td>
<td>315</td>
<td>6.15</td>
<td>807.6</td>
<td>22,220</td>
</tr>
<tr>
<td>8</td>
<td>15</td>
<td>9.69</td>
<td>225</td>
<td>6.42</td>
<td>602.7</td>
<td>14,000</td>
</tr>
<tr>
<td>9</td>
<td>21</td>
<td>6.86</td>
<td>135</td>
<td>6.91</td>
<td>325.2</td>
<td>6,398</td>
</tr>
<tr>
<td>10</td>
<td>27</td>
<td>2.73</td>
<td>45</td>
<td>7.32</td>
<td>58.29</td>
<td>961</td>
</tr>
</tbody>
</table>

\[ \Sigma \bar{M}_y = 5421.11 \quad 191,369 \]

\[ H = \frac{191,369}{5421.18} = 35.3 \text{ tonnes.} \]

**Ex. 11.9.** Calculate the horizontal thrust at the supports of the two-pinned structure shown in Fig. 11.20. It is constant.

**Solution.** Let $H$ be horizontal thrust at $C$.

- Horizontal thrust at $A$ will be
  \[ H_A = 2000R - H \]

- Taking moments about $A$
  \[ R_C \times R + 2000 \times R \times \frac{R}{2} = 0 \]

\[ \therefore R_C = -1000R \text{ (Downwards)} \]

\[ R_A = +1000R \]

In portion $CB$, B.M. at a section at distance $y$ from $C$ is given by

\[ M_y = H \times y - \frac{2000y^2}{3} \]

\[ = H y - 1000y^3 \]

\[ \frac{\partial M_y}{\partial H} = y \]
In curved portion $AB$, bending at a section making an angle $\theta$ with $AC$ is given by,

$$M = 1000R \times R(1 - \cos \theta) - (2000R - H) R \sin \theta$$

$$= 1000R^2(1 - \cos \theta) - 2000R^2 \sin \theta + HR \sin \theta$$

$$\therefore \quad \frac{\partial M}{\partial H} = R \sin \theta.$$

Partial derivative of strain energy with respect to $H$ is zero as there is no yielding of supports.

$$\frac{\partial U}{\partial H} = \frac{1}{EI} \int_0^R (Hy - 1000y^2) dy + \frac{1}{EI} \int_0^{\pi/2} [1000R^2(1 - \cos \theta)$$

$$- 2000R^2 \sin \theta + HR \sin \theta]R \sin \theta \times Rd\theta$$

$$0 = \frac{1}{EI} \int_0^R (Hy - 1000y^2) dy + \frac{R^3}{EI} \int_0^{\pi/2} [1000R(\sin \theta - \sin \theta \cos \theta)$$

$$- 2000R \sin^2 \theta + H \sin^2 \theta]d\theta$$

$$= \left[ \frac{Hy^3}{3} - \frac{1000y^4}{4} \right]_0^R + R^3 \left[ 1000R \left( -\cos \theta + \frac{\cos 2\theta}{4} \right)$$

$$- \frac{2000R}{2} \left( \theta - \frac{\sin 2\theta}{2} \right) + H \left( \frac{\sin 2\theta}{2} \right) \right]_0^{\pi/2}$$

$$= \left[ \frac{HR^3}{3} - \frac{1000R^4}{4} \right] + R^3 \left[ 1000R \left( -\frac{1}{4} + 1 - \frac{1}{4} \right)$$

$$- 1000R \left( \frac{\pi}{2} \right) + H \times \frac{\pi}{2} \right]$$

$$= \frac{H}{3} - 250R + 500R - 500\pi R + \frac{H\pi}{4}$$

$$\therefore \quad \frac{H}{3} + H \times \frac{\pi}{4} = 500\pi R - 250R$$

$$\therefore \quad H = \frac{1321.4R}{1.119}$$

$$= 1180.9R$$

Horizontal thrust at $C$ is $1180.9R$ and at $A$ is $819.1R$.

11.11. Reaction Locus.

When a single concentrated load acts on an arch, two reactions at two supports intersect at a point which lies on the line of action of the concentrated load. The three forces are concurrent. The point of intersection of three forces changes with the position of the load. The curve joining the points of intersection of reactions and load for various load positions is known as reaction locus. Consider a para-
bolic arch of span \( l \) and central rise \( h \). Let the load \( W \) act at distance \( kL \) from one end. It is assumed that moment of inertia at any section of the arch is given by \( I = I_c \) sec \( \theta \), where \( I_c \) is moment of inertia at crown and \( \theta \) is slope at the section.

The horizontal thrust at abutments is given by,

\[
H = \frac{5Wl}{8h} \left( k^4 - 2k^3 + k \right)
\]

Let \( y_o \) be the height at which the reaction intersects the load \( W \).

\[
\frac{y_o}{kL} = \frac{V_A}{H}
\]

\[
= \frac{W(1-k)}{5WL} \frac{8h}{(k^4 - 2k^3 + k)}
\]

\[
y_o = \frac{k(1-k)8h}{5(k^4 - 2k^3 + k)}
\]

\[
y_o = \frac{8h(1-k)}{5(1-k)(1+k-k^2)}
\]

\[
y_o = \frac{8h}{5(1+k-k^2)}
\]

At \( k = 0 \),

\[ y_o = \frac{8h}{5} \]

At \( k = \frac{1}{2} \),

\[ y_o = \frac{32h}{25} \]

The curve is shown in Fig. 11·21 (b).

**Ex. 11·10.** Show that for a two-hinged semi-circular arch, the reaction locus is a straight line at a distance \( \frac{HR}{2} \) from the abutments.
Solution. Consider two hinged semi-circular arch shown in Fig. 11-22. Let the load \( W \) act at point \( D \), at angle \( \alpha \) with horizontal.

Taking moments about \( A \):

\[
V_B \times 2R = WR(1 - \cos \alpha)
\]

\[
\therefore \quad V_B = \frac{WR(1 - \cos \alpha)}{2}
\]

\[
\therefore \quad V_A = W - R_B = \frac{W(1 + \cos \alpha)}{2}
\]

B.M. at a section \( X \), in the portion \( BC \) of the arch is given by,

\[
M = V_B R(1 - \cos \theta)
\]

\[
-\frac{H \times R \sin \theta}{W(1 + \cos \alpha)} \times R(1 - \cos \theta) - HR \sin \theta
\]

\[
\therefore \quad \frac{\partial M}{\partial H} = -R \sin \theta
\]

In portion \( AD \), B.M. at section \( X \) is given by

\[
M = V_A \times R(1 - \cos \theta) - H \times R \sin \theta
\]

\[
\frac{W(1 + \cos \alpha)}{2} \times R(1 - \cos \theta) - HR \sin \theta
\]

\[
\frac{\partial M}{\partial H} = -R \sin \theta
\]

In portion \( DC \), B.M. at section \( X \) is given by

\[
M = V_A - R(1 - \cos \theta) - HR \sin \theta
\]

\[
\frac{W(1 + \cos \alpha)}{2} \times R(1 - \cos \theta) - W \times R(\cos \alpha - \cos \theta)
\]

\[
\cos \theta - HR \sin \theta
\]

\[
-\frac{\partial M}{\partial H} = -R \sin \theta
\]

\[
\frac{\partial U}{\partial H} = 0 \quad \text{as the abutments do not yield.}
\]

\[
0 = \frac{1}{EI} \left[ \int_0^{\pi/2} \left[ \frac{W(1 - \cos \alpha)}{2} \times R(1 - \cos \theta) - HR \sin \theta \right] \frac{\partial M}{\partial H} \, d\theta \right]
\]

\[
+ \frac{1}{EI} \left[ \alpha \left[ \frac{W(1 + \cos \alpha)}{2} \times R(1 - \cos \theta) - HR \sin \theta \right] \left[ -R \sin \theta \right] Rd\theta \right]
\]
\[ + \frac{1}{EI} \int_0^{\pi/2} \left[ \frac{W(1 + \cos \alpha)}{2} \times R(1 + \cos \theta) - HR \sin \theta \right. \\
- WR(\cos \alpha - \cos \theta) \left. \right] (-R \sin \theta) Rd\theta \]

\[ \therefore R^3 \int_0^{\pi/2} \left[ \frac{W(1 + \cos \alpha)}{2} \left( \sin \theta - \sin \theta \cos \theta \right) - H \sin^2 \theta \right] d\theta \]

\[ - R^3 \int_0^{\pi/2} \left[ \frac{W(1 + \cos \alpha)}{2} \left( \sin \theta - \sin \theta \cos \theta \right) - H \sin^2 \theta \right] d\theta \]

\[ - R^3 \int_0^{\pi/2} W(\cos \alpha - \cos \theta) \sin \theta \, d\theta = 0 \]

\[ \therefore \int_0^{\pi/2} \left[ \frac{W(1 + \cos \alpha)}{2} \left( \sin \theta - \frac{\sin 2\theta}{2} \right) \right. \\
- H \times \left( \frac{1 - \cos 2\theta}{2} \right) \left. \right] d\theta + \int_0^{\pi/2} \left[ \frac{W(1 + \cos \alpha)}{2} \left( \sin \theta - \frac{\sin 2\theta}{2} \right) - H \left( \frac{1 - \cos 2\theta}{2} \right) \right] d\theta \\
- \int_0^{\pi/2} W \left( \cos \alpha \sin \theta - \frac{\sin 2\theta}{2} \right) d\theta = 0 \]

\[ \therefore \left[ \frac{W(1 - \cos \alpha)}{2} \right] \left( -\cos \theta + \cos 2\theta \right. \\
- H \left( \frac{\sin 2\theta}{2} \right) \left. \right]_0^{\pi/2} + \left[ \frac{W(1 + \cos \alpha)}{2} \left( -\cos \theta + \cos 2\theta \right)\right]_0^{\pi/2} \\
- \left[ -\cos \alpha \sin \theta + \cos \frac{2\theta}{4} \right]^{\pi/2} = 0 \]

\[ \therefore \left[ \frac{W(1 - \cos \alpha)}{2} \right] \left( 0 - \frac{1}{4} + 1 - \frac{1}{4} \right) - H \left( \frac{\pi}{2} - 0 \right) \\
+ \left[ \frac{W(1 + \cos \alpha)}{2} \right] \left( 0 - \frac{1}{4} + 1 - \frac{1}{4} \right) \\
- H \left( \frac{\pi}{2} - 0 \right) - W \left( 0 - \frac{1}{4} + 1 + \cos^2 \alpha \cos 2\alpha \right) \]

\[ \therefore W(1 - \cos \alpha) = \frac{H \pi}{4} + W(1 + \cos \alpha) - \frac{H \pi}{4} + \frac{W}{4} \\
- W \left[ \frac{1 + \cos 2\alpha - \cos 2\alpha}{4} \right] = 0 \\
\frac{3W}{4} - \frac{H \pi}{2} - \frac{W}{2} - \frac{W \cos 2\alpha}{4} = 0 \]
\[ W(1 - \cos 2\alpha) = \frac{H\pi}{2} \]

\[ H = \frac{W(1 - \cos 2\alpha)}{2\pi} = \frac{W[1 - (2\cos^2\alpha - 1)]}{2\pi} \]

\[ W(1 - \cos^2\alpha) \]

Let \( y_o \) be the height from the abutments where reaction cuts the load.

\[ \frac{y_o}{R(1 - \cos \alpha)} = \frac{V_A}{H} \]

\[ \frac{W(1 + \cos \alpha)}{2} = \frac{W(1 - \cos^2\alpha)}{\pi} \]

\[ \frac{\pi(1 + \cos \alpha)}{2(1 - \cos^2\alpha)} \]

\[ y_o = \frac{\pi R}{2} \left( 1 + \cos \alpha \right) \left( 1 - \cos^2 \alpha \right) \]

\[ = \frac{\pi R}{2} \]

Expression for \( y_o \) is independent of \( \alpha \) and is constant. Therefore reaction locus is a straight line at distance \( \frac{\pi R}{2} \) from abutments.

11'12. Two Hinged Spandrel Arch.

For long span bridges spandrel arches are sometimes used. These types of arches are useful when large rise to span ratio are required.

These types of arches can be analysed by unit load method or Castigliano's theorem.

Analysis by Castigliano's theorem.

The forces in various members of the frame are found in terms of \( H \) and from this total strain energy is found.

In case supports do not yield, partial derivative of strain energy with respect to \( H \) is zero.

\[ \frac{\partial U}{\partial H} = 0 \]

the value of \( H \) can be evaluated.

Analysis by unit load method.

The frame is made determinate by putting end \( B \) on rollers. The forces in all the members of the determinate frame are found.
Let the force in a member be $P$. Next unit horizontal load is applied at the roller end and forces in members are found. Let the force in a member be $k$. The horizontal movement of roller end is given by $\delta = \sum \frac{P kl}{AE}$ where $l$ is the length of the member and $A$ is the area.

The horizontal movement of roller due to horizontal force $H$ is $\sum \frac{(Hk)kl}{AE} + H E \frac{k^2l}{AE}$.

As the end $B$ is hinged, the horizontal movement is zero, so:

$$\sum \frac{P kl}{AE} + H E \frac{k^2l}{AE} = 0$$
Ex. 11.11. The members of the arched frame shown in Fig. 11.24 (a) have the same ratio of length to cross-sectional area. If the supports at A and B are hinged so that no lateral movement is allowed, find the horizontal thrust at the abutments.

**Solution.** Vertical reactions are \( R_A = R_B = 6000 \) kg.

Let \( H \) be the horizontal thrust at abutments.

**Joint A**

\[
P_1 = + \frac{H}{\sin \theta} + \frac{5}{4} H
\]

\[
P_2 = + (6000 - P_1 \cos \theta)
\]

\[
= 6000 - \frac{3}{4} H
\]

**Joint C**

\[
P_3 = - \frac{P_2}{\cos \theta} = - \left(6000 - \frac{3}{4} H\right) \times \frac{5}{3}
\]

\[
= \frac{5}{4} H - 10,000
\]

\[
P_4 = P_3 \sin \theta = \left(\frac{5}{4} H - 10,000\right) \times \frac{4}{5}
\]

\[
= 8000 - H
\]
Joint D.

\[ P_8 = (P_1 \cos \theta + P_2 \cos \theta) \]
\[ = \frac{3}{4} H + 6000 - \frac{3}{4} H \]
\[ = + 6000 \text{ kg.} \]

\[ P_8 = P_1 \sin \theta + P_3 \sin \theta = H + H - 8000 \]
\[ = 2H - 8000 \]

Joint G.

\[ P_7 = - \frac{(P_8 - 4000)}{\cos \theta} = -20( \]
\[ = \frac{-10,000}{3}. \]

Joint E.

\[ P_5 = P_4 - P_7 \sin \theta \]
\[ = 8000 - H + \frac{4}{5} \times \frac{10,000}{3} \]
\[ = 32,000 \text{ kg.} \]

Similarly forces are found in the other half of the frame.

The values of \( P, \frac{\partial P}{\partial H}, P \frac{\partial P}{\partial H} \) and final forces in various members are given in Table 11.3.

Strain Energy

\[ U = \Sigma \frac{P_2 l}{2AE} \]
\[ \frac{\partial U}{\partial H} = \Sigma \frac{P l}{AE} \times \frac{\partial P}{\partial H} \]

As there is no yielding of supports \( \frac{\partial U}{\partial H} = 0 \)

\[ \Sigma \frac{P l}{AE} \times \frac{\partial P}{\partial H} = 0. \]

As all the members have the same ratio of length to cross-sectional area, \( \frac{l}{AE} \) is constant.

\[ \Sigma P \frac{\partial P}{\partial H} = 0 \]
\[ \frac{59}{8} H + 12H - 76,000 - \frac{41,000}{3} \]
\[ = 0 \]

\[ \therefore \quad 19.375H \quad 89,66.6 \text{ kg.} \]
\[ \therefore \quad H \quad 89,666.6 \]
\[ = 4626 \text{ kg.} \]

Final forces in the members are given in Table 11.3.
<table>
<thead>
<tr>
<th>Member</th>
<th>$P$</th>
<th>$\frac{\partial P}{\partial H}$</th>
<th>$\frac{P \cdot \partial P}{\partial H}$</th>
<th>Force $P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{5}{4} H$</td>
<td>$\frac{5}{4}$</td>
<td>$\frac{25}{16} H$</td>
<td>$-5782.5$</td>
</tr>
<tr>
<td>2</td>
<td>$6000 - \frac{3}{4} H$</td>
<td>$-\frac{3}{4}$</td>
<td>$-\frac{4500}{16} + \frac{9}{16} H$</td>
<td>$+2530.5$</td>
</tr>
<tr>
<td>3</td>
<td>$5/4 H - 10,000$</td>
<td>$-1$</td>
<td>$\frac{25}{16} H - 12,500$</td>
<td>$-4217.5$</td>
</tr>
<tr>
<td>4</td>
<td>$8000 - H$</td>
<td>$-1$</td>
<td>$-8000 - \frac{1}{16} H$</td>
<td>$+3374$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{32,000}{3} - H$</td>
<td>$-1$</td>
<td>$-\frac{32,000}{3} + H$</td>
<td>$+6040.7$</td>
</tr>
<tr>
<td>6</td>
<td>$+6000$</td>
<td>$0$</td>
<td>$0$</td>
<td>$+6000$</td>
</tr>
<tr>
<td>7</td>
<td>$-10,000/3$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-3333.3$</td>
</tr>
<tr>
<td>8</td>
<td>$2H - 8000$</td>
<td>$2$</td>
<td>$4H - 16000$</td>
<td>$+1252$</td>
</tr>
<tr>
<td>9</td>
<td>$+6000$</td>
<td>$0$</td>
<td>$0$</td>
<td>$+6000$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{5}{4} H$</td>
<td>$-\frac{5}{4}$</td>
<td>$\frac{25}{16} H$</td>
<td>$+5782.5$</td>
</tr>
<tr>
<td>11</td>
<td>$4000 - \frac{3}{4} H$</td>
<td>$-\frac{3}{4}$</td>
<td>$-\frac{3000}{16} + \frac{9}{16} H$</td>
<td>$4530.5$</td>
</tr>
<tr>
<td>12</td>
<td>$\frac{5}{4} H - 20,000/3$</td>
<td>$\frac{5}{4}$</td>
<td>$\frac{25}{16} H - 25,000/3$</td>
<td>$-884.2$</td>
</tr>
<tr>
<td>13</td>
<td>$16,000/3 - H$</td>
<td>$-1$</td>
<td>$H - 16,000/3$</td>
<td>$+707.3$</td>
</tr>
<tr>
<td>14</td>
<td>$\frac{32,000}{3} - H$</td>
<td>$-1$</td>
<td>$H - \frac{32,000}{3}$</td>
<td>$+6040.7$</td>
</tr>
<tr>
<td>15</td>
<td>$+4,000$</td>
<td>$0$</td>
<td>$0$</td>
<td>$+4000$</td>
</tr>
<tr>
<td>16</td>
<td>$-20,000/3$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-6666.6$</td>
</tr>
<tr>
<td>17</td>
<td>$2H - 16,000/3$</td>
<td>$2$</td>
<td>$4H - 32,000/3$</td>
<td>$+3918.7$</td>
</tr>
</tbody>
</table>

$\Sigma$ $\frac{59}{8} H + 12 H$

$-76,000 + 41,000/3$
Ex. 11.11. The parabolic arches shown in Fig. 11.25 (a) pinned to one another and to a column BD at B, are hinged to abutments at A and C. The arches have variable moment of inertia $I = I_0$ sec $\theta$. If the column restraint is 4,000 kg/cm, for side sway due to the horizontal thrust at B, find the horizontal thrust in each arch and the bending moment at the crown. Neglect any vertical yield of the column at B. $I_0 = 40,000$ cm$^4$. $E = 2 \times 10^8$ kg/cm$^2$.

Solution. Let $\Delta$ be the horizontal displacement of the column B, towards left. Horizontal thrust in arch $AB$ will be

$$ H_{AB} = \int_0^l \frac{M_y dx}{EI_0} + \Delta $$

Horizontal thrust in arch $BC$ will be

$$ H_{BC} = \int_0^l \frac{M_y dx}{EI_0} - \Delta $$

For a parabolic arch

$$ \int_0^l \frac{y^2 dx}{EI_0} = \frac{8}{15} \frac{h^3 I}{EI_0} $$

$$ = \frac{8}{15} \times \frac{10 \times 10 \times 30}{EI_0} $$

$$ = \frac{1600}{EI_0} $$

Rise of the arch at any section is given by

$$ \frac{4h(x(l-x))}{I^2} = \frac{4 \times 10x(30-x)}{30 \times 30} $$

$$ \frac{4}{30}x(30-x) $$
\[ \int_0^l \frac{M_2 y}{EI_0} \, dx = 2 \int_0^{15} \frac{20,000 x}{EI_0} \times \frac{4}{90} \times (30-x) \, dx \]
\[ = \frac{16,000}{9EI_0} \int_0^{15} (30x^3-x^5) \, dx \]
\[ = \frac{16,000}{9EI_0} \left[ 10x^3 - \frac{x^4}{4} \right]_0^{15} \]
\[ = \frac{16,000}{9EI_0} \left[ 10 \times 15^3 - \frac{15^4}{4} \right] \]
\[ = \frac{16,000}{9EI_0} \times 15^3 \left[ 10 - \frac{15}{4} \right] \]
\[ = \frac{37,500,000}{EI_0} \]

For arch AB
\[ \int_0^l \frac{M_2 y}{EI_0} \, dx = \frac{1}{EI_0} \int_0^{15} 6000 x \times \frac{4}{90} \times (30-x) \, dx \]
\[ + \frac{1}{EI_0} \int_0^{15} \left( 18,000 x - \frac{1600x^2}{2} \right) \times \frac{4}{90} \times (30-x) \, dx \]
\[ = \frac{800}{3EI_0} \int_0^{15} (30x^3-x^5) \, dx + \frac{4}{90} \]
\[ \times \frac{200}{EI_0} \int_0^{15} (90x-4x^2)(30-x) \, dx \]
\[ = \frac{800}{3EI_0} \left[ 10x^3 - \frac{x^4}{4} \right]_0^{15} \]
\[ + \frac{80}{9EI_0} \int_0^{15} (2700x^3-210x^4+4x^5) \, dx \]
\[ = \frac{800}{3EI_0} \left[ 10 \times 15^3 - \frac{14^4}{4} \right] \]
\[ + \frac{80}{9EI_0} \left[ 2700x^3 - \frac{210x^4}{4} + \frac{4x^5}{5} \right]_0^{15} \]
\[ = \frac{800}{3EI_0} \times 15^3 \left[ 10 - \frac{15}{4} \right] + \frac{80}{9EI_0} \]
\[ \times 15^3 \left[ 900 - \frac{210 \times 15}{4} \right] + \frac{4}{5} \times 15 \times 15 \]
\[ = \frac{800}{3EI_0} \times 15^3 \times \frac{25}{4} + \frac{80}{9EI_0} \]
\[ \times 15^3 \left[ 900 - \frac{1575}{2} + 180 \right] \]
\[ = \frac{5,625,000}{E_{I_0}} + \frac{30,000}{E_{I_0}} \left[ 2160 - 1575 \right] \]
\[ = \frac{5,625,000}{E_{I_0}} + \frac{8,755,000}{E_{I_0}} \]
\[ = \frac{14,400,000}{EI_0} \]
\[
\frac{14,400,000}{EI_0} + \Delta
\]

\[
H_{AB} = \frac{1600}{EI_0} \cdot \frac{37,500,000}{EI_0} - \Delta
\]

\[
H_{BC} = -\frac{EI_0}{1600}
\]

Net force on the column \(H_{BC} - H_{BA}\).

Displacement of column \(\Delta\)

\[
-\frac{H_{BC} - H_{BA}}{H_{BC} - H_{BA}} = \frac{H_{BC} - H_{BA}}{40,000 \times 100} \text{ metres}
\]

\[
\therefore \quad H_{BC} - H_{BA} = \Delta \times 40,000 \times 100.
\]

Substituting values of \(H_{BC}\) and \(H_{BA}\),

\[
\frac{37,500,000}{EI_0} - \Delta \quad 14,400,000 \quad EI_0 + \Delta
\]

\[
\frac{1600}{EI_0} \quad 1600
\]

\[
= \Delta \times 40,000 \times 100
\]

\[
\therefore \quad \frac{37,500,000 - EI_0 \Delta}{1600} = \frac{14,400,000 + EI_0 \Delta}{1600}
\]

\[
= 4,000,000 \Delta
\]

\[
\therefore \quad 37,500,000 - EI_0 \Delta - 14,400,000 - EI_0 \Delta
\]

\[
= 4,000,000 \times 1600 \Delta
\]

\[
\therefore \quad 23,100,000 = 4,000,000 \times 1600 \Delta + 2 \times 2 \times 10^6 \times 40,000 \times 100 \times 100 \Delta
\]

\[
\therefore \quad 23,100,000 = 4,000,000 \times 1600 \Delta + 16 \times 10^6 \Delta
\]

\[
\therefore \quad 231 = 64,000 \Delta + 160 \Delta
\]

\[
\therefore \quad \Delta \quad 64,160
\]

\[
H_{AB} = \frac{14,400,000 + 2 \times 10^6 \times 40,000}{100 \times 100} \times \frac{231}{64,160}
\]

\[
9000 + 18 = 9018 \text{ kg.}
\]

\[
H_{BC} = \frac{37,500,000 - 2 \times 10^6 \times 40,000}{100 \times 100} \times \frac{231}{64,160}
\]

\[
= 23,438 - 18 = 23,420 \text{ kg.}
\]

B.M. at crown of arch \(BC\)

\[
= 20,000 \times 15 - 23,420 \times 10
\]

\[
= 300,000 - 234,200
\]

\[
= 65,800 \text{ kg. m.}
\]

B.M. at crown of arch \(AB\)

\[
= 6000 \times 15 - 9018 \times 10
\]

\[
= 90,000 - 90,180
\]

\[
= -180 \text{ kg. m.}
\]

Consider a parabolic arch of span \( L \) and central rise \( h \).

Let the unit load act at distance \( kL \) from one support. It is assumed that the moment of inertia of arch at any section is

\[
I = I_c \sec \theta.
\]

Horizontal thrust in the arch for this position of the load is given by

\[
H = \frac{5l}{8h} \left( k^4 - 2k^2 + k \right).
\]

This gives equation for horizontal thrust. As the position of the load changes value of \( H \) changes. Maximum value of \( H \) occurs when load is at the crown, i.e. \( k = \frac{1}{2} \).

Influence line for horizontal thrust is shown in Fig. 11.26.

Influence line for B.M.
The B.M. at any section of the arch is given by

\[ M = M_s - H y_a \]

where \( M_s \) is the B.M. at the section considering the arch as a simply supported beam. \( H \) is horizontal thrust in the arch and \( y_a \) is the rise of the arch at the section. The influence line for B.M. at a section is obtained by superposing influence line due to \( H y_a \) on influence line for \( M_s \). Influence line for \( M_s \) is a triangle with maximum ordinate equal to \( \frac{a(l-a)}{l} \), when the load is at the section. The influence line for \( H y_a \) is obtained by multiplying influence line for \( H \) by \( y_a \), the rise of the arch at the section. The influence line for B.M. is shown in Fig. 11.27 (b).

**Influence line for normal thrust**

Normal thrust at any section of the arch is given by

\[ N = H \cos \theta + V \sin \theta \]

where \( H \) is horizontal thrust, \( V \) is vertical force at the section and \( \theta \) is slope at the section. At any section \( \sin \theta \) and \( \cos \theta \) are constant and, therefore, influence line for normal thrust will be obtained by superposing influence line for \( H \) multiplied by \( \cos \theta \) on influence line for \( V \) multiplied by \( \sin \theta \). Influence line for \( V \) is same as influence line for shear force at a section in a simply supported beam. Influence line for normal thrust at the section is shown in Fig. 11.27 (c).

**Influence line for shear force**

Shear force at any section of arch is given by

\[ P = V \cos \theta - H \sin \theta \]

The influence line for shear force is obtained by superposing influence line for \( V \) multiplied by \( \cos \theta \), on influence line for \( H \) multiplied by \( \sin \theta \). Influence line for shear force is shown in Fig. 11.27 (d).

**PROBLEMS**

1. A two-hinged parabolic arch has a span of 40 m. and central rise of 10 m. The moment of inertia at any section is equal to \( I_c \), sec \( \theta \), \( I_c \) being the moment of inertia at the crown and \( \theta \) the slope at the section. Determine the bending moment at the crown when arch carries a uniformly distributed load of 3000 kg/m, over the central half of the span.

   \[ \text{Ans. } M = 36,000 \text{ kg} \cdot \text{m}. \]

2. If the arch in the problem (1) is prestressed by decreasing the span, determine the necessary movement which will reduce to zero the crown bending moment caused by the given loading. \( I_c = 800,000 \text{ cm}^4 \). \( E = 2 \times 10^6 \text{ kg/cm}^2 \).

   \[ \text{Ans. } \Delta = 4.8 \text{ cm}. \]

3. A two-hinged parabolic arch of span \( l \) has central rise of one-fourth the span. Find the horizontal thrust in the arch when it is loaded with concentrated load of \( W \) at quarter span. \( l = I_c \), sec \( \theta \).

   \[ \text{Ans. } H = \frac{1}{4} W \]
4. Calculate the horizontal thrust at the supports of two hinged arch shown in Fig. 11.28, when it is loaded at the crown with load of 10,000 kg. The arch is of constant flexural rigidity throughout.  

[Ans. \( H = 2800 \text{ kg} \)]

5. A two-pinned arch of constant section throughout consists of two quarter circular arcs connected by a straight beam shown in Fig. 11.29. A uniformly distributed load covers the whole length of the beam. Determine the horizontal thrust at the constraints and sketch the bending moment diagram. Also find B.M. at F.

[Ans. \( H = 0.422wR, M_C = 0.203 wR^2, M_B = 0.078 wR^2, M_F = 0.154 wR^2 \)]

6. Two parabolic arches have hinges at their outer ends and rest on a common roller as shown in Fig. 11.30. Arch AB loaded with load of 16,000 kg. at 5 m. from A. If \( EI \) is same for both arches and \( I = I_0 \sec \theta \), determine the horizontal thrust at hinges and draw B.M. diagram.

[Ans. \( H = 12,035 \text{ kg} \)]
7. The members of the two hinged arch shown in Fig. 11·31 have the same ratio of length to cross-sectional area. If the supports at $A$ and $B$ are hinged so that no lateral movement is allowed, find the horizontal thrust at the abutments.

[Ans. $H = 5333.3$ kg.]

8. In the two-hinged arch shown in Fig. 11·32, all the members have same cross-sectional area. Find the horizontal thrust at abutments when the arch is loaded with concentrated load $W$ at $D$.

[Ans. $H = 0.36W$]

9. A two-hinge circular arch rib of uniform section has a span 20 m. and a rise of 5 m. Determine the maximum intensity of bending stress in the arch due to a rise in temperature of 20°C, if the constant depth of the rib is 50 cm. $E = 2.1 \times 10^8$ kg/cm²; coefficient of linear expansion = $11 \times 10^{-6}$/°C.

10. A two hinged parabolic arch has a span of 30 m and a rise of 7.5 m. The moment of inertia of the arch section is pro-
portional to sec $\theta$, where $\theta$ is the slope of the arch axis at any point with the horizontal. Calculate the horizontal thrust caused in the arch due to a rise of temperature by 14° C. The value of $E = 2 \times 10^6$ kg/cm² and coefficient of expansion $= 10 \times 10^{-6}$ per degree C. The moment of inertia at crown is $125 \times 10^4$ cm units (A M I E. Nov.

11. A symmetrical two hinged parabolic arch-rib has its both supports $l$ distance apart at the same level, the rise upto the crown being $r$. The second moment of area of the section at any point $= I_0 \sec \theta$, where $I_0$ is the corresponding value at the crown and $\theta$ is the inclination of the tangent at the point with the horizontal. Calculate from the first principles, the expression for the horizontal thrust developed at the supports due to a unit load placed on the arch $kl$ distance measured horizontally from the left support.

Hence calculate the S.F. at the left quarter point of a two hinged arch as above due to a concentrated load placed at the crown. Take $l = 30.5$ m and $r = 6.1$ m. (A.M.I.E. May, 1971)
ELASTIC CENTRE METHOD

Consider a structure $AB$ fixed at the ends $A$ and $B$. The structure is indeterminate to third degree. Let $R_B$, $H_B$ and $M_B$ be the vertical reaction, horizontal reaction and moment respectively at $B$ as shown in Fig. 12.1.

Taking $B$ as origin of reference, bending moment at section $X$ having co-ordinates $(x,y)$ is given by

$$M = R_B x + H_B y - M_B + M',$$

where $M'$ is the bending moment due to loads on the structure.

As the ends $A$ and $B$ are fixed and there is no yielding of supports,

$$\int \frac{M \, ds}{EI} = 0$$
$$\int \frac{M x \, ds}{EI} = 0$$
$$\int \frac{M y \, ds}{EI} = 0$$

\[ \text{Fig. 12.2} \]

\[ \text{Fig. 12.3} \]

$$\int \frac{(R_B x + H_B y - M_B + M') \, ds}{EI} = 0 \quad \ldots(12.1)$$
$$\int \frac{(R_B x^2 + H_B xy - M_B x + M' x) \, ds}{EI} = 0 \quad \ldots(12.2)$$
$$\int \frac{R_B xy + H_B y^2 - M_B y + M' y \, ds}{EI} = 0 \quad \ldots(12.3)$$
Three equations are obtained from the above expressions and values of $R_B$, $H_B$ and $M_B$ can be calculated from these three equations. However, it is a tedious procedure to solve the equations. By proper selection of redundant quantities these equations can be obtained, each involving only one unknown. If the end $B$ is free and a moment $M_B$ is applied at $B$, it produces rotation of the cross-section at $B$ and also displacement of $B$. By reciprocal theorem it is evident that a vertical force $R_B$ or horizontal force $H_B$ applied at $B$ will produce displacement of $B$ and also rotation of cross-section at $B$. In order to make the rotation of cross-section $B$ dependent only on moment, a rigid arm $OB$ is attached to end $B$ and vertical force, horizontal force and a moment are applied to end $O$. These forces are applied so that conditions of statics are satisfied and point $O$ is so selected that it remains stationary when a moment is applied. By reciprocal theorem the horizontal or vertical force applied at $O$ will not produce rotation of cross-section at $B$. The point $O$ referred above is known as elastic centre.

If $H_0$, $R_0$ and $M_0$ are the forces and moment applied at a elastic centre.

$$H_0 = H_B, \quad R_0 = R_B,$$

and

$$M_0 = M_B - R_0 \times x_0 - H_B y_0$$

where $x_0$ and $y_0$ are co-ordinates of point $O$ with reference to $B$ as origin.

Let $(X, Y)$ be the co-ordinates of any point with reference to elastic centre as origin.

As $R_0$ and $H_0$ do not have any effect on rotation $\theta$, it evident from equation 12·1 that:

$$\int \frac{X \, ds}{EI} \quad \text{and} \quad \int \frac{Y \, ds}{EI} \quad \text{must each be zero.}$$

$$X = x - x_0$$

$$\therefore \quad \int \frac{X \, ds}{EI} = \int \frac{(x - x_0) \, ds}{EI} = 0$$

$$\therefore \quad \int \frac{x \, ds}{EI} - x_0 \int \frac{ds}{EI} = 0$$

or

$$x_0 = \frac{\int \frac{x \, ds}{EI}}{\int \frac{ds}{EI}} \quad \ldots (12')$$

$$Y = y - y_0$$

$$\therefore \quad \int \frac{Y \, ds}{EI} = \int \frac{(y - y_0) \, ds}{EI} = 0.$$
\[ I_X = \int \frac{Y^2 \, ds}{EI} \]
\[ I_Y = \int \frac{X^2 \, ds}{EI} \]
\[ I_{XY} = \int \frac{XY \, ds}{EI} \]

Equations (12.1), (12.2) and (12.3) can be written as
\[ \int \frac{(M_0 + M') \, ds}{EI} = 0 \]
or
\[ M_0 = \int \frac{M' \, ds}{EI} \]
\[ (R_0 X^2 + H_0 XY + M'X) \, ds = 0 \]
\[ (R_0 XY + H_0 Y^2 + M'Y) \, ds = 0 \]

If axes of reference are principal axes (\(X''-X'\) and \(Y''-Y'\))
\[ I_{X'Y'} = 0 \]
\[ R_0 = \frac{\int \frac{M' \, X' \, ds}{EI}}{I_{Y'}} \]
\[ H_0 = -\frac{\int \frac{M' \, Y' \, ds}{EI}}{I_{X'}} \]
\[ M_0 = -\frac{\int \frac{M' \, ds}{EI}}{\int \frac{ds}{EI}} \]

The orientation of principal axes is given by
\[ \tan 2\beta = \frac{2 I_{XY}}{I_Y - I_X} \]

In case axes of reference are not principal axes equations (12.6), (12.7) and (12.8) give
\[ M_0 = \int \frac{M' \, ds}{EI} \]
\[ R_0 I_Y + H_0 I_{XY} + \int \frac{M'X \, ds}{EI} = 0 \]
\[ R_0 I_{XY} + H_0 I_X + \int \frac{M'Y \, ds}{EI} = 0 \]

The values of \(R_0\) and \(H_0\) can be obtained by solving equations (12.9) and (12.10).
It is evident from the equations (12.4) and (12.5) that the elastic centre is same as centre of analogous column.

Ex. 12.1. Analyze the frame shown in Fig. 12.4 (a) by elastic centre method.

Solution. As the frame is symmetrical about centre line, y axis will pass through the centre of BC.

\[
y_0 = \frac{\int y \, ds}{EI} = \left(\frac{4}{I} + \frac{4}{I}\right) \cdot 2 = \frac{4}{I} + \frac{4}{I} + \frac{6}{3I} = 1.6
\]

\[
I_x = \frac{\int x^2 \, ds}{EI} = 2 \cdot \frac{4}{EI} \times x^3 + \frac{1}{12} \times \frac{1}{3EI} \times 6^3 = \frac{72}{EI} + \frac{6}{EI} = \frac{78}{EI}
\]

\[
I_y = \frac{6}{3EI} \times 1.6^2 + 2 + \frac{1}{12} \times \frac{1}{EI} \times 4^3 + 2 \times \frac{4}{EI} \times (0.4)^2 = \frac{5.12}{EI} + \frac{10.666}{EI} + \frac{1.28}{EI} = \frac{17.066}{EI}
\]

\[
\frac{ds}{EI} = 2 \times \frac{4}{EI} + \frac{6}{3EI} \times 10
\]

\[
\int \frac{Mds}{EI} = \left(\frac{1}{3} \times \frac{36,000 \times 6}{3EI} + \frac{36,000 \times 4}{EI}\right) = \left(\frac{24,000}{EI} + \frac{144,000}{EI}\right) = \frac{168,000}{EI}
\]
\[ M_0 = -\int \frac{M}{{EI}} ds + \frac{163,000}{{EI}} \]
\[ = 16,800 \text{ kg. m.} \]
\[ \int \frac{M yds}{{EI}} = -\frac{1}{3} \times \frac{36,000 \times 6}{{3EI}} (-1.6) - \frac{36,000}{{EI}} \times 4 \times 0.4 \]
\[ = \frac{2400 \times 16}{{EI}} - \frac{16 \times 3600}{{EI}} \]
\[ = -\frac{19,200}{{EI}} \]
\[ H_0 = -\int \frac{M yds}{{EI}} \]
\[ = -\frac{17.066}{{EI}} \]
\[ = 1125 \text{ kg.} \]
\[ \int \frac{M xds}{{EI}} = -\frac{24,000}{{EI}} \times 1.5 - \frac{36,000}{{EI}} \times 4 \times 3 \]
\[ = -\frac{36,000}{{EI}} - \frac{36,000 \times 12}{{EI}} \]
\[ = -\frac{36,000}{{EI}} \times 13 \]
\[ R_n = \frac{M xds}{{EI}} \]
\[ = \frac{36,000 \times 13}{78/EI} \]
\[ = 6000 \text{ kg.} \]

Bending moment at A,
\[ M_{AB} = M_0 + H_0 \times 2.4 - R_n \times 3 \]
\[ = 16,800 + 1125 \times 2.4 - 6000 \times 3 \]
\[ = 16,800 + 2700 - 18,000 \]
\[ = 1500 \text{ kg. m.} \]

Similarly moments can be found at other joints.

**Ex. 12.2.** A link having constant section throughout and consisting of two semi-circles and straight portions is subjected to the action of two equal and opposite axial forces as shown in Fig. 12.5. Assuming that the cross-sectional dimensions of the link are small in comparison with the radius \( R \), determine the bending moment under the load point and at the centre of straight portion.
Solution. The ring is cut at the centre of straight portion as shown in Fig. 12.5 (b). Rigid arm AO is attached to the cut at A. Point O is elastic centre which coincides with C.G. of the section.

Forces \(H_0, R_0\) and moment \(M_0\) are applied to the end \(O\) of the rigid bar.

From symmetry \(H_0 = 0\)

and \(R_0 = \frac{P}{6}\)

\[ M_0 = -\int_{\frac{2\pi R + 2R}{EI}}^{\frac{M'ds}{EI}} ds = \frac{2\pi R + 2R}{EI} \]

\[ M_0 = -\frac{2}{EI} \int_{0}^{\pi/2} P \times R \cos \theta Rd\theta 
+ P \times R \times \frac{R}{2} \]

\[ = -\frac{2}{EI} \left[ P R^2 \left( \sin \theta \right)_{\pi/2}^{0} + \frac{P R^2}{2} \right] \]

\[ = -\frac{2}{EI} \left[ \frac{3PR^2}{2} + \frac{PR^2}{2} \right] \]

\[ = -\frac{3PR^2}{EI} \]

\[ M_0 = \frac{2\pi R + 2R}{EI} \]

\[ = \frac{3PR^2}{2\pi R + 2R} \]

\[ = \frac{3PR}{2\pi + 2} \]

\[ = 0.362 PR. \]

B.M. at loaded point will be

\[ M_0 = 0.362 PR \]

Moment at \(A = M_0 - R_0 \times R = 0.362 PR - \frac{P}{2} \times R \]

\[ = 0.362 PR - 0.5 PR \]

\[ = -0.138 PR. \]

Ex. 12.3. Analyse the frame shown in Fig. 12.6 by elastic centre method.

Solution. Let O be the elastic centre. Co-ordinates of elastic centre are given by