\[ x = \frac{82.7 + 3400 + 12.150 + 5811 + 116, 00 + 1250 - 87.960 + 32.681}{61.334.4} \]

\[ = \frac{92.522}{61.334.4} = 1.509 \text{ m.} \]

**Centre of gravity of pile group.**

Consider piles in 1.5 m. length of retaining wall.

**Taking moments of pile about edge**

\[ x = \frac{2 \times 0.5 + 2 \times 1.25 + 2 \times 2.0 + 1 \times 3 + 1 \times 4}{8} \]

\[ = \frac{1 + 2.5 + 4 + 3 + 4}{8} = \frac{14.5}{8} \]

\[ = 1.8125 \text{ m.} \]

**Eccentricity of resultant force from C.G. of pile group**

\[ = 1.8125 - 1.509 = 0.3035 \text{ m.} \]

**Load on 1.5 m. length of wall**

\[ = 1.5(W + P_V) \]

\[ = 1.5 \times 61.334.4 = 92,001.6 \text{ kg.} \]

**Moment on pile group in 1.5 m. length**

\[ = 1.5(W + P_V) \times 0.3035 \]

\[ = 92,001.6 \times 0.3035 \]

\[ = 27,920 \text{ kg. m.} \]

**Maximum force will be in a pile near the toe**

Due to moment, force in pile,

\[ = F = \frac{Mr_{max}}{\Sigma r^2} \]

\[ r_{max} = 1.942 - 0.5 = 1.442 \text{ m.} \]

\[ \Sigma r^2 = 2 \times 1.442^2 + 2 \times 0.692^2 + 2 \times 0.058^2 + 1 \times 1.058^2 \]

\[ = 4.1586 + 0.9576 + 0.0068 + 1.119 + 4.736 \]

\[ = 10.478 \]

\[ F = \frac{27,920 \times 1.442}{10.478} \]

\[ = 3843 \text{ kg.} \]

**Due to direct load, force in each pile**

\[ = 92,001.6 \]

\[ = 8 \]

\[ = 11,500 \text{ kg.} \]

**Total load on pile near toe**

\[ = 11,500 + 3843 \]

\[ = 15,343 \text{ kg.} \]

Masonry is used as construction material for chimneys. However the use of masonry chimneys is restricted to small heights as for larger heights, sections become heavy, requiring large foundations. This results in uneconomical structure.

The factors governing the design of a chimney are self weight, and wind pressure. The section is designed so that there is no tension anywhere in the section and maximum compressive stress does not exceed the allowable working stress for masonry.

For calculations of wind pressure on chimneys I.S. 875 is followed. Shape factors given in Table 16.2 are used for calculation of wind pressure.

<table>
<thead>
<tr>
<th>Ratio of Height to base width</th>
<th>0 to 4</th>
<th>4 to 8</th>
<th>8 and over</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type of chimney</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Circular</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
</tr>
<tr>
<td>Octagonal</td>
<td>0.8</td>
<td>0.9</td>
<td>1.0</td>
</tr>
<tr>
<td>Square (wind perpendicular to diagonal)</td>
<td>0.8</td>
<td>0.9</td>
<td>1.0</td>
</tr>
<tr>
<td>Square (wind perpendicular to face)</td>
<td>1.0</td>
<td>1.15</td>
<td>1.3</td>
</tr>
</tbody>
</table>


Let $W$ be the weight of chimney above the section considered and $P$ be the resultant wind pressure acting at height '$h$' above the section. The section will be subjected to vertical force $W$ and moment $Ph$. Maximum stress will occur on leeward side and minimum on windward side. Let $A$ be the cross-sectional area of chimney, $I$ moment of inertia and $y_1$ and $y_2$ extreme distances on leeward and windward side respectively.

$$\text{Maximum stress} = \frac{W}{A} + \frac{Phy_1}{I}$$

$$\text{Minimum stress} = \frac{W}{A} - \frac{Phy_2}{I}$$

For no tension,

$$\frac{W}{A} = \frac{Phy_2}{I}$$
Maximum stress \( \frac{W}{A} + \frac{P_{k} y_{1}}{I} \) should not exceed the allowable compressive stress for the masonry.

Ex. 16.15. Find the maximum height for which a circular chimney having 2 m. outside diameter and one metre inside diameter can be used so that no tension develops in the section. The density of masonry is 2400 kg/m\(^3\), and allowable compressive stress in masonry is 30 kg/cm\(^2\). Take intensity of wind pressure as 150 kg/m\(^3\).

Solution. Area of chimney at base

\[
= \frac{\pi}{4} (2^2 - 1^2) = \frac{3\pi}{4} \text{ m}^2.
\]

Self weight of chimney

\[
W = \frac{3\pi}{4} \times H \times 2400 = 1800 \pi H \text{ kg.}
\]

Wind pressure acting on chimney

\[
P = \text{Shape factor} \times \text{projected area} \times \text{intensity} = 0.7 \times 2H \times 150 = 210H \text{, acting at } H/2 \text{ from base.}
\]

Moment at base

\[
= P \times \frac{H}{2} = 210H \times \frac{H}{2} = 105H^2 \text{ kg. m.}
\]

Moment of inertia

\[
= \frac{\pi}{64} (2^4 - 1^4) = \frac{15\pi}{64} \text{ m}^4.
\]

\[
y_1 = y_2 = 1 \text{ m.}
\]

Maximum stress

\[
= \frac{W}{A} + \frac{P_{k} y_{1}}{I} = \frac{1800 \pi H}{3\pi/4} + \frac{105 H^2 \times 1}{15\pi/64}
\]

\[
2400H + \frac{448H^2}{\pi}
\]

Minimum stress

\[
= \frac{W}{A} - \frac{P_{k} y_{2}}{I} = \frac{2400H}{448H^4}
\]
For no tension to occur, in limiting case,

\[ 2400H - \frac{448 \pi H^2}{\pi} = 0. \]

\[ H = \frac{2400 \times \pi}{448} \]

\[ = 16.84 \text{ m.} \]

Maximum stress

\[ 2400 \times 16.84 + 448 \times 16.84 \]

\[ = 80832 \text{ kg/m}^2 \]

\[ = 80832 \text{ kg/cm}^2. \]

**Ex. 16.16.** Check the stability of hollow square chimney section shown in Fig. 16.41. Intensity of wind pressure is 150 kg/m², and density of masonry is 2000 kg/m³.

**Solution.**

**Case I.** Wind acting at right angles to face.

\[ \frac{\text{Height}}{\text{Base}} \frac{14}{27} = 5.58 \]

Shape factor for this \( \frac{H}{B} \) ratio is 1.15

- Wind on top 6 m.
  \[ P_1 = 1.15 \times 150 \times 2.3 \times 6 \]
  \[ = 2380 \text{ kg.} \]

- Wind force on middle 5 m.
  \[ P_2 = 1.15 \times 150 \times 2.5 \times 5 \]
  \[ = 2156 \text{ kg.} \]

- Wind force on bottom 3 m.
  \[ P_3 = 1.15 \times 150 \times 2.7 \times 3 \]
  \[ = 1397 \text{ kg.} \]

Weight of top 6 m.

\[ W_1 = (2.3^2-1.5^2) \times 6 \times 2000 \]

\[ = 38 \times 0.8 \times 12,000 \]

\[ = 32,640 \text{ kg.} \]

Weight of middle 5 m. of chimney

\[ W_2 = (2.5^2-1.5^2) \times 5 \times 2000 \]

\[ = 40,000 \text{ kg.} \]

Weight of bottom 3 m.

\[ W_3 = (2.7^2-1.5^2) \times 3 \times 2000 \]

\[ = 30,240 \text{ kg.} \]

**Check at section 1-1.**

Minimum stress

\[ \frac{W_1}{A_1} - \frac{P_1 \times 3 \times 2.3}{12 (2.3^2-1.5^2)} \]
\[ \text{Maximum stress} \]
\[ = \frac{W_1 + \frac{P_1 \times 3 \times 2.3}{2}}{A_1 + \frac{1}{12} (2.5^4 - 1.5^4)} \]
\[ = \frac{12,000 + 4,297}{12,297 \text{ kg/m}^2}. \text{ Safe} \]

**Check at section 2-2.**

**Minimum stress**

\[ \left[ \frac{W_1 + W_2}{A_2} \right] - \left[ \frac{(P_1 \times 8 + P_2 \times 2.5) \times 2.5}{2} \right] \]
\[ = \frac{32,640 + 40,000}{(2.5^4 - 1.5^4)} - \left[ \frac{(2380.5 \times 8 + 2156 \times 2.5) \times 15}{34} \right] \]
\[ = 18,160 - 10,770 \]
\[ = 7390 \text{ kg/m}^2. \]

**Maximum stress**

\[ = \left[ \frac{W_1 + W_2}{A_2} \right] + \left[ \frac{(P_1 \times 8 + P_2 \times 2.5) \times 2.5}{2} \right] \]
\[ = 18,160 + 10,770 \]
\[ = 28,930 \text{ kg/m}^2. \text{ Safe}. \]

**Check at base**

**Minimum stress**

\[ \left[ \frac{W_1 + W_2 + W_3}{A_3} \right] - \left[ \frac{(P_1 \times 11 + P_2 \times 5.5 + P_3 \times 1.5) \times 2.7}{2} \right] \]
\[ = \frac{32,640 + 40,000 + 30,240}{(2.7^4 - 2.5^4)} - \left[ \frac{(2380.5 \times 11 + 2156 \times 5.5 + 1397 \times 1.5) \times 2.7 \times 6}{8.08} \right] \]
\[ = 20,390 - 13,530 \]
\[ = 7860 \text{ kg/m}^2. \]

**Maximum stress**

\[ = 20,390 + 13,530 \]
\[ = 33,920 \text{ kg/m}^2. \text{ Safe}. \]
**Case II. Wind acting at right angles to diagonal.**

**Height**

\[
\text{Base width} = 2\cdot7\sqrt{2}
\]

\[
= \frac{14}{3.819}
\]

\[
= 3.66
\]

Shape factor = 0.8

Wind force on top 6 m.

\[
P_1 = 0.8 \times 150 \times 3.253 \times 6
\]

\[
= 2342 \text{ kg.}
\]

Wind force on middle 5 m.

\[
P_2 = 0.8 \times 150 \times 3.535 \times 5
\]

\[
= 2121 \text{ kg.}
\]

Wind force on bottom 3 m.

\[
P_3 = 0.8 \times 150 \times 3.819 \times 3
\]

\[
= 1375 \text{ kg.}
\]

**Check at section 1-1**

Minimum stress

\[
= \left[ \frac{W_1}{A_1} \right] \cdot \frac{P_1 \times 3 \times 3.253}{1 \cdot 12 \left( 2 \cdot 3^4 - 1 \cdot 5^4 \right)}
\]

\[
= \left[ \frac{32,640}{2.72} \right] - \left[ \frac{2342 \times 3.253 \times 18}{22.93} \right]
\]

\[
= 12,000 - 5,976
\]

\[
= 6024 \text{ kg/m}^2.
\]

Maximum stress

\[
= \frac{W_1}{A_1} + \frac{P_1 \times 3 \times 3.253}{2 \cdot 12 \left( 2 \cdot 3^4 - 1 \cdot 5^4 \right)}
\]

\[
= 12,000 + 5,976
\]

\[
= 17,976 \text{ kg/m}^2.
\]

**Check at section 2-2.**

Minimum stress

\[
= \left[ \frac{W_1 + W_2}{A_2} \right] - \frac{(P_1 \times 5 + P_2 \times 2.5) \times 3.535}{1 \cdot 12 \left( 2 \cdot 5^4 - 1 \cdot 5^4 \right)}
\]

\[
= \left[ \frac{32,640 + 40,000}{4} \right] - \frac{(2342 \times 8 + 2121 \times 2.5) \times 3.535 \times 6}{34}
\]

\[
= 18,160 - 15,000
\]

\[
= 3160 \text{ kg/m}^2.
\]
Maximum stress \(18,160 + 15,000\) : 33,160 kg/m².

Check at base

Minimum stress

\[
\frac{W_1 + W_2 + W_3}{A_3} \times \frac{(P_1 \times 11 + P_2 \times 5.5 + P_3 \times 1.5) \times 3.819}{2} \left(2.74 - 1.5^4\right) \left[\frac{32,640 + 40,000 + 30,240}{5.04}\right] - \left[\frac{(2342 \times 11 + 2121 \times 5.5 + 1375 \times 1.5) \times 3.819 \times 6}{48.08}\right]
\]

\(= 20,390 - 18,870\)

\(= 1520\) kg/m².

Maximum stress

\(= 20,390 + 18,870\)

\(= 39,260\) kg/m². Safe.

**Ex. 16.17.** Find the resultant lateral pressure and the distance of the point of application from the bottom in the case of retaining wall shown in the Fig. 16.43.

(A. M. I. E. May, 1967)

Weight of soil top portion

\(= 1920\) kg/m², \(\phi = 30^\circ\)

Weight of soil bottom portion

\(= 2680\) kg/m², \(\phi = 30^\circ\).

**Solution.**

\[
k = \frac{1 - \sin \phi}{1 + \sin \phi} = \frac{1 - \sin 30^\circ}{1 + \sin 30^\circ} = \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} = \frac{1}{3}
\]

Pressure in top portion at any depth 'h'

\[p_h = k[\omega h + 1770] = \frac{1}{3}[1920h + 1770]\]

Pressure at top \(= 0 + \frac{1770}{3} = 590\)

Pressure at 3.65 m \(= \frac{1}{3} \times 1920 \times 3.65 + \frac{1770}{3}
\]

\(= 2,336 + 590\).
Total force

\[ P_1 = 590 \times 3.65 \text{ acting at } 2.43 + \frac{3.65}{2} \text{ from base} \]

\[ P_2 = \frac{2.336}{2} \times 3.65 \text{ acting at } 2.43 + \frac{3.65}{3} \text{ from base} \]

Pressure in bottom portion at any depth ‘h’

\[ P_h = \frac{1}{4} [1920 \times 3.65 + 1770] + \frac{1}{4} [2680(h - 3.65)] \]

\[ = 2,926 + \frac{1}{4} \times 2680 \times (h - 3.65) \]

pressure at 3.65 m = 2926 kg/m²

pressure at bottom = \[ 2926 + \frac{2680 \times 2.43}{3} \]

\[ = 2926 + 2170.8 \]

Total force

\[ P_3 = 2926 \times 2.43 \text{ acting } \frac{2.43}{2} \text{ from bottom} \]

\[ P_4 = \frac{2170.8}{2} \times 2.43 \text{ acting at } \frac{2.43}{3} \text{ from bottom} \]

Total force \( P = P_1 + P_2 + P_3 + P_4 \)

\[ = 590 \times 3.65 + \frac{2336}{2} \times 3.65 + 26 \times 2.43 + \frac{2170.8}{2} \times 2.43 \]

\[ = 2154 + 4263 + 7111 \times 0 + 2637 \]

\[ = 16,165 \text{ kg.} \]

C.G. of the pressure from base.

\[ = \frac{2154[2.43 + 1.825] + 4263[2.43 + 1.217] + 7111 \times 1.215 + 2637 \times 0.81}{16,165} \]

\[ = \frac{9164 + 15,830 + 8,640 + 2,136}{16,165} \]

\[ = \frac{35,770}{16,165} = 2.14 \text{ m.} \]

**Ex. 16.18.** A mass concrete dam shown in Fig. 16.14 has a trapezoidal cross-section. The height above the foundation is 61.5 m. and its water face is vertical. The width at top is 45 m. Calculate the necessary minimum width to ensure that no tension shall be developed when water is stored up to 60 m. Draw the pressure diagram at the base of the dam for this condition and indicate the maximum pressure developed.

Density of concrete = 2400 kg/m³.

Density of water = 1010 kg/m³.  
(A.M.I.E. Nov. 1967)
Solution. Let $B$ be width at the base. Take 1 m. length of dam,

$W_1 = 4.5 \times 61.5 \times 2400$

acting at 2.25 from $C$.

$W_2 = (B-4.5) \times \frac{61.5}{2} \times 2400$

acting at $4.5 + \frac{B-4.5}{3}$ from $C$.

$P = \frac{\omega H^3}{2} = 1010 \times \frac{60^3}{2} = 18,180,000$

acting at 20 m. from $C$.

Let the resultant strike the base at $'x'$ metres from $C$.

Take moments about $C$ of all forces.

$W_1 \times 2.25 + W_2 \left(4.5 + \frac{B-4.5}{3}\right) + P \times 20$

$= (W_1 + W_2) x$

\[ x = \frac{W_1 \times 2.25 + W_2 \left(\frac{B}{3} + 3\right) + P \times 20}{W_1 + W_2} \]

\[ \therefore x = \frac{B}{2} + \frac{B}{6} = \frac{2B}{3} \]

For no tension to occur at $C$, eccentricity should be $B/6$,

\[ i.e. \quad x = \frac{B}{2} + \frac{B}{6} = \frac{2B}{3} \]

\[ 4.5 \times 61.5 \times 2400 \times 2.25 + (B-4.5) \times \frac{61.5}{2} \times 2400 \left(\frac{B}{3} + 3\right) \]

\[ + 18,180,000 \times 20 \]

\[ = \frac{2B}{3} \]

\[ = \frac{4.5 \times 61.5 \times 2400 + (B-4.5) \times \frac{61.5}{2} \times 2400}{4.5 \times 61.5 + (B-4.5) \times \frac{61.5}{2}} + 15,150 \]

\[ 6 \times 61.5 \times B + 20.5 B \times (B-4.5) \]

\[ = \frac{61.5}{2} \left[ 20.25 + \frac{B^2}{3} + 1.5 \times 20 - 13.5 \right] + 15,150 \]

\[ 20.5 B^2 + 276.75 B \]

\[ = 10.25 B^2 + 48.125 B + 16,976 \]

\[ 10.25 B^2 + 230 \times 6B - 16976 = 0 \]

\[ B^2 + 22.5 B - 1657 = 0 \]

\[ B = -22.5 \pm \sqrt{5255 + 4 \times 1657} \]

\[ = -11.35 + 42.25 \]

\[ = 31 \text{ m.} \]
\[ W_1 + W_2 = 4.5 \times 61.5 \times 2400 + \left( B - 4.5 \right) \times \frac{61.5}{2} \times 2400 \]
\[ = 61.5 \times 2400 \left[ 4.5 + \frac{(31 - 4.5)}{2} \right] \]
\[ = 61.5 \times 2400 \times 17.25 \]

**Max. Pressure**
\[ = 2 \left[ \frac{W_1 + W_2}{B} \right] = 16.5 \times 2400 \times 17.25 \times \frac{31}{2} \times 2 \]
\[ = 2 \times 82,130 \text{ kg/m}^2. \]
\[ = 164,260 \text{ kg/m}^2. \]

**Ex. 16.19.** A masonry pier of 3.6 m x 4.8 m supports a vertical load of 80 t as shown in Fig. 16.45.

(a) Find the stress developed at each corner of the pier.

(b) What additional load should be placed at the centre of the pier so that there is no tension anywhere in the pier section?

(c) What are the stresses at the corners with the additional load in the centre. (A.M.I.E. May 1968)

**Solution.**

\[ P = 80 \text{ t} \]
\[ M_x = 80 \times 0.6 = 48 \text{ t.m.} \]
\[ M_y = 80 \times 1.2 = 96 \text{ t.m.} \]

\[ I_x = \frac{4.8 \times 3.6^3}{12} = 1.2^3 \times 108 \]
\[ I_y = \frac{3.6 \times 4.8^3}{12} = 1.2^3 \times 192 \]
\[ A = 4.8 \times 3.6 = 1.2^2 \times 12 \]

**Stress at (x, y)**

\[ f = \frac{P}{A} + \frac{M_x}{I_x} x + \frac{M_y}{I_y} y \]
\[ f = \frac{80}{1.2^2 \times 12} + \frac{96}{1.2^3 \times 192} x \]
\[ + \frac{48}{1.2^3 \times 108} y \]
\[ = 4.629 + 2.894 x + 2.571 y \]
\[ f_A = 4.629 - 2.894 \times 2.4 - 2.571 \times 1.8 \]
\[ = 4.629 - 6.944 - 4.629 \]
\[ = -6.944 \text{ t/m}^2. \quad (x = -2.4, \quad y = -1.8) \]
\[ f_B = 4.629 + 2.892 \times 2.4 - 2.571 \times 1.8 \]
\[ = 4.629 + 6.944 - 4.629 \]
\[ = +6.944 \text{ t/m}^2. \quad (x = +2.4, \quad y = -1.8) \]
DAMS, RETAINING WALLS AND CHIMNEYS

\[ f_c = 4.629 + 6.944 + 4.629 \]
\[ = 16.202 \text{ t/m}^2 \]
\[ f_D = 4.629 - 6.944 + 4.629 \]
\[ = +2.314 \text{ t/m}^2. \]

If no tension is to occur, additional force \( P_1 \), should be such as to cause compressive stress of 6.944 t/m², so that stress at \( A \) be zero.

\[ P_1 = 6.944 \times 3.6 \times 4.8 = 120 \text{ tonnes.} \]

With additional load stresses will be—

\[ f_A = 0 \]
\[ f_b = 6.944 + 6.944 = 13.888 \text{ t/m}^2 \]
\[ f_c = 16.202 + 6.944 = 23.146 \text{ t/m}^2 \]
\[ f_D = 2.314 + 6.944 = 9.258 \text{ t/m}^2. \]

PROBLEMS

1. A masonry dam of trapezoidal section is proposed to be made 10 m. high, 1 m. thick at top and 5 m thick at base, to store water against its vertical face. Upon scrutiny of the design it is found that if water is stored up to the top of the dam there will be considerable tension in the masonry at the heel. In order to avoid tension in the masonry, calculate:

   (a) Upto what height water can be stored without making any alteration in the dam?

   (b) What should be the width of the base if water is to be stored up to the top of the dam?

   Weight of masonry = 2000 kg/m³.

2. A retaining wall having a batter 1 in 12 on the back is 10 m. high. The backfill slopes upward from the top of the wall at 15° with the horizontal. Weight of backfill material is 1600 kg/m² and angle of repose 25° and co-efficient of friction between the wall and backfill = 0.35. Find the magnitude, direction and point of application of the total earth pressure on the wall.

3. A retaining wall has top width of 1 m., bottom width 3 m. and its height is 6 m. The front batter is 1 in 12. The earth at the back has surcharge of 18°. If the angle of repose of the soil is 30° and the weight of the earth-fill is 1800 kg/m³, and the density of masonry is 2400 kg/m³, determine maximum and minimum pressures at base.

4. A trapezoidal retaining wall, 8 m. high, 1 m wide at top and 4 m. wide at bottom retains earth against its vertical face. The
earth is in level at top and supports a road pavement carrying a uniform load of 1200 kg/m². If the earth is assumed granular and weighs 1800 kg/m² with angle of internal friction of 30°, and the masonry weighs 2400 kg/m² find,

(a) the maximum horizontal pressure on the wall and its position,

(b) the maximum and minimum intensities of pressure at the base,

(c) the factor of safety against sliding if coefficient of friction between the base of the wall and soil below is 0.6.

5. A masonry retaining wall, 5 m. high, has its earth face with a batter of 1 in 7.5. Its top and bottom widths are 1 m. and 2 m. respectively. The earth sustained positive surcharge of 20° and the angle of repose of the soil is 30°. The angle of friction between the wall and the soil is 30°. The weights of masonry and the soil are 2000 kg/m² and 1800 kg/m², respectively. Determine the total thrust on the wall and pressure distribution at base.

6. A retaining wall is vertical on the earth side and is 15 m. high. It supports a bank of dry loose earth sloping upwards from the top of the wall at an angle of 30° to the horizontal. The angle of repose of the earth is 40°. Find the total pressure on the wall by wedge theory. Assume the coefficient of friction along the plane of rupture and that along the back of the wall equal. Take the density of soil as 1600 kg/m³.

7. A retaining wall 10 m. high has back slope of 20° with vertical. It retains earth at surcharge of 30°. The angle of repose of soil is 30° and angle of friction between wall and earth is 20°. Find total pressure against the wall by Rebhann's construction.

8. Assuming uniformly varying stress across the base, find the limit of height of a triangular masonry dam with water up to top of the vertical face in order that vertical compressive stress across the base shall not exceed 9 kg/cm². Masonry weighs 2 kg/litre.

(A.M.I.E. May 1969)

9. An R.C.C. footing rectangular in plan 2 m × 3 m. carries 4 vertical concentrated loads of 10 t, 20 t, 30 t and 40 t which are located as shown in Fig. 16.46. Neglecting the self weight of the footing.

(i) Calculate the intensity of loading on the foundation at each of the corners A, B, C and D.

(ii) Determine the location of another 50 t load with reference to X and Y axes which will make the intensity of loading uniform at all corners.

(A.M.I.E. Nov. 1970)
10. A concrete dam has its upstream face vertical and a top width of 3.05 m. Its downstream face has a uniform batter. It stores water to a depth of 14.6 m, with a free board of 1.825 m, as given in Fig. 16.47. The weights of water and concrete may be taken as 1000 kg/m³ and 2570 kg/m³.

(a) Calculate the minimum width at bottom for no tension in concrete. Neglect uplift.

(b) Calculate the extreme intensities of pressure on the foundation when reservoir is empty. (A.M.I.E. May 1971)
PLASTIC ANALYSIS OF STRUCTURES

17.1. Previous chapters of this volume dealt with analysis of linear elastic structures i.e., structures obeying Hooke's law and strained within elastic limit. Sometimes it is required to know the behaviour of structure when loaded beyond elastic limit. Also it becomes economical to design the structures based on ultimate strength and hence it is necessary to know the behaviour of structure beyond elastic limit.

For ductile materials like mild steel the stress-strain curve has form as shown in Fig. 17.1. \( B \) is upper yield point, \( C \) is lower yield point, \( CD \) is plastic range in which strain increases at constant stress. After point \( D \), on further loading strain increases and the deformation is partly elastic and partly plastic. Point \( E \) corresponds to ultimate stress. After this point, specimen starts forming neck and load decreases till it fractures at point \( F \).

Ideal elastic-plastic material is defined as one which has definite elastic range and after this range material becomes plastic as shown in Fig. 17.2. \( AC \) is elastic range and \( CD \) is plastic range. Plastic analysis of structures is based on this ideal elastic-plastic stress-strain curve.

Ductile materials like mild steel follow to certain extent elastic-plastic behaviour up to point \( D \). For analysis upper yield point is not considered and the material is said to be elastic upto lower yield point \( C \) and then plastic upto \( D \).

17.2. Assumptions in Bending Beyond Yield Point.
The following assumptions are made in theory of bending after yield point.
1. The cross-section plane before bending remain plane after bending.
2. Longitudinal fibres are free to expand and contract without affecting the fibres in lateral direction.
3. Modulus of Elasticity has same value in tension and compression.
4. The material is homogeneous and isotropic.
5. There are no axial forces on the section.
6. The cross-section is symmetrical about axis at right angles to axis of bending.

17.3. Plastic Moment of a Section.

Consider a cantilever of rectangular section and subjected to increasing moment \( M \) at the free end. When the moment is such that stress in the extreme fibres does not reach yield stress, variation of stress will be from zero at neutral axis to maximum in the extreme fibres. When the moment is increased extreme fibres will reach yield and the stress in fibres towards neutral axis will be less than yield stress. The stress diagram will be a triangle with zero stress at neutral axis and yield stress in extreme fibres. At this point, yield moment \( M_y = f_y \times \frac{bd^2}{6} \), where \( f_y \) is yield stress.

With further increase in moment, the stress in extreme fibres will remain same i.e., yield stress but some portion from extreme fibre towards neutral axis will also be stressed to yield stress as it will have reached the plastic stage. If \( y_0 \) is the distance from extreme fibre upto which stress is yield stress, value of moment at this point will be,

\[
M = f_y \times y_0 \times b \times (d - y_0) + f_y \times b(d - 2y_0) \times \frac{d}{2}(d - 2y_0)
\]

\[
= f_y [d(y_0 - y_0^2) + \frac{1}{4}(d^2 - 4y_0 - 4y_0^3)]
\]

\[
= f_y \left[ \frac{d^2}{3} - \frac{dy_0}{3} + \frac{y_0^2}{3} \right]
\]

\[
= f_y \frac{b}{3} \left[ d^2 - dy_0 + y_0^2 \right]
\]
With further increase in the moment, the stress in the entire section reaches yield stress and whole section will be plastic. Plastic moment will be

\[
M_p = \frac{f_y b}{3} \left[ d^3 - d \times \frac{d}{2} \times \left( \frac{d}{2} \right)^2 \right] \\
= \frac{f_y b}{3} \times \frac{3}{4} d^3 \\
= \frac{f_y b d^3}{4}
\]

This is called fully plastic moment of the section.

\[
\frac{M_p}{M_y} = \frac{4}{f_y b d^2} = 1.5.
\]

17.4. Shape Factor.

Shape factor is defined as the ratio of fully plastic moment \( M_p \) of a section to the yield moment \( M_y \) of the section.

(a) The ratio \( \frac{M_p}{M_y} = 1.5 \) found in article 17.3 for a rectangular section is shape factor.

(b) Circular section

\[
M_y = f_y \times \frac{I}{d^2/2} \\
= f_y \times \frac{\pi}{32} d^3
\]

To find fully plastic moment consider a strip of thickness \( b \) at distance \( y \) from N.A.

\[
y = \frac{d}{2} \sin \theta
\]

\[
dy = \frac{d}{2} \cos \theta d\theta
\]

\[
b = \frac{2d}{2} \cos \theta = d \cos \theta
\]

Fully plastic moment \( M_p \)

\[
= 2 \int_0^{\pi/2} \left( f_y b dy \right) \times y \\
= 2 \int_0^{\pi/2} f_y \times d \cos \theta \times \frac{d}{2} \cos \theta \sin \theta d\theta \\
= \frac{f_y d}{2} \int_0^{\pi/2} \cos^3 \theta \sin \theta d\theta
\]
\[ M_p = f_p d^3 \left( \frac{BD^3}{6} - \frac{bd^3}{6D} \right) + \frac{f_p d}{2} (B-b) \frac{d}{2} \]

Putting \( d = kD \) and \( b = k'R \)

\[ M_p = f_p \left( \frac{BD^3}{4} + \frac{d^3}{4} - \frac{d^3}{4} \right) \]

Taking \( k = k' = 0.9 \)

\[ \frac{M_p}{M_v} = \frac{3}{2} \left[ \frac{1 - 0.729}{1 - 0.6561} \right] = 1.182 \]

For rolled sections, in common use shape factor is between 1.15 to 1.17.

\[ M_p = f_p \times \frac{I}{n} \]

\[ = \frac{f_p}{n} \left[ \frac{t_x n^3}{3} + \frac{t_x (D-n)^3}{3} + \frac{(B-t_x) t_1^3}{12} \right] + (B-t_x) t_1 \left( D-n - \frac{t_1}{2} \right)^2 \]

(d) Unsymmetrical section.

The neutral axis passes through C.G. of the section. Take T-section having N.A. at \( n \) distance from top.
For fully plastic condition the area will be axis which divides the area equally as stress throughout will be \( f_p \) and for total compression to be equal to total tension, the two areas should be equal.

The value of \( M_p \) is equal to \( f_p \times \frac{\text{area of section}}{\text{distance between C.G. of compression area and C.G. of tension area.}} \)

Ex. 17·1. Find the shape factor of the T-section shown in Fig. 17·8. Also find fully plastic moment if \( f_p = 2400 \text{ kg/cm}^2 \).

Solution. Taking moments about top, depth of N.A. is given by

\[
n = \frac{10 \times 1 \times 5 + (10 - 1) \times 1 \times 9.5}{10 \times 1 + (10 - 1) \times 1} = \frac{50 + 85.5}{10 + 9} - \frac{135.5}{19} = 7.13 \text{ cm.}
\]

\[
\frac{1 \times 10^3}{12} + 1 \times 10 \times (7.13 - .5)^2 + \frac{0 \times 1^2}{12} + 9 \times 1 \times (2.37)^2
\]

\[
= \frac{250}{3} + 10 \times 2.13^2 + 0.75 + 9 \times 2.37^2
\]

\[
= 83.33 + 45.37 + 0.75 + 50.55
\]

\[
= 180.00
\]

\[
M_p = f_p \times \frac{I}{y_{max}} = f_p \times \frac{180}{7.13}
\]

Total area = \( 10 \times 1 + 9 \times 1 = 19 \text{ cm}^2 \)

Half area will be \( 9.5 \text{ cm}^2 \)
The N.A. will pass through the horizontal seat at 0·95 cm from bottom

\[ M_p = 9 \times f_v \left( 4·5 + 0·05 + \frac{0·95}{2} \right) \]

\[ + 10 \times 0·05 \left( \frac{0·05}{2} + \frac{0·95}{2} \right) f_v \]

\[ = \frac{9 \times 10 \times 0·05}{2} \times f_v + \frac{0·5 \times 1}{2} \times f_v \]

\[ = 45·475 f_v \]

Shape factor: \[ \frac{180}{7·13 \times f_v} \]

\[ M_p = 45·475 \times 2400 \]

\[ = 109,140 \text{ kg cm.} \]

17·5. Plastic Hinges.

In a structure, fully plastic section is subjected to fully plastic moment \( M_p \). If the load is increased on the structure, there will be no change in moment at the fully plastic section but the structure can rotate about this section. Therefore a fully plastic section acts like a hinge and is called plastic hinge.

(a) Simply supported beam loaded with concentrated load \( W \).

Consider a simply supported beam loaded with a concentrated load \( W \). Maximum B.M. occurs under the load and has value \( \frac{Wab}{l} \).

If the load is increased, at some value of load \( \lambda W \) the section \( X \) reaches fully plastic state and will be subjected to fully plastic moment \( M_p \). The adjoining sections will not be fully plastic, but some portions of these sections will be plastic as shown shaded in Fig. 17·9 (c).

With the formation of plastic hinge under the concentrated load, the structure can undergo any rotation as it transforms into mechanism and will collapse.

\[ M_p = \frac{\lambda Wab}{l} \]

\[ \therefore \quad \lambda = \frac{M_p l}{Wab} \]

where \( \lambda \) is called collapse load factor.

For central load, load factor

\[ \lambda = \frac{M_p l}{W \times \frac{l}{2} \times \frac{l}{2}} = \frac{4M_p}{Wl} \]

\[ (a), (b), (c) \]

\[ g \text{ 17·9} \]
If \( M_p \) is the moment when yielding has first commenced, corresponding to load \( \lambda' W \),

\[
\lambda' = \frac{4M_p}{Wl}
\]

\[
\lambda = \frac{M_p}{M_p} = \text{Shape factor.}
\]

(b) Propped cantilever loaded with concentrated load.

Consider a propped cantilever subjected to concentrated load \( W \) as shown in Fig. 17-10 (a).

Maximum \(-ve\) B.M. occurs at the fixed end and maximum \(+ve\) M.M. under the load. If the load is increased, at some value of load first plastic hinge is formed either at fixed end \( A \) or under load point \( X \), depending on where B.M. is numerically maximum. That section will have fully plastic moment \( M_p \). The structure does not collapse at this stage as it has not formed into mechanism.

In case first plastic hinge is formed at \( A \), the beam will act as a simply supported beam for further loading till second hinge is formed at \( X \). This turns the beam into mechanism and collapse occurs.

In case first plastic hinge is formed at \( X \), the structure will be determinate and \( XB \) will act as supported span with end \( X \) supported on free end of cantilever \( AX \). With further increase in the load, the moment will increase only in portion \( AX \) and hinge will be formed at \( A \). This will transform the beam into mechanism and collapse occurs.

At collapse there will be two plastic hinges, one at \( A \) and the other at \( X \). Let the collapse load be \( \lambda W \).

Collapse load can be found by principle of virtual work.

In the mechanism formed, let \( \lambda W \) be given virtual displacement, defined by rotation \( \theta_1 \) and \( \theta_2 \) at \( A \) and \( B \) respectively. Rotation at \( X \) will be \( (\theta_1 + \theta_2) \).
Work done by plastic moments at A and X will be
\[-M_p \theta_1 - M_p (\theta_1 + \theta_2)\]. The work done by external load \(\lambda W\) will be
\(\lambda W \times y\).

Total work done is zero.

\[
\lambda \quad -M_p \theta_1 - M_p (\theta_1 + \theta_2) + \lambda W \times y = 0
\]
\[M_p (2 \theta_1 + \theta_2) = \lambda W \times y\]
\[\therefore \quad \lambda = \frac{M_p}{W} \times \frac{(2 \theta_1 + \theta_2)}{y} \]
\[y = a \theta_1 = b \theta_2\]
\[\therefore \quad y = \frac{M_p}{W} \left( \frac{2 \theta_1}{a \theta_1} + \frac{\theta_2}{b \theta_2} \right) = \frac{M_p (2b + a)}{Wab}\]

If load is central load, \(a = b = l/2\)
\[\therefore \quad \lambda = \frac{M_p (l + l/2)}{W \frac{l}{2} \times \frac{l}{2}}\]
\[M_p \times \frac{3}{2} l = \frac{6 M_p}{W l}\]
\[= \frac{W}{l} \times \frac{M_p}{3 l}\]

Let \(\lambda' W\) be the load when yielding has first commenced
\[M_A = \lambda' W ab (2b + a)\]
\[\lambda = \frac{\lambda'' W}{2 l^2} = \frac{l}{2}\]
\[M_p = \\frac{\lambda' W}{2 l^2} \times \frac{l}{2} \left( l + \frac{l}{2} \right) = \lambda' \times 3 W l\]
\[\lambda' = \frac{16}{\frac{M_p}{W l}} \times \frac{M_p}{W l}\]
\[\frac{\lambda}{\lambda'} = \frac{6 M_p}{\frac{M_p}{W l}} - \frac{9}{8} \times \frac{M_p}{M_p}\]
\[= \frac{9}{8} \times \text{Shape factor.}\]

Collapse load factor can also be found from B.M. diagram at collapse. At failure B.M. at support A and under load will be \(M_p\).

\[
\lambda \quad M_p = \frac{\lambda W ab}{l} - \frac{M_p \times b}{l}\]
\[\therefore \quad \lambda = \frac{M_p (2b + a)}{W ab}\]
3. Fixed beam loaded with concentrated load

Consider a fixed beam loaded with concentrated load \( W \). Maximum negative B.M. occurs at the support near the load. Maximum +ve B.M. occurs under the load. At other support B.M. is -ve.

When the load is increased first hinge is formed at the support near the load. For further increase in load, the structure

![Diagram showing beam with concentrated load and bending moment diagram](image)

will act as propped cantilever and second hinge will be formed either under the load or at second support. At this stage beam is indeterminate structure. With further increase in the load third hinge is formed thus transforming beam into mechanism. At collapse there will be two plastic hinges at supports and one under the load. Let \( \lambda W \) be the load at collapse.

In the mechanism formed, let \( \lambda W \) be given virtual displacement defined by rotations \( \theta_1 \) and \( \theta_2 \) at \( A \) and \( B \) respectively, as shown in Fig. 17.11 (c).

Work equation gives
\[
\lambda Wy - M_p \theta_1 - M_p \theta_2 - M_p (\theta_1 + \theta_2) = 0
\]
\[
y = a \theta_1 = b \theta_2
\]
\[ \therefore \lambda W = M_p \frac{\theta_1}{a \theta_1} + M_p \frac{\theta_2}{b \theta_2} + M_p \left( \frac{\theta_1}{a \theta_1} + \frac{\theta_2}{b \theta_2} \right) \]

\[ = \frac{M_p}{a} + \frac{M_p}{b} + M_p \left( \frac{1}{a} + \frac{1}{b} \right) \]

\[ = 2M_p \left( \frac{1}{a} + \frac{1}{b} \right) \]

\[ = \frac{2M_p}{ab} (a + b) \]

1. \[ \lambda = \frac{2M_p l}{W ab} \]

If \( a = b = l/2 \)

\[ \lambda = \frac{2M_p l}{W \times \frac{l}{2} \times \frac{l}{2}} \]

\[ = \frac{8M_p}{Wl} \]

Let \( \lambda' W \) be the load when yielding has just commenced.

Yield moment,

\[ M_p = \frac{\lambda' W l}{8} \text{ (for central load)} \]

\[ \lambda' = \frac{8M_p}{Wl} \]

\[ \frac{\lambda'}{\lambda} = \frac{M_p}{M_p} = \text{Shape factor}. \]

Collapse load factor can also be found from B.M. diagram at failure. At failure B.M. at each of supports and under the load will be \( M_p \) as shown in Fig. 17.11 (c).

\[ \therefore M_p = \frac{\lambda W ab}{l} = -M_p \]

or \[ M_p = \frac{\lambda W ab}{2l} \]

\[ \therefore \lambda = \frac{2M_p l}{W ab} \]

4. Continuous beam.

Consider continuous beam of two equal spans with two equal loads at mid span as shown in Fig. 17.12. In the elastic range B.M. diagram will be as shown in Fig. 17.12 (b). Maximum negative B.M. at sup, crt B being \( \frac{3Wl}{16} \) and minimum positive B.M. being under the loads equal to \( \frac{5}{32} Wl \).
As the load is increased, first plastic hinge will be formed at B. With the formation of plastic hinge at B, structure becomes determinate and spans \( AB \) and \( BC \) will be as simply supported spans for further load. With further increase in load, plastic hinges will be formed simultaneously at \( D \) and \( E \), centres of \( AB \) and \( BC \) respectively. With the formation of hinges structure turns into mechanism and collapses. Let the collapse load be \( \lambda W \). In the mechanism formed, let loads \( \lambda W \) be given virtual displacement defined by rotation \( \theta \).

Work equation gives
\[
\lambda W \times y + \lambda W \times y - 2M_{2} \theta - 2M_{3} \theta - 2M_{3} \theta = 0
\]

Thus
\[
2\lambda W \times \frac{l}{2} \theta = 6M_{3} \theta
\]
\[ \lambda = \frac{6M_p}{Wl} \]

Collapse load can also be found from B.M. diagram at col.

\[ \text{At collapse B.M. at } B, D \text{ and } E \text{ will be fully plastic moments } M_p \text{ as shown in Fig. 17.12 (c).} \]

\[ \text{B.M. at } D: \quad M_p = \frac{AWl}{4} - \frac{M_p}{2} \]

\[ \frac{6M_p}{Wl} \]

11.1. Effect of uniformly distributed loads

In all previous cases the loads acting were concentrated loads and the maximum +ve B.M. was occurring under the loads. Hence plastic hinges formed were at supports or under the loads. However if the load acting on the span is uniformly distributed, the plastic hinge will be formed at the point of maximum +ve B.M.

Consider a beam having

moments as \( M_L \) and \( M_R \)

and carrying uniformly distributed load throughout the span. Let \( M_C \) be the B.M. at centre and maximum B.M. be at distance \( x_0 \) from the centre. At point of maximum B.M., S.F. is zero.

\[ w x_0 = \frac{M_R - M_L}{l} \]

\[ x_0 = \frac{M_R - M_L}{wl} \]

\[ M_C = M_{max} - \frac{w x_0^2}{2} \]

\[ M_{max} = M_C + \frac{w}{2} \left( \frac{M_R - M_L}{wl} \right)^2 \]

\[ = M_C + \frac{1}{2wl^2} (M_R - M_L)^2 \]

\[ M_C = \frac{wl^2}{8} + \frac{(M_R + M_L)}{2} \]
Consider a propped cantilever loaded with uniformly distributed load throughout. Let \( \lambda W \) be the collapse load. Moment at fixed end \( A \) will reach value of fully plastic moment \( M_p \) and also at some section near the centre it will reach \( M_p \). Plastic hinges will be formed at these sections to transform structure into mechanism.

\[
x_0 = \frac{O - (\lambda M_p)}{\lambda w l}
\]

\[
\frac{M_p}{\lambda w l}
\]

\[
M_C = \frac{\lambda w l^3}{8} - \frac{M_p}{2}
\]

But

\[ M_p = M_C + \lambda w x_0^3 \]

Substituting \( M_C \) and \( x_0 \)

\[
M_p = \left( \frac{\lambda w l^3}{8} - \frac{M_p}{2} \right) + \frac{\lambda w}{2} \times \frac{M_p^2}{\lambda w l^3}
\]

\[
\frac{\lambda w l^3}{8} - \frac{M_p}{2} + \frac{M_p^2}{2\lambda w l^3} = 0
\]

\[
\frac{\lambda w l^3}{8} - \frac{3}{2} M_p + \frac{\lambda w l^3}{8} = 0
\]

\[
M_p^3 - 3\lambda w l^3 M_p + \frac{\lambda^2 w^2 l^4}{4} = 0
\]

\[
M_p - 3\lambda w l^3 M_p + (1.5\lambda w l^3)^2 - \frac{\lambda^2 w^2 l^4}{4} + (1.5\lambda w l^3)^2
\]

\[
(M_p - 1.5\lambda w l^3)^2 = 2\lambda^2 w^2 l^4
\]

\[
M_p = \pm \lambda w l^3 \sqrt{2 + 1.5\lambda w l^3}
\]

\[
= \lambda w l^3 (1.5 - \sqrt{2}) \text{ (neglecting +ve sign as it gives value of } x_0 \text{ greater than } l)
\]

\[
M_p = \lambda w l^3 (1.5 - 1.414)
\]

\[
= 0.086 \lambda w l^3
\]

\[
\lambda = \frac{M_p}{0.086 w l^3}
\]

\[
= \frac{11.628}{w l^3} M_p
\]
Ex. 17-2. Find the collapse load factor for the fixed beam loaded with uniformly distributed load of \( w \) unit length as shown in Fig. 17-15. Fully plastic moment of section is \( M_p \).

Sol. Maximum B.M. occurs at supports \( A \) and \( B \) in the elastic stage and hence plastic hinges will be formed at these points simultaneously. As the load is further increased third plastic hinge will be formed at the centre and thus forming the beam into mechanism.

Let \( \lambda w \) be the collapse load. Let the mechanism formed be given virtual displacement defined by angle \( \theta \) at ends.

Work equation gives,

\[
\lambda wl \times \frac{y}{2} - M_p \theta - 2M_p \theta - M_p \theta = 0
\]

\[
\frac{\lambda wl}{2} \times \frac{\theta}{2} - 4M_p \theta = 0
\]

\[
\lambda = \frac{16M_p}{wl^2}
\]

This value can also be obtained from B.M. diagram at collapse.

From B.M. diagram,

B.M. at centre = \( M_p \)

\[
\frac{\lambda wl^2}{8} - M_p
\]

\[
2M_p : \frac{\lambda wl^2}{8}
\]

\[
\lambda = \frac{16M_p}{wl^2}
\]

Ex. 17-3. Find the collapse load factor for the continuous beam loaded as shown in Fig. 17-16 (a). Fully plastic moment of each pane is \( M_p \).
Solution. In the elastic stage maximum B.M. occurs at support B. Hence first plastic hinge will be formed at support B. As the load is increased further plastic hinges will be formed in spans AB and BC at points where B.M. is maximum. With the formation of plastic hinges in spans AB, BC and support B, the structure transforms into mechanism and collapses. Let collapse load factor be \( \lambda \). Let plastic hinges be formed in spans AB and BC at distance \( x_0 \) from the centre of span.

\[
\frac{M_R - M_L}{\lambda wl} = 0 - \frac{(-M_P)}{\lambda wl} = \frac{M_P}{\lambda wl}
\]

\[
M_P = M_c + \lambda w \frac{(x_0)^2}{2} = \left( \frac{\lambda wl^2}{8} - \frac{M_P}{2} \right) + \lambda w \left( \frac{M_P}{\lambda wl} \right)^2
\]

\[
M_P = \frac{\lambda wl^2}{8} \left( 2 - \frac{M_P}{\lambda wl} + \frac{M_P^2}{2\lambda wl^2} \right)
\]

\[
\frac{M_P^2}{2\lambda wl^2} - \frac{3M_P}{2} + \frac{\lambda wl^2}{8} = 0
\]

\[
\lambda^2 - \frac{12M_P}{w^2l^2} \lambda + \frac{4M_P^3}{w^3l^4} = 0
\]

\[
\lambda = \frac{12M_P}{w^2l^2} \pm \sqrt{\frac{144M_P^2}{w^2l^4} - 16M_P^3}
\]

\[
\therefore \lambda = \frac{6M_P \pm 2\sqrt{9M_P^2 - M_P^3}}{w^2l^2}
\]

\[
= \frac{6M_P \pm 2\sqrt{2M_P^3}}{w^2l^2}
\]

\[
= \frac{2M_P}{w^2l^2} \left( 3 \pm 2\sqrt{2} \right)
\]

\[
= \frac{11.656M_P}{w^2l^2}
\]

\[
\text{or} \quad = \frac{0.344M_P}{w^2l^2}
\]
Substituting value of $\lambda$ in (1)

$$x_0 = \frac{M_p}{11.656 \frac{wl}{M_p \times wl}}$$

$$= \frac{l}{11.656}$$

or

$$x_0 = \frac{M_p}{0.344 \frac{wl}{M_p \times wl}}$$

$$= \frac{l}{0.344}$$

This is greater than $l$, hence not possible.

$$\therefore \lambda = \frac{11.656 M_p}{wl^4}.$$  

Ex. 17.4. Find the collapse load factor for the continuous beam shown in Fig. 17.17 (a). All spans have same value of fully plastic moment $M_p$.

Solution. B.M. diagram in the elastic stage is shown in Fig. 17.17 (b).

Absolute maximum B.M. occurs somewhere near the centre of spans $AB$ and $CD$. Thus first two plastic hinges will be formed at these sections. With further increase in the load, plastic hinges will be formed at supports $B$ and $C$ and thus transforming the structure into mechanism.
Let $\lambda$ be the collapse load factor.

\[
x_0 = \frac{M_R - M_I}{\lambda w f} = \frac{\lambda x_0}{6 \times 1 \times \lambda} \quad \frac{M_p}{6 \lambda}
\]

\[
M_C = \frac{\lambda \times 1 \times 6 \times 6 \times M_P}{8} = \frac{9 \lambda - M_P}{2}
\]

B.M. at plastic hinge

\[
M_p = M_c + \frac{\lambda w x_0}{2}
\]

\[
= \frac{9 \lambda}{2} \lambda - \frac{M_P}{2} + \frac{\lambda}{2} \left( \frac{M_P}{6 \lambda} \right)^2
\]

\[
M_P = 4.5\lambda - \frac{M_P}{2} + \frac{M_P}{72\lambda}
\]

\[
\frac{M_P^2}{72\lambda} - \frac{3}{2} M_P + 4.5\lambda = 0
\]

\[
324\lambda^2 - 108 M_P \lambda + M_P^2 = 0
\]

\[
\lambda = \frac{108 M_P \pm \sqrt{(108 M_P)^2 - 4 \times 324 M_P^2}}{2 \times 324}
\]

\[
= \frac{M_P}{a} \left\{ \frac{72 \sqrt{2 M_P}}{64} \right\}
\]

\[
= \frac{M_P}{18} \left( 3 \pm 2 \sqrt{2} \right)
\]

Only $+ve$ sign is to be taken as by taking $-ve$ sign and substituting the value of $\lambda$ in (1), value of $x_0$ comes to be greater than $l$.

\[
\lambda = \frac{M_P}{18} \left( 3 + 2 \sqrt{2} \right)
\]

\[
= 0.3235 \quad M_P
\]

17.7. Statical and Mechanism Methods of Analysis of Continuous Beams.

Consider a continuous beam of same section throughout and loaded as shown in Fig. 17.18 (a). The beam is subjected to proportional loading i.e., loads bear constant ratio to one another as the loads on the beam are increased.

When the loads are within elastic range, bending moment diagram can be drawn by using theorem of three moments or any other method.

Applying theorem of three moments to spans $AB$ and $BC$.
PLASTIC ANALYSIS OF STRUCTURES

(a)

(b)

ELASTIC BENDING MOMENTS

(c)

MOMENT AT FORMATION OF FIRST HINGE

(d)

(e)

(f)

(g)

Fig. 17-18
\[ 2M_B(l+l) + M_C \]

\[ = -6 \left( \frac{2Wl}{4} \times \frac{l}{2} \times \frac{l}{2} + \frac{Wl}{8} \times l \times l \right) \]

\[ = -6 \left( \frac{Wl^3}{8} + \frac{Wl^3}{16} \right) \]

\[ = -\frac{9}{8} Wl^2 \]

\[ 4M_B + M_C = -\frac{9}{8} Wl \quad \text{(1)} \]

Applying theorem of three moments to spans BC and CD,

\[ M_B \times l + 2M_C(l+l) \]

\[ = -6 \left( \frac{Wl}{4} \times \frac{l}{2} \times \frac{l}{2} + \frac{Wl}{4} \times \frac{l}{2} \times \frac{l}{2} \right) \]

\[ = -\frac{3}{4} Wl^2 \]

\[ M_B + 4M_C = -\frac{3}{4} Wl \quad \text{(2)} \]

Solving Eqs. (1) and (2)

\[ M_C = -\frac{Wl}{4} \]

\[ M_C = -Wl \]

The B.M. diagram is shown in Fig. 17-18 (b).

If the loads are increased in same proportion, first hinge will be formed at E. Let the loads be multiplied by \( \lambda_1 \) to have the first hinge

\[ \therefore \quad \frac{3}{5} Wl \lambda_1 = M_P \]

\[ \lambda_1 = \frac{8M_P}{3Wl} \]

Value of bending moments at other points in terms of \( M_P \) is shown in Fig. 17-18 (c).

With the formation of a hinge at E, the structure does not collapse as it does not transform into mechanism.

For further load structure will act as continuous beam EBCD with overhang EB as shown in Fig. 17-18 (d).

Let the additional load be \( \lambda_3 \) times the original load to form second plastic hinge.
Applying theorem of three moments,

\[ M_b l + 2M_c (l + l) = -6 \left( \frac{W \lambda_2 l}{4} \times \frac{l}{2} \times \frac{l}{2} \right) \times 2 \]

\[ -\lambda_2 WL^2 + 4MC_l = -6 \left( \frac{W \lambda_2 l^2}{8} \right) \]

\[ = \frac{3}{4} W \lambda_2 l^2 \]

The B.M. diagram will be as shown in Fig. 17.18 (c). Maximum B.M. occurs at B. Thus second plastic hinge will be formed at B.

\[ \therefore \lambda_2 WL + \frac{2}{3} M_P = M_P \]

\[ \lambda_2 = \]

Bending moments at various points in terms of \( M_P \) due to additional load are shown in Fig. 17.18 (f). Final B.M. diagram is shown in Fig. 17.18 (g). The span \( AB \) will collapse, causing collapse of whole structure.

Collapse load factor = \( \lambda_1 + \lambda_2 \)

\[ = \frac{3M_P + M_P}{3WL} + \frac{3M_P}{3WL} \]

\[ = \frac{3M_P}{WL} \]

Collapse load factor can also be found by mechanism method. Load factor for each of the independent collapse mechanisms is found. Least value of the collapse load factor thus obtained will be actual collapse load factor of the structure. Three independent mechanisms of three spans are shown in Fig. 17.19.

**Mechanism of span AB.**
Equation of virtual work gives

\[ 2WL \times \frac{l}{2} \theta - M_P \times 2\theta - M_P \times \theta = 0 \]

\[ \therefore \lambda = \frac{3M_P}{WL} \]

**Mechanism of span BC.**
Equation of virtual work gives

\[ \lambda WL \times \frac{l}{2} \theta - M_P \times \theta - M_P \times 2\theta - M_P \times \theta = 0 \]

\[ \lambda = \frac{3M_P}{WL} \]
Mechanism of span CD.

Equation of virtual work gives

$$\lambda W \times \sum \theta - M_p \theta - M_p \times 2 \theta = 0$$

$$\lambda = \frac{6M_p}{Wl}$$

Mechanism of span AB gives least value of collapse load factor. Hence $\lambda = \frac{3M_p}{Wl}$.

As the collapse takes place only in the span AB, the moments in the remaining portion are indeterminate.

If in a structure 'N' is the degree of indeterminacy and 'M' is the number of plastic hinges formed, the degree of indeterminacy $I$ at collapse is given by

$$I = N - (M - 1) = N - M + 1$$

In the above case $N = 2, M = 2$

$$I = 1$$
The moments in remaining portion of structure can be found by any method of elastic structural analysis, the moments at points where plastic hinges are formed being taken as fully plastic moments $M_P$.

Consider collapse mechanism of above case. Remaining portion of structure $BCD$ has not collapsed. Moment at support $B$ is fully plastic moment $M_P$.

Applying theorem of three moments to spans $BC$ and $CD$.

$$M_B + 2M_C(l + l) = -6\left(\frac{\lambda Wl}{4} \times \frac{l}{2} \times \frac{l}{2} \times 2\right)$$

$$-M_B + 4M_C = -6 \times \frac{\lambda Wl^2}{8}$$

$$-M_B + 4M_C = -\frac{5}{4} \times Wl^2 \times \frac{\lambda M_P}{Wl}$$

$$-M_B + 4M_C = -\frac{9}{4} M_P l$$

$$M_C = -\frac{5}{16} M_P$$

Final B.M. diagram will be as shown in Fig. 17.18 (q). The B.M. at any section other than plastic hinges is less than the fully plastic moment.

A mechanism that satisfies the condition that B.M. at every section is less than the fully plastic moment of the section, excepting at plastic hinges, the mechanism will represent true collapse mechanism of the structure.

Whether assumed mechanism is true collapse mechanism or not can also be checked by assuming B.M. at any section as fully plastic moment of the section and drawing B.M. diagram for the structure with moments at plastic hinges and assumed section as fully plastic moments. If B.M. at any other section is less than fully plastic moment, the mechanism represents true collapse mechanism.

Number of independent mechanisms is given by the following formula—

$$X = n - N$$

where

- $X$ = number of independent mechanisms.
- $n$ = number of possible formations of plastic hinges
- $N$ = degree of indeterminancy.

Ex. 17.5. A continuous beam of uniform section throughout, carries central loads in all spans as shown in Fig. 17.21 (a). Find the collapse load factor.

Solution. Consider independent mechanism of span $AB$, let $\lambda$ be collapse load factor.

Equation of virtual work gives,

$$\lambda W \times l \theta - M_{P2} - M_P \times 2 \theta = 0$$

$$\lambda = \frac{3M_P}{Wl}$$
To check whether this represents true collapse mechanism or not, bending moments are found in remaining portion of structure.

\[ M_{bl} + 2M_C(l + 1.5l) = -6 \left( \frac{\lambda WL}{4} \times \frac{l}{3} \times \frac{l}{9} \right) \]

\[ + \frac{\lambda WL \times 1.5l}{4} \times \frac{1.5l}{2} \times \frac{1.5l}{2} \]

\[ -M_P \times l + 5M_Cl = -6 \left[ \frac{\lambda WL^2}{16} + \frac{9\lambda WL^2}{64} \right] \]

\[ \frac{39}{32} \lambda WL^2 \]
\[
\begin{align*}
\frac{30}{32} Wl^3 \times \frac{3M_P}{Wl} &= -\frac{117}{32} M pl \\
5M_{Cl} &= -\frac{117}{32} M pl + M pl \\
\therefore \quad M_C &= \frac{85}{32 \times 5} M_P \\
&= -\frac{17}{32} M_P.
\end{align*}
\]

The B M. diagram will be as shown in Fig. 17.21 (d).

The B M. at every section but for points of plastic hinges is less than fully plastic moment of the section. Thus the mechanism is true collapse mechanism.

**Ex. 17.6.** Find the collapse load factor for the continuous beam shown in Fig. 17.22 (a). Span AB has fully plastic moment 2M_P, span BC has M_P and span CD, 1.5 M_P.

**Solution.** Independent mechanisms of spans AB, BC and CD are shown in Fig. 17.22. Collapse load factor is calculated for each of the mechanisms. Least value of load factors will be actual collapse load factor of the structure.

**First mechanism of AB**

\[
2\theta_1 = 7\theta_2
\]

\[
\therefore \quad \theta_1 = \frac{7}{2} \theta_2
\]

Equation of virtual work gives,

\[
10\lambda \times 2\theta_1 + 5\lambda \times 3\theta_2 - 2M_P(\theta_1 + \theta_2) - M_P\theta_1 = 0
\]

\[
10\lambda \times 7\theta_2 + 5\lambda \times 3\theta_2 - 2M_P \times \frac{9}{2} \theta_2 - M_P \times \theta_2 = 0
\]

\[
\therefore \quad \lambda = \frac{10M_P}{85}
\]

\[
= \frac{2M_P}{17}
\]

**Second mechanism of AB**

\[
6\theta_1 = 3\theta_2
\]

\[
\therefore \quad \theta_1 = \frac{\theta_2}{2}
\]

\[
10\lambda \times 2\theta_1 + 5\lambda \times 3\theta_2 - 2M_P(\theta_1 + \theta_2) - M_P\theta_1 = 0
\]

\[
10\lambda \theta_1 + 15\lambda \theta_2 - 2M_P \times \frac{9}{2} \theta_2 - M_P \times \theta_2 = 0
\]

\[
\therefore \quad \lambda = \frac{4M_P}{45}
\]
Mechanism of BC

\[ 6\lambda \times 4\theta - M_p \times 2\theta - M_p \times \theta - M_p \times \theta = 0 \]

\[ \lambda = \frac{4M_p}{6 \times 4} \]

\[ = \frac{M_p}{6} \]

Mechanism of CD

\[ 8\lambda \times 4\theta - M_p \times \theta - 1.5M_p \times 2\theta = 0 \]

\[ \lambda = \frac{4M_p}{4 \times 8} \]

\[ = \frac{M_p}{8} \]
Least value of \( \lambda \) is \( \frac{2M_P}{17} \), hence collapse load factor is \( \frac{2M_P}{17} \).

**Check for B.M. at other sections**

Assume moment at \( C \) as \( -M_P \). Bending moments at \( A \) and \( B \) are \( 2M_P \) and \( M_P \) respectively.

**B.M. at centre of \( BC \)**

\[
\begin{align*}
6\lambda \times \frac{2}{4} &= \frac{12\lambda}{2} - M_P \\
&= \frac{12}{17} \times \frac{2M_P}{17} - M_P \\
&= \frac{7}{17} M_P
\end{align*}
\]

**B.M. at centre of \( CD \)**

\[
\begin{align*}
8\lambda \times \frac{8}{4} &= \frac{16\lambda}{2} - \frac{M_P}{2} \\
&= \frac{16 \times 2}{17} M_P - \frac{M_P}{2} \\
&= \frac{47}{34} M_P < 1.5 M_P
\end{align*}
\]

Bending moments at various sections are shown in Fig. 17.23 (b). Bending moment at every section is less than fully plastic moment.

Next, with mechanism of \( AB \), assume moment at centre of \( CD = 1.5 M_P \).

**Moment at \( C \)**

\[
\begin{align*}
\text{Moment at } C &= 2 \left[ \frac{8\lambda \times 8}{4} - 1.5 M_P \right] \\
&= 2 \left[ \frac{16 \times 2}{17} M_P - 1.5 M_P \right] \\
&= 2 \times \frac{6 M_P}{17} = \frac{13}{17} M_P
\end{align*}
\]

**Bending moment at centre of \( BC \)**

\[
\begin{align*}
\frac{6\lambda \times 8}{4} &= \frac{12 \times \frac{2}{17} M_P - \frac{15}{17} M_P}{2} \\
&= \frac{9}{17} M_P
\end{align*}
\]
Bending moment diagram is shown in Fig. 17.23 (c). Bending moment at every section is less than or equal to fully plastic moment of the section. Hence the assumed mechanism is collapse mechanism.

17.8. Portal Frames and Gable Frames

**Portal Frames**

A single bay portal frame will have two types of independent mechanism—beam mechanism and sway mechanism as shown in Fig. 17.24. If the frame is multistoreyed frame there will be sway mechanism for each storey and in addition there will be joint mechanism, i.e. plastic hinges forming at junction of three or more members as shown in Fig. 17.25.

Collapse may take place due to these independent mechanisms or combination of these mechanisms. Load factor is found for each of the mechanism and least value of load factors gives collapse load factor of the structure.
PLASTIC ANALYSIS OF STRUCTURES

Fig. 17-24

Gable Frames

For a gable frame independent mechanisms are

(a) Beam mechanism,
(b) Sway mechanism,
(c) Gable mechanism,
(d) Joint mechanism.
Fig. 1725
Independent mechanisms for the gable frame of Fig. 17.26 are shown in Fig. 17.26 (a) to (k). Some combined mechanisms are shown in Fig. 17.26 (i) to (l). Load factor for each mechanism is found. Least value gives collapse load factor.
Ex. 17.7. Find the collapse load factor for the frame shown in Fig. 17.27.

Solution. Independent beam mechanism, sway mechanism and combined mechanism are shown in Fig. 17.27.

**Beam mechanism**

\[ 10 \lambda \times 4 \theta - M_P \theta - M_P \theta - 2 M_P \times 2 \theta = 0 \]

\[ \lambda = \frac{6 M_P}{40} = \frac{3}{20} \]

**Sway mechanism**

\[ 3 \lambda \times 6 \theta = 4 M_P \]

\[ \lambda = \frac{4 M_P}{18} = \frac{2}{9} M_P \]

\[ \text{Fig. 17.27 (a), (b)} \]

**Combined mechanism**

\[ 10 \lambda \times 4 \theta + 3 \lambda \times 6 \theta - M_P \times \theta - 2 M_P \times 2 \theta - M_P \times \theta - M_P \times 2 \theta = 0 \]

\[ 58 \lambda = 8 M_P \]

\[ \therefore \quad \lambda = \frac{4 M_P}{29} = 0.134 M_P \]

Collapse load factor = 0.134 \( M_P \).

\[ \text{Fig. 17.27 (c), (d)} \]
Ex. 17·8. The portal frame shown in Fig. 17·28 (a) is subjected to ultimate loads as shown in the figure. Find the plastic moment required if it is of uniform section throughout.

Solution. There are three possible beam mechanisms for BC. Only one mechanism with hinge at centre of BC is considered of the three beam mechanisms, this mechanism will give maximum value of $M_p$.

Beam mechanism of BC

$$16·8 \times 3\theta + 16·8 \times 6\theta + 16·8 \times 3\theta - M_p \times \theta - M_p \times 2\theta$$

$$- M_p \times \theta = 0$$

Fig. 17·28
201.6 = 4M_p

M_p = 50.4 t.m.

(ii) Beam mechanism of AB

Let plastic hinge be formed at distance y_0 from centre,

Moment at centre

\[ M_C = \frac{1.4 \times 6^2}{8} - \frac{M_p}{2} \]

\[ 6.3 - \frac{M_p}{8} \]

\[ 0 - (-M_p) = \frac{M_p}{8} \]

\[ \frac{1.4 \times 6}{8} = \frac{M_p}{4} \]

\[ M_p = M_C + \frac{1.4 y_0^2}{4} \left( 6.3 - \frac{M_p}{8} \right) \]

\[ \frac{M_p}{8} \]

\[ M_p = 6.3 - \frac{M_p}{2} + \frac{M_p^2}{100.8} \]

\[ M_p^2 - 151.2 M_p + 635.04 = 0 \]

This gives \( M_p = 147 \) t.m. or \( 4.2 \) t.m.

\( M_p \) cannot be 147 t.m, as this will give value of \( y_0 \) greater than 6 m.

\( \therefore M_p = 4.2 \) t.m.

(iii) Sway mechanism

\[ 1.4 \times 6 \times 3\theta - M_p \times \theta - M_p \times \theta = 0 \]

\[ M_p = \frac{1.4 \times 18}{2} \]

\[ = 12.6 \) t.m.

(iv) Combined mechanism

\[ 16.8 \times 3\theta + 16.8 \times 6\theta + 16.8 \times 3\theta + 1.4 \times 6 \times 3\theta \]

\[ - M_p \times 2\theta - M_p \times 2\theta = 0 \]

\[ \therefore 201.6\theta + 25.2\theta = 4M_p\theta \]

\[ M_p = \frac{226.8}{4} \]

\[ = 56.7 \) t.m.

Maximum value of \( M_p \) from the above four mechanisms is 56.7 t.m. hence section provided should have the value of fully plastic moment of 56.7 t.m.
Ex. 17.9. Explain in brief the following terms —

(i) Shape factor and (ii) load factor.

A fixed beam of span 6 m. carries a uniformly distributed load of 6.56 t/m. on the right hand 4.5 m. as shown in Fig. 17.29. The load factor is 1.75 and shape factor is 1.15, the yield stress is 2.32 t/cm². Calculate the sectional modulus of the beam and locate the positions of the plastic hinges.

\[ \text{Fig. 17.29} \]

Sol.

(i) Shape factor is ratio of fully plastic moment \( M_P \) of the section to the yield moment \( M_Y \) of the section.

(ii) Load factor is the factor by which applied load is multiplied to cause collapse of the structure.

At collapse hinges will be formed at supports \( A \) and \( B \) and in the loaded portion say at \( C \), \( x \) distance from \( B \).

\[ \text{Fig. 17.30} \]

\[ \theta_1 = \frac{\nu_2}{x}, \quad \nu_2 = \frac{6-x}{1.5} \cdot \]

\[ M_P \theta_1 + M_P (\theta_1 + \theta_2) + M_P \theta_1 = 6.6\lambda (4.5-x) \cdot \]

\[ \left( \frac{6-x+1}{2} \right) + 6.6\lambda x \times \frac{1}{2}. \]

\[ \therefore \quad 2 M_P \left[ \frac{1}{x} + \frac{1}{6-x} \right] = \frac{6.6\lambda}{3} (4.5-x) (7.5-x) + \frac{6.6\lambda x}{2}. \]

\[ 2 M_P \times \frac{6}{x, (6-x)} = \frac{2.2\lambda (4.5-x)(7.5-x)}{6-x} + 3.3\lambda x. \]

\[ \frac{12 M_P}{x} = \lambda \left[ -3.8x^2 - 6.6x + 74.25 \right] \]

\[ \therefore \quad M_P = \frac{\lambda}{12} \left[ -3.8x^2 - 6.6x + 74.25 \right] \]

\[ \frac{\partial M_P}{\partial x} = 0 = \frac{\lambda}{12} \left[ -11.4x^2 - 13.2x + 74.25 \right] \]
ANALYSIS OF STRUCTURES

\[-11.4x^3 - 13.2x + 74.25 = 0\]

\[x^3 + \frac{13.2}{11.4} x - \frac{74.25}{11.4} = 0\]

\[x^3 + 1.158x - 6.513 = 0\]

\[x = \frac{-1.158 + \sqrt{1.158^2 + 25.052}}{2} = \frac{-1.158 + 5.139}{2}\]

\[= 1.99 \text{ m.}\]

\[M_P = \frac{\lambda}{12} \left[-3.8 \times 1.99^3 - 6.6 \times 1.99^2 + 74.25 \times 1.99\right]\]

\[\lambda = 1.75.\]

\[M_P = \frac{1.75 \times 1.99}{12} \left[-3.8 \times 1.99^3 - 6.6 \times 1.99^2 + 74.25\right]\]

\[= \frac{1.75 \times 1.99 \times 48.20}{12} = 13.28 \text{ t.m} = 13.28 \times 100 \text{ t. cm.}\]

\[M_Y \text{ Shape factor} = \frac{13.28 \times 100}{1.15}.\]

\[Z = \frac{M_Y}{f_Y} = \frac{13.28 \times 100}{1.15 \times 2.32} = 497.8 \text{ cm}^3.\]

**Ex. 17.10.** Define the terms—(i) plastic hinge, (ii) shape factor, (iii) load factor and plastic moment.

A fixed beam carries two point loads at 1.5 m and 3 m from support A as shown in Fig. 17.31. Assuming the beam of constant section, calculate the collapse load for the beam if the plastic moment is \(M_P\). The point loads are \(P\) tonnes each.

**Indicate the position of the plastic hinges.**

\((A.M.I.E. \ May \ 1968)\)

---

*Fig. 17.31*
Sol.

(i) 1st Mechanism. Let \( P_c \) be the collapse loads

\[
M_p \theta_1 + M_p \theta_2 + M_p (\theta_1 + \theta_2) = P_c y_1 + P_c y_3
\]

\[
\theta_1 = \frac{y_1}{1.5}, \quad \theta_2 = \frac{y_1}{4.5}
\]

\[
y_3 = \frac{2y_1}{3}
\]

\[
\therefore M_p \frac{y_1}{1.5} + M_p \frac{y_1}{4.5} + M_p \left( \frac{y_1}{1.5} + \frac{y_1}{4.5} \right) = P_c y_1 + P_c y_3
\]

\[
= P_c y_1 + P_c \times \frac{2y_1}{3}
\]

Multiplying each term by \( \frac{4.5}{y_1} \),

\[
3M_p + M_p + 4M_p = P_c [4.5 + 3] = 7.5 \ P_c
\]

\[
\therefore P_c = \frac{8}{7.5} M_p
\]

\[
\ldots (i)
\]

(ii) 2nd Mechanism. Let \( P_c \) be the collapse load

\[
\theta_1 = \theta_2 = \frac{y_3}{3}
\]

\[
y_1 = \frac{y_3}{2}
\]

\[
M_p \theta_1 + M_p \theta_2 + M_p (\theta_1 + \theta_2) = P_c y_1 + P_c y_3
\]

\[
\therefore M_p \frac{y_3}{3} + M_p \frac{y_3}{3} + M_p \frac{2y_3}{3} = P_c \frac{y_3}{2} + P_c y_3
\]

Multiplying by \( \frac{6}{y_3} \) each term,

\[
2M_p + 2M_p + 2M_p = 3P_c + 6P_c
\]

\[
\therefore P_c = \frac{8 M_p}{9}
\]

Therefore collapse will take place by second mechanism and collapse load will be \( \frac{8 M_p}{9} \).

Problems

1. A uniform beam of span 3 m. and fully plastic moment \( M_p \) is simply supported at one end and rigidly clamped at other end. A concentrated load of 1000 kg. may be applied anywhere within the span. Find the smallest value of \( M_p \) such that collapse would first occur when the load is in its most unfavourable position.

[Ans. 516 kg.m.]
2. In the portal frame shown in Fig. 17·32, stanchions have fully plastic moment of 2000 kg·m and beam 1000 kg·m. The frame is loaded as shown in the figure. Find the permissible value of $W$. 

$$\text{Fig. 17·32}$$

[Ans. $W = 3000$ kg]

3. Find the collapse load factor for the continuous beam shown in Fig. 17·33.

$$\text{Fig. 17·33}$$

4. The fixed beam shown in Fig. 17·34 has fully plastic moment of 2000 kg·m. Find permissible value of $W$.

$$\text{Fig. 17·34}$$

[Ans. 3375 t·m]

5. Find fully plastic moment required for the frame shown in Fig. 17·35 if all members have same value of $M_P$.

$$\text{Fig. 17·35}$$

[Ans 3375 t·m]
6. Find fully plastic moment $M_p$ for the frame shown in Fig. 17.36. Columns have plastic moment $M_p$ and beams $2M_p$.

[Ans. 11 t.m.]

7. Find the fully plastic moment for the frame shown in Fig. 17.37. All members have same value of $M_p$.

[Ans 2.3 t.m.]

8. A beam of uniform section, of span $L$, is fixed at one end, and is simply supported at the other end. A point load $W$ can act on the beam at any section. Using the principles of limit design, calculate the fully plastic moment of resistance of the beam to be able to support the load $W$ at failure.

9. A rectangular portal frame of span \( L \) and height \( L/2 \) is fixed to the ground at both ends and has a uniform section throughout with its fully plastic moment of resistance equal to \( M_p \). It is loaded with a point load \( W \) at centre of span as well as a horizontal force \( \frac{W}{2} \) at its top right corner. Calculate the value of \( W \) at collapse of the frame. 

(A.M.I.E. May 1970)
18.1. Concurrent forces in space—If several forces meet at the same point, these can be reduced to a single resultant force by successive use of parallelogram of forces.

Thus $F_1$, $F_2$, $F_3$ are the three forces in space acting at $O$. The resultant of forces $F_1$ and $F_3$ i.e. $R_1$ is given by diagonal $OD$ of parallelogram $AOBD$. The resultant of $R_1$ and $F_2$ i.e. $R_2$ is given by the diagonal $OE$ of the parallelogram $OCED$. Thus $OA$, $OB$ and $OC$ are three sides of the parallelopiped and the resultant is given by diagonal $OE$.

The resultant can be obtained by the closing side $OE$ of the space polygon $OADE$ where $OA$, $AD$ and $DE$ are drawn parallel and equal to $F_1$, $F_2$ and $F_3$. Since the space polygon can be drawn for any number of forces, therefore resultant of any number of concurrent forces in space is obtained by closing line of the space polygon and the resultant will pass through the point of concurrence and its line of action will be parallel to closing line of the space polygon.

If a system of concurrent forces is projected on any plane orthogonally, the resultant of these projected forces will be projection of the resultant of forces on this plane. In particular if the given system of forces in space is in equilibrium, their projection orthogonally on any plane will represent a coplanar forces in equilibrium.
Similarly the resultant of projections of all forces in space on any axis will be the projection of the resultant of forces on the axis. Therefore if the forces in space are in equilibrium then the algebraic sum of projections on any three orthogonal axes will be zero. Hence, \( \Sigma x = 0, \Sigma y = 0, \Sigma z = 0 \), where \( x, y, z \) are the projections of the forces on the three mutually perpendicular axes and these are equations of equilibrium.

The following conclusions can be drawn —

(1) Three concurrent forces which do not lie in one plane cannot be in equilibrium unless all three are zero. As if the forces are resolved on a normal to the plane containing the other two, there will be only projection of one force which will have to be zero if there is equilibrium.

(2) If two of four concurrent forces in space, that are not in one plane, are collinear, equilibrium can exist if the other two are zero and the collinear forces are equal and opposite. This can be proved by projecting the forces on a plane normal to the two collinear forces on which only the other two forces will have projections which cannot be in equilibrium unless they are zero.

(3) In case all forces but one are in one plane and they are in equilibrium the force not in the plane will be zero. This can be proved by projecting the forces on the normal to the plane in which all but one force lie. Then there will be projection of one force which for equilibrium should be zero.

(4) If the lines of action of all excepting two of number of concurrent forces, are known to be coplanar, then if one force of the two is known the other can be obtained by projecting them on an axis normal to the coplanar forces.

18-2. Moment of a force

To obtain the moment of the force \( F \) with respect to axis \( Z \), the force is projected on a plane at rt. angles to the axis as shown and then taking moment about the point through which axis \( Z \) passes, Thus \( A'B' \) i.e. \( F' \) is the projection of force \( AB \) and moment is given by \( F' \times OO' \) where \( OO' \) is the lever arm. The moment will be positive as the axis of the moment will be in the positive direction of \( Z \)-axis. If \( \theta \) is the angle made by \( F \) with the plane, then moment will be \( F \cos \theta \times OO' \). Thus the moment will be zero if \( \theta \) is 90° or \( OO' \) is zero. This will happen if the force is parallel to the axis or it intersects the axis.

If a system of concurrent forces in space is in equilibrium, the algebraic sum of their moments about any axis will be zero.
18.3. Constraint of a point in space

By attaching a point $O$ to rigid foundation by two bars $OA$ and $OB$, will constraint the point in the plane $AOB$, but there will be no constraint as for rotation about axis $AB$ is concerned. By putting third bar $OC$ which should not be in same plane as $AOB$ the full constraint can be had.

Thus complete and satisfactory constraint of a point in space can be obtained by attaching it to a foundation by three bars the axis of which do not lie in one plane.

![Fig. 18.3](image_url)

If this constrained point is subjected to an external force $P$, forces in the three bars can be obtained. If the point is attached to the foundations by more than three bars then the system will be redundant as the three equations of equilibrium, $\Sigma x = 0$, $\Sigma y = 0$, $\Sigma z = 0$, will enable to solve only three unknowns.

18.4. Tension Coefficient method

Let the force $F$ assumed to be tensile, be represented by $AB$ length $l'$ in space. The coordinates of $A$ and $B$ are $(x_A, y_A, z_A)$ and $(x_B, y_B, z_B)$ respectively.

Now the components in the direction of three axes $X_{AB}, Y_{AB}, Z_{AB}$ will be force $F$ multiplied by the direction cosines, thus

$$ F \frac{(x_B - x_A)}{l} $$

![Fig. 18.4](image_url)
\[ Y_{AB} = \frac{F(y_B - y_A)}{l} \]
\[ Z_{AB} = \frac{F'(z_B - z_A)}{l}. \]

Denoting \( t_{AB} \) where \( t_{AB} \) is known as tension coefficient,
\[ X_{AB} = t_{AB} (x_B - x_A) \]
\[ Y_{AB} = t_{AB} (y_B - y_A) \]
\[ Z_{AB} = t_{AB} (z_B - z_A) \]

Now if there are a number of members \( AB, AC, AD \ldots \) meeting at \( A \) and there are external forces \( X_A, Y_A, Z_A \) acting at \( A \) in directions along axes \( x, y \) and \( z \) respectively and if \( A \) is in equilibrium under the forces in members and external forces then,
\[ t_{AB} (x_B - x_A) + t_{AC} (x_C - x_A) + \ldots + X_A = 0 \]
\[ t_{AB} (y_B - y_A) + t_{AC} (y_C - y_A) + \ldots + Y_A = 0 \]
\[ t_{AB} (z_B - z_A) + t_{AC} (z_C - z_A) + \ldots + Z_A = 0 \]

In case there are not more than three members in which forces are not known, meeting at a joint, the tension coefficients in these can be found from the above equations.

If the tension coefficient works out to be negative, then the force in the member will be compressive. The force in member is obtained by multiplying the tension coefficient by its length. The length \( t_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} \)

**Ex. 18.1.** A force \( P \) in applied at end \( A \) of the strut \( OA \) which is supported by two guys \( AB \) and \( AC \). \( O, B, C \) are in the same plane and \( OA \) is normal to this plane. Find the forces in \( OA, AB \) and \( AC \).

**Sol.**

![Diagram](image-url)
Taking axes $X, Y, Z$ as shown. Let $S_1, S_2, S_3$ be the forces in $OA, AB$ and $AC$ respectively.

**Method 1.**

**Project the forces on axis $Z$**

$OC = 2.5$ m, $CA = 6.5$ m.

$S_3 \times \frac{1.5}{AC} = P$

$\therefore \quad S_3 = \frac{P \times 6.5}{1.5} = \frac{13P}{3} \text{(tensile)}.$

**Project the forces on axis $Y, AB = 6.5$ m.**

$S_3 \times \frac{2}{AC} = S_3 \times \frac{2.5}{AB}$

$\therefore \quad S_3 \times \frac{2}{6.5} = S_3 \times \frac{2.5}{6.5}.$

$\therefore \quad S_3 = S_3 \times \frac{4}{5} = \frac{13P}{3} \times \frac{4}{5} = \frac{52P}{15} \text{(tensile)}.$

**Project the forces on axis $X$**

$S_3 \times \frac{6}{AC} + S_2 \times \frac{6}{AB} = S_1$

$\therefore \quad S_1 = S_3 \times \frac{6}{6.5} + S_2 \times \frac{6}{6.5} = \frac{6}{6.5} \times \left[ \frac{13P}{3} + \frac{52P}{15} \right]$

$= \frac{6}{6.5} \times 17P = \frac{36P}{7} = 5.2P \text{ (compressive).}$

**Method 2.**

Project the forces on plane $YOZ$.

Resolving along $Z$ axis,

$S_3 \times \frac{2.5}{6.5} \times \frac{1.5}{2.5} = P$.

$S_3 = \frac{13}{5} P \text{(tensile)}.$

Resolving along $Y$-axis,

$S_3 \times \frac{2.5}{6.5} \times \frac{2}{2.5} = S_3 \times \frac{2.5}{6.5}$

$S_2 = \frac{4}{5} S_3 = \frac{4}{5} \times \frac{13P}{3} = \frac{52P}{15} \text{ (tensile)}$

Projecting on plane, $OXY$, 

![Fig. 18.6](image-url)
Method 3.

Tension coefficients

Let \( t_{AO}, t_{AB}, t_{AC} \) be the tension coefficients. Co-ordinates of \( A, B, C \) are shown in Fig. 18-5.

Resolving in Z direction,
\[
t_{AO}(0-0) + t_{AB}(0-0) + t_{AC}(1.5-0) - P = 0
\]
\[
\therefore \quad t_{AC} = \frac{P}{1.5} - \frac{2P}{3} \quad \ldots (i)
\]

Resolving in Y direction,
\[
t_{AO}(0-0) + t_{AB}(2.5-0) + t_{AC}(-2-0) + 0 = 0
\]
\[
t_{AB} = \frac{2}{2.5} t_{AC} - \frac{8P}{15} \quad \ldots (ii)
\]

Resolving in X direction,
\[
t_{AO}(0-6) + t_{AB}(0-6) + t_{AC}(0-6) + 0 = 0
\]
\[
t_{AO} = -t_{AC} - t_{AB}
\]
\[
= -\left[ \frac{2P}{3} + \frac{8P}{15} \right] = -\frac{6P}{5} \quad \ldots (iii)
\]

\[
S_1 = \frac{6P}{5} \times 6 = \frac{-36P}{5} \quad 7.2P \text{ (compressive)}
\]

\[
S_2 = \frac{8P}{15} \times 6.5 = \frac{52P}{15} \quad \text{(tensile)}
\]

\[
S_3 = \frac{2P}{3} \times 6.5 = \frac{13P}{3} \quad \text{(tensile)}
\]

Method 4.

Moments.

Take moments about Y-axis,

The forces which will give moment are \( P \) and \( S_4 \). \( S_4 \) is resolved at \( A_s \) into two forces along \( AC_1 \) and \( OC_1 \). Force along \( AC_1 \) will not give any moment about Y-axis.
\[ S_2 \times \frac{1.5}{6.5} \times 6 : P \times 6. \]

\[ \frac{13P}{3} \quad \text{(tensile)} \]

Take moments about Z-axis. The forces which will give moments will be \( S_2 \) and \( S_3 \). \( S_2 \) is resolved at \( B \), the component parallel to \( X \) axis will give moment. \( S_3 \) is resolved at \( C \), the component in \( X \) direction will give moment.

\[ S_2 \times \frac{6}{6.5} \times 2.5 = S_2 \times \frac{6}{6.5} \times 2. \]

\[ S_3 = \frac{4}{5} \times S_3 \quad \frac{4}{5} \times \frac{13P}{3} = \frac{52P}{15} \quad \text{(tensile)} \]

Take moments about axis passing through \( C \) and in \( Z \) direction i.e. \( OC_1 \). Moments will be given by \( S_1 \) and \( S_2 \)

\[ S_2 \times \frac{6}{6.5} \times (2.5 + 2) = S_1 \times 2. \]

\[ S_1 \quad \frac{6}{6.5} \times \frac{4.5}{2} \times S_2 \quad \frac{3 \times 9}{13} \times \frac{52P}{15} \]

\[ = \frac{36P}{5} \quad 7.2P \quad \text{(compressive)} \]

**Ex. 18.2.** Three bars of equal lengths 'l' are joined together at \( A \) and supported at \( B, C \) and \( D \) as shown in Fig. 18.8. Find the axial forces induced in all bars due to a vertical load \( P \) at \( A \), if \( OB = OC = OD = l \).

**Sol.**

Let the coordinates of \( A \) be \( x, y, z \).

\[ AB^2 = (x-l)^2 + y^2 + z^2 \]
\[ = l^2 \quad \text{(i)} \]

\[ AC^2 = (x-0)^2 + (y-l)^2 \]
\[ + z^2 = l^2 \quad \text{(ii)} \]

\[ AD^2 = (x-0)^2 + y^2 \]
\[ + (z-l)^2 = l^2 \quad \text{\( (iii) \)} \]

\[ \therefore \quad x^2 + y^2 + z^2 - 2xl = 0 \]
\[ x^2 + y^2 + z^2 - 2yl = 0 \]
\[ x^2 + y^2 + z^2 - 2zl = 0 \]
\[ \therefore \quad x = y = z = k. \]

\[ \therefore \quad (k-l)^2 + k^2 + k^2 = l^2 \quad \text{\( (iv) \)} \]

\[ 3k^2 - 2kl = 0. \quad k = \frac{2l}{3} \]

The coordinates of \( A \) are \( \left( \frac{2l}{3}, \frac{2l}{3}, \frac{2l}{3} \right) \).
Along $X$-axis
\[ t_{AB} \left( 0 - \frac{2l}{3} \right) + t_{AC} \left( 0 - \frac{2l}{3} \right) + t_{AD} \left( 0 - \frac{2l}{3} \right) + 0 = 0. \]
\[ \therefore t_{AB} - 2t_{AC} - 2t_{AD} = 0 \quad \text{(i)} \]

Along $Y$-axis
\[ t_{AB} \left( 0 - \frac{2l}{3} \right) + t_{AC} \left( l - \frac{2l}{3} \right) + t_{AD} \left( 0 - \frac{2l}{3} \right) + 0 = 0. \]
\[ \therefore -2t_{AB} + t_{AC} - 2t_{AD} = 0 \quad \text{(ii)} \]

Along $Z$-axis
\[ t_{AB} \left( 0 - \frac{2l}{3} \right) + t_{AC} \left( 0 - \frac{2l}{3} \right) + t_{AD} \left( l - \frac{2l}{3} \right) - P = 0 \]
\[ -2t_{AB} - 2t_{AC} + t_{AD} - \frac{3P}{l} = 0 \quad \text{(iii)} \]

Adding 2 (ii) to equation (i) and (iii),
\[ t_{AB} - 2t_{AC} - 2t_{AD} - 4t_{AB} + 2t_{AC} - 4t_{AD} = 0. \]
\[ -6t_{AB} - 2t_{AC} + t_{AD} - \frac{3P}{l} = 0 \quad \text{(iv)} \]
\[ \therefore -6t_{AB} - 3t_{AD} - \frac{3P}{l} = 0. \quad \text{(v)} \]

Putting value of $t_{AB}$ in (v)
\[ 12t_{AD} - 3t_{AD} - \frac{3P}{l} = 0 \]
\[ t_{AD} = \frac{P}{3l}. \]
\[ t_{AB} = \frac{-2P}{3l}. \]

Putting these values in equation (ii)
\[ t_{AC} = 2 \times \frac{-2P}{3l} + 2 \times \frac{P}{3l} = -\frac{2P}{3l}. \]

Therefore forces in three members will be,
\[ F_{AB} = \frac{-2P}{3l} \times l = -\frac{2P}{3} \quad \text{(compressive)} \]
\[ F_{AC} = \frac{-2P}{3l} \times l = -\frac{2P}{3} \quad \text{(compressive)} \]
\[ F_{AD} = \frac{r}{3l} \times l = \frac{P}{3} \quad \text{(tensile)} \]

2nd Method
Taking moments about $Z$-axis, moments will be given by forces $F_{AB}$ and $F_{AC}$ these are resolved orthogonally at $B$ and respectively.
\[ F_{AB} \left( \frac{2l}{3} - 0 \right) x l = F_{AC} \left( \frac{2l}{3} - 0 \right) x l. \]

\[ F_{AB} = F_{AC} \quad \text{...(i)} \]

Taking moments about \( X \)-axis, moments will be given by \( P \), \( F_{AC} \) and \( F_{AD} \). \( F_{AC} \) and \( F_{AD} \) are resolved at \( C \) and \( D \) respectively

\[ P \times \frac{2l}{3} = F_{AD} \left( \frac{2l}{3} - 0 \right) \times \frac{l}{l} + F_{AC} \left( \frac{2l}{3} - 0 \right) \times \frac{l}{l} \]

\[ \therefore \quad F_{AD} + F_{AC} = P \quad \text{...(ii)} \]

Projecting the forces on \( XY \)-plane,

\[
A_1B = A_1C = \sqrt{\left( \frac{2l}{3} - l \right)^2 + \left( \frac{2l}{3} - 0 \right)^2} = \frac{l}{3} \sqrt{5}
\]

\[
A_1D_1 = \sqrt{\left( \frac{2l}{3} \right)^2 + \left( \frac{2l}{3} \right)^2} = \frac{2l}{3} \sqrt{2}
\]

\[ \tan \theta = \frac{l/3}{2l/3} = \frac{1}{2}, \quad \cos \theta = \frac{2}{\sqrt{5}}, \quad \sin \theta = \frac{1}{\sqrt{5}} \]

\[ \cos (\theta + 45) = \cos \theta \cos 45 - \sin \theta \sin 45 \]

\[= \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{5}} \times \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{10}} \]

\[
F_{AD} \times \frac{2l \sqrt{2}}{3} \frac{1}{l} = F_{AB} \times \frac{1}{\sqrt{10}} + F_{AC} \times \frac{1}{\sqrt{10}} \times \frac{1}{l} \times \frac{1}{3} \sqrt{2} + \frac{1}{3} \sqrt{2}.
\]
\[ F_{AD} = \frac{4}{4} F_{AC} \]

Putting this value in (ii),

\[ \frac{F_{AB}}{4} + \frac{F_{AC}}{4} + F_{AC} \]

As \[ F_{AB} = F_{AC} \]

\[ \frac{2}{2} F_{AC} = P \therefore F_{AC} = F_{AB} = 2P \quad \text{(compressive)} \]

\[ F_{AD} = 2 \times \frac{2P}{5} \times \frac{1}{x} \times \frac{P}{3} \quad \text{(tensile)} \]

18:5. Simple Space Trusses. Space truss consists of a number of bars joined together to form a rigid space structure. The connection at the joints is formed by either riveting or by welding. Due to rigidity at the joints, besides the axial forces in the bars, secondary bending of the bars is induced. In most cases the effect of secondary bending is neglected as it does not affect the results materially. Transmission line towers, cranes etc. are instances of space trusses.

As discussed previously, complete constraint of a point is obtained by connecting it by three bars to the foundation such that the point and three supports do not lie in one plane as shown in Fig. 18:10. The point A is constrained completely by three bars \( AB, AC \) and \( AD \). Further points can be constrained by three bars joined to A and two supports. The point E is constrained by three bars \( EA, EB \) and \( EF \). The point G is constrained by three bars \( GA, GE \) and \( GD \). It should be seen that in each case, three bars such as \( EA, EB, EF \) or \( GA, GE \) and \( GD \) do not lie in one plane. The three bars after the first joint may be connected to foundation or already established points to form a new joint.

The analysis of space trusses is done by applying successively to each joint the conditions of equilibrium. The first joint to be analysed shall have not more than three members in which forces are to be found.

Thus from above discussion it follows that if \( j \) is number of joints and \( m \) is number of members,

\[ m = 3j \]

Ex. 18:3. Find the forces in the members of a braced crane in the Fig. 18:11.
Solution.

Fig. 18.11 (a)

The coordinates have been chosen as shown. Centre of BO below A is chosen as origin.

Joint G

\[ t_{GD} (6 - 12) + t_{GE} (6 - 12) + t_{GF} (5 - 12) = 0 \]

\[ 6t_{GD} + 6t_{GB} + 7t_{GF} = 0 \] ... (i)

\[ t_{GD} (2 - 0) + t_{GB} (-2 - 0) + t_{GF} (0 - 0) = 0. \]

\[ t_{GD} = t_{GB} \] ... (ii)

\[ t_{GD} (4 - 6) + t_{GE} (4 - 6) + t_{GF} (7 - 6) - 10 = 0. \]

\[ -2t_{GD} - 2t_{GE} + t_{GF} - 10 = 0. \] ... (iii)

Fig. 18.11 (b)

From (i) and (ii), \( 7t_{GF} = -12t_{GB} \)

\[ t_{GF} = \frac{-12}{7} t_{GB}. \]

From (iii), \( -2t_{GE} - 2t_{GB} - \frac{12}{7} t_{GB} - 10 = 0. \)

\[ \frac{-40t_{GE}}{7} = 10. \]

\[ t_{GB} = \frac{-7}{4} = t_{OD}. \]

\[ t_{GF} = \frac{-12}{7} \times \frac{-7}{4} + 3 \]
Joint F
\[ t_{FA} (0-5) + t_{FD} (6-5) + t_{FB} (6-5) + t_{GF} (12-5) = 0. \]
\[ \therefore -5t_{FA} + t_{FD} + t_{FB} + 21 = 0, \quad (t_{GF}=3) \quad \ldots (i) \]
\[ t_{FA} (0-0) + t_{FD} (2-0) + t_{FB} (-2-0) + t_{GF} (0-0) = 0. \]
\[ t_{FD} = t_{FE} \quad \ldots (ii) \]
\[ t_{FA} (5-7) + t_{FD} (4-7) + t_{FB} (4-7) + t_{GF} (6-7) = 0. \]
\[ \therefore -2t_{FA} - 3t_{FD} - 3t_{FE} - 3 = 0, \quad (t_{GF}=3) \quad \ldots (iii) \]
From (i) and (ii),
\[ -5t_{FA} = -21 - 2t_{FE}. \]
\[ \therefore t_{FA} = \frac{21}{5} + \frac{2t_{FE}}{5}. \]
Putting these values in (iii)
\[ -42 \cdot \frac{4t_{FE}}{5} - 3t_{FE} - 3 = 0. \]
\[ \frac{34}{5} t_{FE} = \frac{-57}{5}. \]
\[ t_{FE} = \frac{-57}{34} = t_{FD}. \]
\[ t_{FA} = \frac{21}{5} - \frac{57}{5} \times \frac{34}{34} \]
\[ = \frac{21}{5} - \frac{57}{85} \]
\[ = \frac{357 - 57}{85} = \frac{+60}{17} \]

Joint E
\[ t_{EA} (0-6) + t_{EC} (0-0) + t_{ED} (6-6) + t_{EF} (5-0) + t_{EG} (12-6) = 0. \]
\[ -6t_{EA} - 6t_{EC} + \frac{57}{34} - \frac{21}{2} = 0. \]
\[ t_{EA} + t_{EC} = \frac{-25}{17} \quad \ldots (i) \]
\[ t_{EA} (0+2) + t_{EC} (-4+2) + t_{ED} (2+2) + t_{EF} (0+2) + t_{EG} (0+2) = 0 \]
\[ 2t_{EA} - 2t_{EC} + 4t_{ED} - \frac{57}{17} - \frac{7}{2} = 0. \]
\[ t_{EA} - t_{EC} + 2t_{ED} = \frac{233}{63} \quad \ldots (ii) \]
\[ t_{EA} (5-4) + t_{EC} (0-4) + t_{ED} (4-4) + t_{EF} (7-4) + t_{EG} (6-4) = 0 \]
\[ \therefore t_{EA} - 4t_{EC} - \frac{171}{34} - \frac{7}{2} = 0 \]
\[ \therefore t_{EA} - 4t_{EC} = \frac{135}{17} \quad \ldots (iii) \]
\[(iii)-(i) \quad -3t_{EC} = \frac{110}{17}\]

\[\therefore \quad t_{EC} = \frac{-110}{51}\]

\[t_{BA} = \frac{-25}{17} + \frac{110}{51} = \frac{+35}{51}\]

\[2t_{ED} : \quad \frac{333}{68} - \frac{35}{51} - \frac{110}{51} = \frac{419}{17 \times 12}\]

\[t_{ED} : \quad \frac{+419}{17 \times 24}\]

**Joint D**

\[t_{DA} (0-6) + t_{DB} (0-6) + t_{DC} (0-6) + t_{DB} (6-6) + t_{DF} (5-6) + t_{DG} (12-6) = 0\]

\[-6t_{DA} - 6t_{DB} - 6t_{DC} + \frac{57}{34} - \frac{21}{2} = 0\]

\[t_{DA} + t_{DB} + t_{DC} = -\frac{25}{17}. \quad \text{(i)}\]

\[t_{DA} (0-2) + t_{DB} (4-2) + t_{DC} (4-2) + t_{DE} (2-2) + t_{DF} (0-2) + t_{DG} (0-2) = 0\]

\[-2t_{DA} + 2t_{DB} - 6t_{DC} + \frac{419}{17 \times 6} + \frac{57}{17} + \frac{7}{2} = 0\]

\[t_{DA} - t_{DB} + 3t_{DC} = \frac{70}{51}. \quad \text{(ii)}\]

\[t_{DA} (5-4) + t_{DB} (0-4) + t_{DC} (0-4) + t_{DE} (4-4) + t_{DF} (7-4) + t_{DG} (6-4) = 0\]

\[t_{DA} - 4t_{DB} - 4t_{DC} - \frac{171}{34} - \frac{7}{2} = 0\]

\[\therefore \quad t_{DA} - 4t_{DB} - 4t_{DC} = \frac{145}{17} \quad \text{...(iii)}\]

**Equation (i)-(ii),**

\[2t_{DB} - 2t_{DC} = -\frac{145}{51}\]

\[\therefore \quad t_{DB} - t_{DC} = \frac{-145}{17 \times 6}\]

**Equation (ii)-(iii),**

\[5t_{DB} + 5t_{DC} = -\frac{170}{17}\]

\[t_{DB} + t_{DC} = -\frac{2}{5}\]

\[2t_{DB} = \frac{-145}{17 \times 6}\]
\[ t_{DB} = \frac{-349}{17 \times 12} \]
\[ t_{DC} = \frac{-59}{17 \times 12} \]
\[ t_{DA} = \frac{-25}{17} + \frac{349}{17 \times 12} + \frac{59}{17 \times 12} = \frac{9}{17} \]

**Forces in members**

\[ F_{GD} = \frac{-7}{4} \sqrt{0^2 + 2^2 + 2^2} \]
\[ = \frac{-7 \sqrt{11}}{2} = -11.62 \text{ (compressive)} \]

\[ F_{GB} = \frac{-7}{4} \sqrt{44} \]
\[ = -7 \sqrt{11} = -11.62 \text{ (compressive)} \]

\[ F_{GF} = 3 \sqrt{7^2 + 0^2 + 1^2} \]
\[ = 15 \sqrt{2} = 21.21 \text{ (tensile)} \]

\[ F_{FA} = \frac{60}{17} \times \sqrt{5^2 + 0^2 + 2^2} \]
\[ = \frac{60 \sqrt{29}}{17} = 19.01 \text{ (tensile)} \]

\[ F_{FD} = \frac{-57}{34} \sqrt{1^2 + 2^2 + 3^2} \]
\[ = \frac{-57 \sqrt{14}}{34} = -6.273 \text{ (compressive)} \]

\[ F_{FB} = \frac{-57}{34} \sqrt{14} \]
\[ = \frac{-57 \sqrt{14}}{34} = -6.273 \text{ (compressive)} \]

\[ F_{EA} = \frac{35}{51} \sqrt{6^2 + 2^2 + 1^2} \]
\[ = \frac{35 \sqrt{41}}{51} = 4.394 \text{ (tensile)} \]

\[ F_{EC} = \frac{-110}{51} \sqrt{6^2 + 2^2 + 4^2} \]
\[ = \frac{-110 \sqrt{56}}{51} \text{ = } -16.14 \text{ (compressive)} \]

\[ F_{ED} = 419 \sqrt{4^2} \]
\[
\frac{419}{17 \times 6} = 4.108 \text{ (tensile)}
\]
\[
F_{DA} = \frac{9}{17} \times \sqrt{6^2 + 2^2 + 1^2}
\]
\[
\frac{9\sqrt{41}}{17} = 3.39 \text{ (tensile)}.
\]
\[
F_{DB} = \frac{-349}{17 \times 12} \sqrt{6^2 + 2^2 + 4^2}
\]
\[
\frac{-349\sqrt{56}}{17 \times 12} = -12.8 \text{ (compressive)}
\]
\[
F_{DC} = \frac{-59}{17 \times 12} \sqrt{6^2 + 6^2 + 4^2}
\]
\[
\frac{-59\sqrt{88}}{17 \times 12} = -2.716 \text{ (compressive)}
\]

18.6. Method of Sections. The previous methods are applicable if forces in all members are to be worked out. In case forces only in a few members are to be worked out, then method of sections will give the solution quicker. The truss is cut by a plane such that not more than six members are cut including the members in which forces are to be found. From six equations of equilibrium,

\[
\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0
\]

and \( \Sigma M_x = 0, \quad \Sigma M_y = 0, \quad \Sigma M_z = 0 \)

the forces in these six members can be worked out.

18.7. Compound Space Trusses. Any simple space truss can be rigidly attached to the foundation by six bars which are not in one plane or are parallel or do not meet at one point. A rigid tetrahedron \( ABCD \) is a simple space truss which cannot be distorted. Thus truss can be extended by attaching new joints to the existing system by means of three bars whose axes do not lie in one plane. Thus \( ABCDE \) is extended simple truss connected to foundation by six bars. This is a compound truss which cannot be solved by method of joints. This can be solved by tension coefficients but it will involve solution of a set of simultaneous equations. To solve this truss the six bars are cut by a plane and by six equations of equilibrium, the forces in these bars are worked out and then the forces in simple truss \( ABCDE \) can be evaluated. Thus a self constrained space truss which is to be connected to foundation will satisfy the equation \( m = 3j - 6 \). The compound truss may consist of
two simple trusses or even plane trusses connected by six bars.

\[ \text{Fig. 16'14} \]

In Fig. 18'14 two tetrahedrons \(ABCD\) and \(EFGH\) have been connected by six bars. In Fig. 18'15 two triangles \(ABC\) and \(DEF\) in different planes are connected by six bars to form a compound truss.

\[ \text{Fig. 18'15} \]

**Ex 18'4. Prove that the system of bars shown in Fig. 18'16 constitutes a compound space truss. The system has the form of a regular octahedron. Analyse the system under two equal and opposite forces \(P\) acting as shown.**

**Solution.**

This truss can be looked upon as two triangles \(ABE\) and \(CDF\), connected by six bars \(AD, AF, BC, BF, CE\) and \(DE\). Cut these six bars by passing a plane and study one half of the truss.

\[
AB = BC = CD = DA = AE = BE = CE = DE = AF = BF = CF = DF = l.
\]

Take moments about \(CD\), the resolved parts of forces in \(FA\) and \(FB\) and \(P\) will give moments about \(CD\). The forces in \(FA\) and \(FB\) will be tensile. \(\bar{F}_{FA} = \bar{F}_{FO} + \bar{F}_{AO}\)

\[
2 \times F_{FA} \times \frac{OF}{FA} \times \frac{l}{2} = P \times \frac{l}{2}.
\]

\[
OF = \sqrt{FD^2 - OD^2}; \\
\sqrt{l^2 - \left(\frac{\sqrt{2} \cdot l}{2}\right)^2},
\]

\[
\sqrt{l^2 - \frac{l^2}{2}} = \frac{l}{\sqrt{2}}.
\]

\[
\therefore 2 \times \frac{F_{FA}}{\frac{\sqrt{2}}{l}} = P.
\]

\[
F_{FA} \frac{P}{\sqrt{2}} = r_{FB}.
\]
Similarly, $F_{CE} = F_{DE} \sqrt{2}$ (tensile).

By taking moments about vertical, passing through $EF$ it can be proved that the forces in $AD$ and $BC$ will be equal

$$F_{AD} = F_{BC} = 2 \times \frac{P}{\sqrt{2}} \times \frac{AD/2}{AF} = \frac{P}{\sqrt{2}}.$$  (compressive)

Take the primary truss $CDF$.

At joint $C$, resolved part of $CE$ in vertical direction will balance resolved part of $CF$ in vertical direction. Their inclinations being same,

$$F_{CF} = F_{CE} = \frac{r}{\sqrt{2}}.$$

Similarly $F_{DF}, F_{DE} = \frac{r}{\sqrt{2}}$ (tensile).

Force in $CD$ will be compressive and equal to sum of resolved parts of $F_{DE}$ and $F_{EF}$ along direction $DC$ and will be equal to as shown previously $\frac{P}{\sqrt{2}}$ compressive.

Therefore, forces in $AB, BC, CD$ and $DA$ will be $\frac{r}{\sqrt{2}}$ compressive and in all eight inclined bars the forces will be $\frac{P}{\sqrt{2}}$ tensile.

18.8. Complex Trusses. The trusses which do not fall under the categories simple trusses and compound trusses and follow the rule $m = 3j$, are called complex trusses. These trusses will be determinate if they are stable which can be tested by zero load test. In this zero load is assumed at all joints. Then forces in members will be zero. If any other set of values different from zero satisfy the conditions of equilibrium at the joints, then the truss will have a critical form and will be unstable. Fig. 18·17 shows a complex truss. The number of members is 15 and number of joints is 5. The truss cannot be classified as compound truss as $ABCDE$ is not a simple truss. Here method of joints cannot be applied and method of sections will not be possible as any plane will cut more than six bars. The solution can be obtained by tension co-efficients.
in which 15 simultaneous equations will be obtained which when solved will give the forces in various members. The other method is Henneberg's method. In this one member is replaced by another member which will make the truss a compound one and which can be evaluated. At the joints, where the member is removed forces \( X \) are applied along the direction of the member. The analysis of assumed truss with assumed member is carried out. Finally the force in the assumed member is equated to zero and this will give the value of \( X \), the force in the member which is removed. Thus forces in all members can be calculated. The complex truss in Fig. 18.18 is made compound by removing the bar \( HD \) and introducing the bar \( DB \). \( ABCDE \) is a simple truss, connected by six bars to foundation forming a compound truss which can be analysed first by method of sections to get forces in six bars connecting the simple truss to foundations and then in the remaining members of the truss by method of joints or tension co-efficients or any other method.

Take the complex truss of Fig. 18.19. There are twelve members and four joints. Under zero load test, a tension is assumed in bar \( AJ \). To balance joint \( A \), there will be compression in \( AE \). The resultant of these two forces will be a force in plane of \( ABCD \) at \( A \), which will produce forces in \( AB \) and \( AD \). Thus forces other than zero will be possible and the truss is critical and unstable. Fig. 18.20 shows in plan the nonrigidity of the truss.
STATICALLY INDETERMINATE PIN-JOINTED SPACE STRUCTURES

19.1. In a space structure if number of members 'm' is more than 3 times number of joints 'j', the structure will be indeterminate. The degree of indeterminacy will be by how much number the members are in excess than 3j. To analyse such structure the deformation of bars is considered. Using Castiglione's theorem of least work, the forces in the redundant bars are assumed as \( X, Y \ldots \ldots \) and total strain energy including that in the redundant bars is differentiated with respect to these forces \( X, Y \ldots \ldots \) and equated to zero, thus obtaining simultaneous equations, which on solving will give the forces in the redundant bars and thus forces in all other members can be obtained.

Unit load method can also be used. The redundant bars are removed and unit force is introduced along these members one at a time and forces in all members are found for external forces and unit loads. Let the forces due to external forces be \( F \) in any member. Due to unit load let the forces in the member be \( k_1, k_2 \) respectively. If actual forces in redundant members are \( X_1, X_2 \).

Force in member, \( P = F + X_1k_1 + X_2k_2 \ldots \)

\[
\frac{\partial U}{\partial X_1} = \sum \frac{Plk_1}{AE} = - \frac{X_1L_1}{A_1E_1}
\]

Thus

\[
\sum \frac{Plk_1}{AE} + \frac{X_1L_1}{A_1E_1} = 0.
\]

In case there is change in lengths due to temperature in bars or there is possible error \( \Delta \) in the bars

\[
\sum \left( \frac{Pl}{AE} + at + \Delta \right) k_1 + \frac{X_1L_1}{A_1E_1} + atL_1 + \Delta_1 = 0.
\]

Ex. 19.1. Analyse the space structure shown in Fig. 19.1.
Sol.
\( AD \) is taken as redundant bar.
Let the force in \( AD \) be \( X \) tensile

\[
t_{AD} = \frac{X}{L_1}.
\]

In \( X \)-direction

\[
t_{AB} (0-6) + t_{AC} (0-6) + t_{AD} (0-6) + t_{AO} (0-6) = 0
\]
\[ t_{AB} + t_{AC} + t_{AO} = 0. \]

\[ t_{AB} = \frac{4}{5} t_{AC} = \cdot 8 t_{AC} \quad \text{.. (ii)} \]

**In Y-direction**

\[ t_{AB} (2.5 - 0) + t_{AC} (-2 - 0) + t_{AD} (0 - 0) + t_{AO} (0 - 0) = 0. \]

\[ t_{AB} = \frac{4}{5} t_{AC} = \cdot 8 t_{AC} \quad \text{.. (ii)} \]

**In Z-direction**

\[ t_{AB} (0 - 0) + t_{AC} (1.5 - 0) + t_{AD} (2.5 - 0) + t_{AO} (0 - 0) - P = 0. \]

\[ 1.5 t_{AC} = -P - 2.5 t_{AD} + P - \frac{2.5X}{6.5} \]

\[ t_{AC} = \frac{-2P}{3} \quad \frac{10X}{39} \]

\[ t_{AB} = \frac{8P}{15} \quad \frac{8X}{39} \]

\[ t_{AO} = \frac{-X}{6.5} - \left[ \frac{8P}{15} - \frac{8X}{39} \right] - \left[ \frac{2P}{3} - \frac{10X}{39} \right] \]

\[ = \frac{-2X}{13} - \frac{8P}{15} + \frac{8X}{39} - \frac{2P}{3} + \frac{10X}{39} \]

\[ = \frac{4X}{13} - \frac{6P}{5}. \]

\[ F_{AB} = 6.5 \left[ \frac{8P}{15} - \frac{8X}{39} \right] = \frac{52P}{15} - \frac{4X}{3} \]
\[
\frac{\partial F}{\partial X} = -\frac{4}{3}.
\]

\[
F_{AC} = 6.5 \left[ \frac{2P}{3} \cdot \frac{10X}{39} \right] = \frac{13P}{3} - \frac{5X}{3}
\]

\[
\frac{\partial F}{\partial X} = -\frac{5}{3}.
\]

\[
F_{AD} = X
\]

\[
\frac{\partial F}{\partial X} = 1.
\]

\[
F_{AO} = 6 \left[ \frac{4X}{13} - \frac{6P}{5} \right] = \frac{24X}{13} - \frac{36P}{5}
\]

\[
\frac{\partial F}{\partial X} = \frac{24}{13}.
\]

\[
\frac{\partial U}{\partial X} = 0 = \Sigma \frac{F_{lk}}{AE} \text{ where } k = \frac{\partial F}{\partial X}
\]

**AE is constant**

\[
\Sigma F_{lk} = 0.
\]

\[
\left( \frac{52P}{15} - \frac{4X}{3} \right) 6.5 \times \frac{4}{3} + \left( \frac{13P}{3} - \frac{5X}{3} \right) 6.5 \times \frac{5}{3}
\]

\[
+ 6.5X \left( \frac{24X}{13} - \frac{36P}{5} \right) 6 \times \frac{24}{13} = 0.
\]

\[
\frac{-26}{3 \times 15} \left[ 52P - 20X \right] - \frac{32.5}{3 \times 3} \left( 13P - 5X \right) + 6.5X + \frac{144}{13 \times 65}
\]

\[
\left[ 120X - 468P \right] = 0.
\]

\[
-30.04P + 11.55X - 46.95P + 18.06X + 6.5X
\]

\[
+ 26.45X - 79.75P = 0.
\]

\[
-156.74P + 56.56X = 0.
\]

\[
X = \frac{156.74P}{56.56} = 2.77P.
\]

\[
F_{AB} = \frac{52P}{15} - \frac{4}{3} \times 2.77P = \frac{52P - 55.4}{15} \times P
\]

\[
= -\frac{3.4P}{15} \text{ (compressive)}
\]

\[
F_{AC} = \frac{13P}{3} - \frac{5 \times 2.77P}{3} = \frac{13 - 13.85}{3} \times P
\]
\[ F_D = 2.77P \text{ (tensile)} \]
\[ F_O = \frac{24 \times 2.77P \times 36P}{13} - \frac{5}{5} = (5.11 - 7.20)P \]
\[ = -2.09P \text{ (compressive)} \]

Ex. 19.2. Determine, by tension coefficient method, the forces in the members of the ball and socket connected truss shown in Fig. 19.2. The cross-section areas of members CO and DO are twice the areas of members AO and BO.

As the joint O is connected by four bars to the foundation the space frame is redundant.

Let \( X \) be the force in member OD. Coordinates of the members are shown. The force is resolved along \( x \) and \( y \) direction as 8 \( t \) and 6 \( t \). There is no force in \( z \) direction.

\[ t_{OD} = \frac{X}{\sqrt{4.5^2 + 3^2 + 10^2}} \]
\[ = \frac{X}{11.37} \]

Along \( x \)-direction
\[ t_{OA}(-7.5 - 0) + t_{OB}(-7.5 - 0) + t_{OC}(4.5 - 0) + t_{OD}(4.5 - 0) + 8 = 0. \]
\[ \therefore -7.5 t_{OA} - 7.5 t_{OB} + 4.5 \]
\[ t_{OC} + 4.5 \times \frac{X}{11.37} + 8 = 0 \]
\[ \ldots (i) \]

Along \( y \)-direction
\[ t_{OA}(-3 - 0) + t_{OB}(6 - 0) + t_{OC}(6 - 0) \]
\[ + t_{OD}(4.5 - 0) + 6 = 0. \]
\[ \therefore -3t_{OA} + 6t_{OB} + 6t_{OC} + 4.5 \times \frac{X}{11.37} + 6 = 0. \]
\[ \ldots (ii) \]
Along x-direction

\[ t_{OA} (-10\cdot0) + t_{OB} (-10\cdot0) + t_{OC} (-10\cdot0) + t_{OD} (-10\cdot0) = 0 \]

\[ \therefore t_{OA} + t_{OB} + t_{OC} + \frac{X}{11.37} = 0 \quad \text{...(iii)} \]

Multiply (iii) by 3 and add to (ii)

\[ 9t_{OB} + 9t_{OC} + \frac{7.5X}{11.37} + 6 = 0. \quad \text{...(iv)} \]

Multiply (iii) by 7.5 and add to (i),

\[ 12t_{OC} + \frac{12X}{11.37} + 8 = 0. \quad \text{...(v)} \]

\[ \therefore t_{OC} = \left( \frac{-12X}{11.37} - 8 \right) \times \frac{1}{12} = \frac{-X}{11.37} - \frac{2}{3} \]

From equation (iv),

\[ t_{OB} = \left[ \left( \frac{-7.5X}{11.37} - 6 \right) + \left( \frac{9X}{11.37} + 6 \right) \right] \times \frac{1}{9} \frac{11.37 \times 6}{11.37 \times 6} \]

From equation (iii)

\[ t_{OA} = \frac{-X}{11.37} - \frac{X}{11.37 \times 6} + \frac{X}{11.37} + \frac{2}{3} \]

\[ = \frac{-X}{11.37 \times 6} + \frac{2}{3} \]

\[ F_{OA} = \left( \frac{-X}{11.37 \times 6} + \frac{2}{3} \right) \sqrt{7.5^2 + 3^2 + 10^2} \]

\[ = -1.885X + 8.570 \]

\[ \frac{\partial F}{\partial X} = -1.885 \]

\[ F_{OB} = \frac{X}{11.37 \times 6} \times \sqrt{7.5^2 + 6^2 + 10^2} = 2032X \]

\[ \frac{\partial F}{\partial X} = 0.2032 \]

\[ F_{OC} = \left( \frac{-X}{11.37} - \frac{2}{3} \right) \sqrt{4.5^2 + 6^2 + 10^2} \]

\[ = -1.00X - 7.578. \]
\[ \frac{\partial F}{\partial X} = -1.00. \]

\[ F_{OD} = X \]

\[ \therefore \quad \frac{\partial F}{\partial X} = 1. \]

\[ \frac{\partial U}{\partial X} = 0 = \sum \frac{F_{kl}}{AE} \]

\[ AE \text{ is constant, } k = \frac{\partial F}{\partial X}. \]

\[ (-1885X + 8.570) \times 13.85 \times (-1885) + (-2032X) \times 13.87 \times 2032 \]
\[ + (-X - 7.578) \times 11.37 \times (-1) + X \times 11.37 = 0. \]

\[ \therefore \quad (45.65X - 20.75) + 5724X + 11.37X + 86.18 + 11.37X = 0. \]

\[ \therefore \quad 237689X - 6543 = 0. \]

\[ X = \frac{6543}{237689} = 0.2752 \text{ t.} \]

\[ F_{OA} = -5187 + 8.570 = +0.0513 \text{ t (tensile).} \]

\[ F_{OB} = +5591 \text{ t (tensile).} \]

\[ F_{OC} = -2.752 - 7.578 = -10.330 \text{ t (compressive).} \]

\[ F_{OD} = 2.752 \text{ t (tensile).} \]
20.1. Curved beams are used to support circular reservoirs, curved balconies and curved ramps. In curved beams the centre of gravity of loads acting on a span lies outside the line joining the supports. This causes the overturning of the beams unless it is fixed at the supports or continuous over the supports. As the reactions and the loads do not lie along the axis at any point of the beam, there will be a torsional moment in addition to bending moment and shear force. The beam is to be designed for torsion as well as bending moment and shear force.

In case of circular beams supported by symmetrically placed columns the columns will give vertical reaction. Due to symmetry the torsion at the centre of curved beam between two consecutive supports will be zero. Also the torsional moment at the supports will be zero. There will be only shear force and bending moment at the supports.

20.2. Circular beams loaded uniformly and supported on symmetrically placed columns.

Consider portion $AB$ of beam between two consecutive columns.

Let the angle subtended at the centre by two consecutive columns $A$ and $B$ be $\theta$.

The load on the portion $AB$ of the beam $= wR\theta$, where $w$ is the load per unit length.

The centre of gravity of the load will lie at distance

$$\frac{R \sin \theta/2 \theta/2}{from the centre.}$$

Let $M_0$ be the bending moment and $F_0$ be the shear force at the supports.

From symmetry $F_0 = \frac{wR\theta}{2}$ (20.1)

Representing the moments at supports by vectors, the direction of moments at supports is shown in Fig. 20.2.

The moment of $M_0$ at support can be resolved along the chord $AB$ and at right angles to it. The two components will be $M \sin \theta/2$ and $M \cos \theta/2$ respectively.
Taking moments of all forces about line $AB$

$$2M_0 \sin \theta/2 = wR \theta \left[ \frac{R \sin \theta/2}{\theta/2} - R \cos \theta/2 \right]$$

$$M_0 = \frac{wR^2}{2} \left[ \frac{\theta}{2 \sin \theta/2} \times \frac{\sin \theta/2}{\theta/2} - \theta \times \frac{\cos \theta/2}{2 \sin \theta/2} \right]$$

$$= \frac{wR^2}{2} \left[ 1 - \frac{\theta \cot \theta/2}{2} \right]$$  \hspace{1cm} \text{...(20.2)}

To find S.F., B.M. and torsional moment at a point $P$, at an angle $\phi$ from one support.

Load between $AP = wR \phi$

S.F. at $P = \frac{wR \theta}{2} - wR \phi$

$$= wR(\theta/2 - \phi)$$  \hspace{1cm} \text{...(20.3)}

The distance of centre of gravity of load between $AP$ from centre $O = \frac{R \sin \theta/2}{\phi/2}$.

The direction of vector representing B.M. at $P$ will act along $PO$ and direction of vector representing torsion at $P$ will act at right angles to $PO$.

$$M_\phi = F_0 R \sin \psi - M_0 \cos \phi$$

$$= wR \frac{R \sin \theta/2}{\phi/2} \times \sin \phi/2$$

$$= \frac{wR \theta}{2} \times R \sin \phi - wR^2 \left( 1 - \frac{\theta \cot \theta/2}{2} \right) \cos \phi$$

$$- wR^2 \times 2 \sin^2 \phi/2$$

$$= wR^2 \left[ \frac{\theta}{2} \sin \phi - \cos \phi + \frac{\theta}{2} \cot \theta/2 \cos \phi - 2 \sin^2 \phi/2 \right]$$

$$= wR^2 \left[ -1 + \frac{\theta}{2} \sin \phi + \frac{\theta}{2} \cot \theta/2 \cos \phi \right]$$  \hspace{1cm} \text{...(20.4)}

\begin{align*}
\therefore - \cos \phi - 2 \sin^2 \phi/2 &= -1
\end{align*}
\[
\varphi = M_y \sin \varphi - \frac{wR}{2} \times 2R \sin^2 \varphi \frac{2}{2} + wR^2 \left( R - \frac{R \sin \frac{\varphi}{2}}{\frac{\varphi}{2}} \times \cos \frac{\varphi}{2} \right) \\
- \sin \varphi - wR^2 \theta \sin^2 \frac{\varphi}{2} + wR^2 \left( \varphi - 2 \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} \right)
\]

\[
= wR^2 \left[ \sin \varphi - \theta \cot \frac{\theta}{2} \sin \varphi - \theta \sin^2 \frac{\varphi}{2} + \varphi - \sin \varphi \right] \\
= wR^2 \left[ \varphi - \theta \sin^2 \frac{\varphi}{2} + \theta \cot \frac{\theta}{2} \sin \varphi \right] \\
= wR^2 \left[ \varphi - \theta \sin^2 \frac{\varphi}{2} + \theta \cos \frac{\theta}{2} \sin \varphi \right] \quad \ldots ('0'5)
\]

To obtain maximum value of torsional moment \( \frac{dT_\varphi}{d\varphi} \) is put equal to zero. This gives value of \( \varphi \) for maximum value of \( T \). Substituting in (20.5), maximum value of \( T \) is obtained.

Let, support moment \( = kwR^2 \times \theta \)
Mid span moment \( = k'wR^2 \times \theta \)
Maximum torsion \( = k''wR^2 \times \theta \)

\( \varphi' \) be the angle from support where maximum torsion occurs.

Table 20.1 gives value of \( k, k', k'' \) and \( \varphi' \) for circular beams resting on number of supports.

**Table 20.1. Coefficient for B.M. and Torsion in circular beams**

<table>
<thead>
<tr>
<th>Number of supports</th>
<th>( \varphi )</th>
<th>( k )</th>
<th>( k' )</th>
<th>( k'' )</th>
<th>Value of ( \varphi' ) for maximum twisting moment</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>90°</td>
<td>0.137</td>
<td>0.07</td>
<td>0.021</td>
<td>19.4°</td>
</tr>
<tr>
<td>5</td>
<td>72°</td>
<td>0.108</td>
<td>0.054</td>
<td>0.014</td>
<td>15.4°</td>
</tr>
<tr>
<td>6</td>
<td>60°</td>
<td>0.089</td>
<td>0.045</td>
<td>0.009</td>
<td>15.8°</td>
</tr>
<tr>
<td>7</td>
<td>51.3°</td>
<td>0.077</td>
<td>0.037</td>
<td>0.007</td>
<td>10.3°</td>
</tr>
<tr>
<td>8</td>
<td>45°</td>
<td>0.066</td>
<td>0.03</td>
<td>0.005</td>
<td>9.5°</td>
</tr>
<tr>
<td>9</td>
<td>40°</td>
<td>0.060</td>
<td>0.027</td>
<td>0.004</td>
<td>8.5°</td>
</tr>
<tr>
<td>10</td>
<td>36°</td>
<td>0.054</td>
<td>0.023</td>
<td>0.003</td>
<td>7.5°</td>
</tr>
<tr>
<td>12</td>
<td>30°</td>
<td>0.045</td>
<td>0.017</td>
<td>0.002</td>
<td>6.5°</td>
</tr>
</tbody>
</table>
20.3. **Semicircular beam simply supported on three supports equally spaced.**

Let semicircular beam $ABC$ be supported on three equally spaced supports $A$, $B$ and $C$. Let $R$ be the radius of the centre line of the beam and $w$ be the load per unit length of the beam.

Total load on the semicircular beam $= wR\pi$ as $q = w$.

![Fig. 20.4](image)

**Distance of C.G. of load from centre $O$**

$$\frac{R \sin \phi/2}{\phi/2} = \frac{R \sin \pi/2}{\pi/2} = \frac{2R}{\pi}$$

Let $R_1$ be the reaction at support $A$ and $C$ and $R_2$ be the reaction at $B$.

Taking moment of reactions about line passing through $B$ and parallel to $AC$.

$$2R_1 \times R = wR\pi \left( R - \frac{2R}{\pi} \right)$$

$$\therefore \quad R_1 = \frac{wR}{2} (\pi - 2)$$

$$\therefore \quad R_2 = wR\pi - 2 \times \frac{wR}{2} (\pi - 2)$$

$$= 2wR.$$  

**B.M. at a point $P$ at $\theta$**

Load on portion $AP = wR\theta$

Distance of C.G. of load from $O$

$$= \frac{R \sin \theta/2}{\theta/2}$$

$$M_0 = R_1 \times R \sin \theta - wR\theta \left[ \frac{R \sin \theta/2}{\theta/2} \times \sin \theta/2 \right]$$

$$= \frac{wR}{2} (\pi - 2) \times R \sin \theta - 2wR^2 \sin^2 \theta/2$$

$$= wR^2 \left[ \frac{(\pi - 2)}{2} \sin \theta - 2 \sin^2 \theta/2 \right]$$

At $\theta = \frac{\pi}{2}$

$$M_B = wR^2 \left[ \frac{\pi - 2}{2} - 2 \times \frac{1}{2} \right]$$

$$= wR^2 \times \frac{\pi - 4}{2} = -0.429 \ wR^2$$