PROPPED CANTILEVERS

1.1. When a cantilever is supported at any point in the span, the structure becomes indeterminate. Under vertical loads, there will be two unknown reactions at the fixed end and one at support-end. Two equations of statics i.e., $\Sigma V = 0$ and $\Sigma M = 0$ are available. This type of structure cannot be analysed by mere equations of statics. One more equation besides two equations of statics is required to solve three unknowns. Therefore, this structure is said to be indeterminate to first degree. The third equation can be obtained by considerations of deflections or slopes.

Consider the propped cantilever shown in Fig. 1.1. The support at C has been removed and force equal to unknown reaction $R_c$ is applied [Fig. 1.1(b)]. The third equation is obtained from the consideration that the deflection at C is zero. In [Fig. 1.1(c)] the fixity at A has been removed and moment $M_A$ is applied equal to fixed moment. The third equation is obtained from the consideration that slope at A is zero.

1.2. Analysis of Propped Cantilevers

Statically indeterminate structures can be analysed by several methods. In this chapter, analysis of propped cantilevers by methods of consistent deformation and moment area is described.

METHOD OF CONSISTENT DEFORMATION

(a) The support at C is removed thus making the structure indeterminate and the deflection at C is worked out, let it be $S_C$. 
The loading is removed and force $R_c$ equal to unknown reaction at $C$, is applied at $C$ and the deflection at $C$ is worked out. Let the deflection be $\delta_{c1}$.

$\delta_{c1} + \delta_{c2} = 0$, in case support $C$ remains at the same level when the beam is loaded.

$\delta_{c1} + \delta_{c2} = \delta$, in case the support $C$ sinks by $\delta$.

From this equation the value of unknown reaction $R_c$ can be obtained and thus the structure can be analysed.

(b) The structure is made determinate by removing fixity at $A$ and thus the structure will be a simply supported beam with over-hang. Under the external loading, the slope at $A$ is worked out. Let the slope be $\theta_1$.

The load is removed and a moment $M_A$ equal to fixed end moment is applied at $A$. The slope at $A$ is worked out. Let it be $\theta$

$\theta_1 + \theta_2 = 0$, in case there is no rotation of support.

In case support at $A$ rotates by $\theta$,

$\theta_1 + \theta_2 = \theta$.

With the help of this equation the value of unknown moment $M_A$ can be found out.
II. MOMENT AREA METHOD

(a) Taking \( R_c \) as indeterminate reaction.
The B.M. diagrams due to external loading and \( R_c \) are drawn
considering \( ACB \) as a cantilever.

\[ \text{M/EI Diagram due to load} \]

\[ \text{M/EI Diagram due to } R_c \]

Fig. 1-4

As the end \( A \) is fixed, the tangent to the deflection curve at
will pass through \( C \) in case \( A \) and \( C \) are at the same level. Thus
moment of \( \text{M/EI diagram between } A \) and \( C \) about \( C \) will be
\( \gamma \). If there is change of level equal to \( \delta \) between \( A \) and \( C \), the
moment of \( \text{M/EI diagram between } A \) and \( C \) about \( C \) will be \(-\delta\).

(b) Taking \( M_A \) as indeterminate moment.
The B.M. diagrams are drawn for the load and fixed end
ment \( M_A \), considering \( ABC \) as simply supported beam with

\[ \text{M/EI Diagram due to load} \]

\[ \text{M/EI Diagram due to } M_A \]

Fig. 1-5

In case \( A \) and \( C \) are at the same level, the moment of \( \text{M/EI di}-
grams between \( A \) and \( C \) about \( C \) will be zero. If there is change
level \( \delta \) between supports \( A \) and \( C \) after loading, the moment of
\( \text{EI diagrams between } A \) and \( C \) about \( C \) will be \(-\delta\).
Ex. 1.1. Draw B.M. diagram for the propped cantilever loaded as shown in Fig. 1.6 (a). The supports A and B remain at the same level after the load.

Solution. The support at B is removed and B.M. diagram for the cantilever is drawn as shown in Fig. 1.6 (c).

The deflection $\delta_{B1}$ at B will be equal to moment of $M/E$ diagram about B.

$$
\delta_{B1} = \frac{Wl^3}{24EI} \times \frac{l}{4} \times \frac{5}{6} l
$$

$$
\delta_{B1} = \frac{5Wl^3}{48EI}
$$

Unknown reaction $R_o$ applied at B and B.M. diagram is drawn.

Upward deflection of end B.

$$
\delta_{B2} = - \frac{R_B l^3}{4EI}
$$

$$
\delta_{B2} = - \frac{R_B l^3}{3EI}
$$

$$
\delta_{B1} + \delta_{B2} = 0
$$

$$
\frac{5Wl^3}{48EI} - \frac{R_B l^3}{3EI} = 0
$$

Fig. 1.6

$$
R_B = \frac{5}{16} W
$$

$$
M_A = \frac{5}{16} Wl - W \times l/2
$$

$$
= - \frac{3Wl}{16}
$$

Maximum +ve B.M. will be at centre.

Maximum +ve B.M. = \frac{5}{16} W \times \frac{l}{2} = \frac{5Wl}{32}

The B.M. diagram is shown in Fig. 1.6 (f).

Ex. 1.2. Find the support moment for the cantilever loaded shown in Fig. 1.7 (a), if the support A rotates clockwise by 0° radian; $EI = 1 \times 10^6$ kg m²
Solution. The B.M. diagram for the beam will consist of that to fixed end moment at $A$ that due to simply supported at $AB$. The B.M. diagrams is shown in Fig. 1.7 (b) and (c). A tangent drawn at $A$ to the selected form, will give intercept on vertical at $B$ equal to $0.2 \times 4$. Therefore moment of $EI$ diagrams between $A$ and about $B$ will be $4 \times 0.002$

$$\frac{10,000 \times \frac{4}{2} \times 2}{EI}$$

$$M_A \times \frac{4}{2} \times \frac{2}{3} \times 4$$

$$\frac{4 \times 0.002}{EI}$$

$$40,000 - \frac{16}{3} M_A = 0.008 \times 1 \times 10^8$$

$$\frac{16}{3} M_A = 40,000 - 8,000 = 32,000 \text{ kg. m.}$$

$$\therefore M_A = 6,000 \text{ kg. m.}$$

Ex. 1.3. A beam $B$ is supported on beam $A$ as shown in Fig. 1.8. The flexural rigidity of beam $B$ is twice that of $A$. Determine deflection at $C$.

Solution. Let $R_c$ be the reaction at support $C$ and $\delta$ be the downward deflection of point $C$.

Fig. 1.8

Beam $B$

Net downward deflection of end $C$ of beam $B = \delta$

$$\delta = \frac{wl^4}{8EI_B} - \frac{R_c \times l^3}{3EI_B}$$

$$= \frac{1000 \times 2^4}{8EI_B} - \frac{R_c \times 2^3}{3EI_B}$$

$$= 2000 - \frac{8}{3} R_c$$

$$\delta = \frac{2000 - \frac{8}{3} R_c}{EI_B}$$
Beam A
Downward deflection of end C of beam
\[ A = \delta \]
\[ = \frac{R_C \times l^3}{3EI_A} = \frac{R_C}{3EI_A} \]
\[ \therefore \frac{R_C}{3EI_A} = \frac{2000 - \frac{8}{3} R_C}{EIB} \]
\[ EIB = 2EI \]
\[ \frac{R_C}{3} = \frac{2000 - \frac{8}{3} R_C}{2} \]
\[ \therefore \frac{2}{3} R_C = 2000 - \frac{8}{3} R_C \]
\[ \frac{10}{3} R_C = 2000 \]
\[ R_C = 600 \text{ kg.} \]
\[ \delta = \frac{600 \times l^3}{3EI_A} = \frac{200}{EI_A} \]

EXAMPLES

1. A beam of length \( l \) carrying uniformly distributed load \( w \) per unit length, is fixed at one end and simply supported at other end. The supports remain at the same level during loading. Determine the reaction at simply supported end and moment at fixed end.

\[
\begin{bmatrix}
\text{Ans. Reaction} \\
\text{Moment}
\end{bmatrix}
= \begin{bmatrix}
3wl \\
-\frac{wl^2}{8}
\end{bmatrix}
\]

2. A beam \( AB \) of length \( l \) carries a uniformly distributed load of \( w \) per unit length throughout its length. The beam is fixed at end \( A \) and supported on a spring at end \( B \). Show that the reaction at end \( B \) is \( \frac{3wl}{8\left(1 - \frac{3EI\lambda}{l^4}\right)} \) where \( EI \) is the flexural rigidity of the beam and \( \lambda \) is the compression of the spring produced by unit force.

2. The beam shown in Fig. 1-9 is fixed at one end and when unloaded there is a clearance of 1 cm. between the other end and the support at that end. Determine the reactions on the supports
when the beam is loaded with load of 10,000 kg./m. run.

\[ EI = 1 \times 10^4 \text{ kg. m}^2. \]

\[ \text{Fig. 1.9} \]

[Ans. \( R_A : 16,250 \text{ kg.} \)]

[Ans. \( R_B : 3750 \text{ kg.} \)]

[Ans. \( M_A : 12,500 \text{ kg.m.} \)]

4. A timber beam 12 cm. wide, 20 cm. deep and 4 m. long is loaded with uniformly distributed load. It is fixed at the left end and simply supported at the right end. If the maximum allowable fibre stress is 100 kg/cm² and right support settles by an amount equal to \( wL^4/24EI \), where \( w \) is the load per metre run, \( L \) is span and \( EI \) is flexural rigidity, determine the permissible value of load \( w \).

[Ans. \( w = 200 \text{ kg./m.} \)]

5. A cantilever \( AB \) of length 2 m. is fixed at end \( A \) and end \( B \) rests at the centre of simply supported beam of span 4 m. Find the support moment at \( A \) and deflection at \( B \) when the cantilever is loaded with uniformly distributed load of 2000 kg/m. \( EI \) for cantilever = \( 1\cdot0 \times 10^4 \text{ kg. m}^2 \) and \( EI \) for simply supported beam = \( 2 \times 10^8 \text{ kg. m}^2 \).

[Ans. \( M_A = 1600 \text{ kg. m.} \)]

[Ans. \( \delta_B = 0\cdot08 \text{ cm.} \)]

6. Determine the value of moment \( M \) at end \( A \) of the beam shown in Fig. 1.10, which will cause the slope of the elastic curve of the beam to be zero at the right support.

[Ans. \( M = 1000 \text{ kg.m.} \)]
FIXED BEAMS

2.1. Fig. 2.1 (a) shows a fixed beam $AB$ of uniform section and span $l$, loaded as shown in the figure. As the ends of the beam are fixed the slope at the supports after loading will be zero as shown in deflected form in Fig. 2.1 (b).

Let $M_A$ and $M_B$ be the fixing moments at the supports $A$ and $B$ respectively.

The angle between the two tangents drawn at $A$ and $B$ on the deflected curve is zero, therefore, the total area of $M/EI$ diagram between $A$ and $B$ will be zero. For a beam of constant moment of inertia $A/EI = 0$, where $A$ is the total area of B.M. diagram.

The fixed beam can be looked upon as simply supported beam with end moments $M_A$ and $M_B$ such that the slopes at the supports are zero. Due to simply supported condition the loading will cause $+$ve B.M. Due to fixing moments the B.M. diagram will vary from $M_A$ at $A$ to $M_B$ at $B$. As the total area of $M/EI$ diagram is to be zero, the area of $M/EI$ diagram due to fixing moments will be equal to area of $M/EI$ diagram as a simply supported beam.

For a beam of constant moment of inertia, if $A_s$ is the area of B.M. diagram considering beam as simply supported and $A_t$ is the area of B.M. diagram due to fixing moments,

$$\frac{A}{EI} = \frac{A_s}{EI} + \frac{A_t}{EI} = 0 \quad \ldots \quad (2.1)$$

$$\therefore \quad A_t = -A_s$$
where \[ A_4 = \frac{(M_A + M_B)}{2} \times l \].

\[ \therefore \frac{M_A + M_B}{2} \times l = -A_4 \]

\[ M_A + M_B = -\frac{2A_4}{l} \] \hspace{1cm} (3.2)

The tangent drawn at \( A \) will pass through \( B \), therefore, the intercept on the vertical at \( A \) by the tangents drawn at \( A \) and \( B \) will be zero. Therefore, moment of \( M/EI \) diagram about \( A \) will be zero. Similarly moment of \( M/EI \) diagram about \( B \) will be zero.

\[ \therefore \frac{A_4}{EI} \times x + \frac{A_4}{EI} \bar{x} = 0 \] \hspace{1cm} (2.3)

\( x \) and \( \bar{x} \) are distances of centre of gravities of \( A_4 \) and \( A_4 \) respectively, from end \( A \).

\[ A_4 \bar{x} = M_A \times \frac{l}{2} \times \frac{l}{3} + M_B \times \frac{l}{2} \times \frac{2l}{3} \]

\[ = \frac{l^3}{6} (M_A + 2M_B) = -A_4 \bar{x} \]

\[ M_A + 2M_B = -\frac{6A_4 \bar{x}}{l^3} \] \hspace{1cm} (2.4)

Solving (2.2) and (2.4)

\[ M_B = \frac{6A_4 \bar{x}}{l^3} + \frac{2A_4}{l} \]

\[ = -\frac{2A_4}{l} [3 \bar{x} - l] \]

\[ M_A = -\frac{2A_4}{l} + \frac{2A_4}{l} [3 \bar{x} - l] \]

\[ = -\frac{2A_4}{l^4} [2l - 3\bar{x}]. \]

**Summarising**

(i) The total area of \( \frac{M}{EI} \) diagram is zero.

(ii) Moment of area of \( \frac{M}{EI} \) diagram about any support is zero.

(iii) From the above two conditions it is evident that C.G. of \( A_4 \) diagram and C.G. of \( A_4 \) diagram will be equidistant from the same support and thus will lie on the same vertical for beam of constant moment of inertia \( I \).
2.2. Fixing moments for a fixed beam of uniform section due to

(1) Concentrated load at the centre of span.

As a simply supported beam, \( \frac{M}{EI} \) diagram will be a triangle with maximum ordinate \( \frac{WL}{4EI} \) at the centre. As the beam is symmetrical, fixing moment \( M_A \) will be equal to \( M_B \).

\[
\frac{WL}{4} \times \frac{l}{2} + (M_A + M_B) \times \frac{l}{2} = 0
\]

\[
M_A + M_B = -\frac{WL}{4}
\]

But \( M_A = M_B \)

\[
M_A = M_B = -\frac{WL}{8}
\]

\[
R_A = R_B = \frac{W}{2}
\]

B.M. at a section \( x \) distance from \( A \), between \( A \) and \( C \) will be

\[
M_x = \frac{W}{2} x - \frac{WL}{8} = \frac{W}{8} (4x - l).
\]

At point of contraflexure

\[
M_x = 0
\]

\[
x = \frac{l}{4}.
\]

The points of contraflexure will be at \( l/4 \) from either end.

Maximum +ve B.M. will be at the centre.

Maximum +ve B.M. = \( \frac{WL}{4} - \frac{WL}{8} = \frac{WL}{8} \).

Maximum -ve B.M. will be at the supports and will be \( -\frac{WL}{8} \).
(2) Uniformly distributed load of \( w \) per unit length throughout the span.

As the loading is symmetrical about the centre, \( M_A \) will be equal to \( M_B \).

Simply supported B.M. diagram will be a parabola with central ordinate \( \frac{wL^3}{8} \)

\[
\frac{2}{3} \times \frac{wL^3}{8} \times l = 0
\]

\[
+ \left( \frac{M_A + M_B}{2} \right) \times l = 0
\]

\[
\therefore M_A + M_B = -\frac{wL^2}{6}
\]

But \( M_A \) \[ \quad M_B \]

\[
\therefore M_A \cdot M_B = \frac{wL^2}{12}
\]

But \( R_A \cdot R_B = \frac{wl}{2} \)

B.M. at a section \( x \) distance from \( A \) will be

\[
M_x = \frac{wl}{2} x - \frac{wx^2}{2} - \frac{wL^2}{12}
\]

At point of contraflexure B.M. is zero

\[
\frac{wl}{2} x - \frac{wx^2}{2} - \frac{wL^2}{12} = 0
\]

\[
x^2 - lx + \frac{L^2}{6} = 0
\]

\[
x = \frac{l \pm \sqrt{l^2 - \frac{L^2}{6}}}{2}
\]

\[
= \frac{l}{2} [1 \pm 0.577] = 0.212l \text{ or } 0.788l.
\]

\[
\therefore \text{Points of contraflexure are at a distance of } 0.212l \text{ from either end.}
\]

\[
\text{Maximum } +ve \text{ B.M. will be at the centre.}
\]
Maximum $+ve$ B.M. \[ \frac{w l^2}{8} - \frac{w l^2}{12} \] \[ w l^2 \]

Maximum $-ve$ B.M. will be at the supports equal to \[ \frac{-w l^2}{12} \]

(3) *Unsymmetrical concentrated load.* B.M. diagram as a simply supported beam will be a triangle with maximum ordinate under the load equal to \( \frac{W a b}{l} \)

\[ M_A + M_B = -\frac{2A_i}{i} \]
\[ M_A + M_B = -\frac{2}{l} \times \frac{1}{2} \times \frac{W a b}{l} \times l \]
\[ = -\frac{W a b}{l} \] \( \ldots \) \( (1) \)

Let \( \bar{x}_a \) be the distance of C.G. of simply supported B.M. diagram from \( A \).

\[ \bar{x}_a = \frac{a + l}{3} \]
\[ M_A + 2M_B = -\frac{6A_i\bar{x}_a}{l^2} \]
\[ \therefore M_A + 2M_B \]
\[ 6 \times \frac{W a b}{l} \times \frac{l}{2} \times \frac{a + l}{2} \]
\[ = \frac{W a b(a + l)}{l^2} \]
\( \ldots \) \( (2) \)

Subtracting (1) from (2)

\[ M_B = -\frac{W a b}{l^2}(a + l) + \frac{W a b}{l} \]
\[ -\frac{W a b}{l^2}(a + l - l) \]
\[ = \frac{-W a b}{l^2} \].
\[ M_A = - \frac{W_{ab}}{l} M_B = - \frac{W_{ab}}{l} + W_{ab^2} \]

\[ = \frac{W_{ab}}{l^2} (a - l) \]

\[ = - \frac{W_{ab}}{l^2}. \]

Maximum +ve B.M. will be under the load and maximum -ve B.M. will be either at support A or B depending on relative values of \( a \) and \( b \).

(4) **Varying load on the beam.** The fixed beam carries a load of varying intensity as shown in Fig. 2-5.

Take a small length \( \delta_x \) of the beam at \( x \) distance from A. Let the intensity of load be \( w_x \). Fixing moments \( \delta M_A \) and \( \delta M_B \) due to the load \( w_x \delta_x \) of elementary strip are

\[ \delta M_A = - \frac{w_x \delta_x \times x \times (l - x)^2}{l^2} \]

\[ \delta M_B = - \frac{w_x \delta_x \times (l - x) \times x^2}{l^2} \]

Total fixing moments will be

\[ M_A = - \sum \frac{w_x x (l - x)^3}{l^2} \delta x \]

\[ - \sum \frac{w_x x^2 (l - x)}{l^2} \delta x. \]

If the loading curve is given by \( w_x = f(x) \)

\[ M_A = - \int_0^l \frac{f(x) x (l - x)^3}{l^2} dx \]

\[ M_B = - \int_0^l \frac{f(x) x^2 (l - x)}{l^2} dx. \]

When the beam is partially loaded as shown in Fig. 2-6.

\[ M_A = - \int_a^b \frac{f(x) x (l - x)^3}{l^2} dx \]

\[ M_B = - \int_a^b \frac{f(x) x^2 (l - x)}{l^2} dx. \]

**Fig. 2-6**

Ex. 2-1. Find the fixed end moments for the beam loaded as shown in Fig. 2-7.
**Solution.** The B.M. diagram for the beam as simply supported shown in Fig. 2.9. The beam is symmetrically loaded,
\[ M_A + M_B = -\frac{2A_s}{l} \]
\[ M_A = M_B \]
\[ M_A = M_B = -\frac{2}{l} \left[ \frac{1}{2} \times \frac{3WL}{8} \times \frac{l}{2} + \frac{3WL}{8} \times \frac{l}{4} \right] \]
\[ = -\frac{2}{l} \left[ \frac{3WL^2}{64} + \frac{7WL}{64} \right] \]
\[ = -\frac{5}{16} \frac{WL}{l}. \]

![Simply Supported B.M. Diagram](image)

The B.M. diagram for the beam is shown in Fig. 2.7 (c).

**Ex. 2.2.** Find the fixed end moments and plot the S.F. and BfM. diagrams for the beam loaded as shown in Fig. 2.8 (a).

**Solution.** Simply supported B.M. diagram for the beam is shown in Fig. 2.8 (b).
\[ M_A + M_B = -\frac{2A_s}{l} \]
\[ = -\frac{2}{l} \left[ \frac{1}{2} \times 6200 \times 1 + \frac{1}{2} (6200 + 8800) + \frac{1}{2} (8800 + 10500) + \frac{1}{2} (10500 + 5900) \times 2 + \frac{1}{2} \times 5900 \times 1 \right] \]
\[ M_A + M_B = -\frac{1}{2} [3,100 + 7,500 + 9,650 + 16,400 + 2,950] \]
\[ M_A + M_B = -\frac{1}{2} \times 39,600 \]
\[ = -13,200 \]
\[ M_A + 2M_B = -\frac{6A_sx_s}{l^3} \]
\[ = -\frac{6}{6} \times \left[ \frac{1}{2} \times 6,200 \times \frac{2}{3} + 6,200 \times 1 \times \frac{3}{2} \right] \]
\[ + \frac{1}{2} \times 2,600 \left( 1 + \frac{2}{3} \right) + 8,200 \times \frac{5}{2} + \frac{1}{2} \times 1700 \]
\[ \times \left( 2 + \frac{2}{3} \right) + 5,900 \times 2 \times 4 + \frac{1}{2} \]
\[ \times 2 \left( 10,500 - 5,900 \right) \left( 3 + \frac{2}{3} \right) + \frac{1}{2} \times 5,900 \times \left( 5 + \frac{1}{3} \right) \]
\[
\begin{align*}
\mathcal{E} &= -\frac{1}{6} \left[ \frac{6.200}{3} + 9,300 + 1,300 \times \frac{5}{3} + 22,000 \\
+ 850 \times \frac{8}{3} + 47,200 + 4,600 \times \frac{11}{3} + 5,900 \times \frac{8}{3} \right] \\
&= -\frac{1}{6} \times \frac{1}{2} \left( 6,200 + 27,900 + 6,500 + 6,600 + 6,800 \\
+ 141,600 + 50,600 + 47,200 \right)
\end{align*}
\]

SIMPLY SUPPORTED B.M. DIAGRAM.

S.F. DIAGRAM

\[ 6800 \]

\[ \begin{array}{c}
6200 \\
\hline
8800
\end{array} \]

\[ -7 \times 6400 \]

Fig. 2-8

\[ M_A + 2M_6 = -\frac{1}{18} \times 352,800 = 19,600 \quad (2) \]

(2) - (1) gives \( M_B = -6,400 \text{ kgm.} \)

\[ M_A = -6,800 \text{ kgm.} \]
Taking moments about \( A \),
\[
R_B \times 6 + 6800 - 6400 - 3600 \times 1 - 3600 \times 5 - 900 \times 2 - 4000 \times 3 = 0
\]
\[
R_B = 3600 + \frac{900}{3} + 2000 - \frac{400}{6}
\]
\[
= 5900 - 66.67 = 5833.33 \text{ kg.}
\]
\[
R_A = 6266.67 \text{ kg.}
\]
S.F. and B.M. diagrams are shown in Fig. 2-8.
Maximum +ve B.M. = 6800 kg. m.
Maximum -ve B.M. at centre of span
\[
= 10,500 - \frac{6800 + 6400}{2}
\]
\[
= 10,500 - 6600
\]
\[
= 3900 \text{ kg. m.}
\]

Ex. 2-3. Find the fixed end moments and plot the B.M. diagram for the beam loaded as shown in Fig. 2-9(a).

**Solution. First Method.**

Consider a small width \( \delta x \) of the beam in loaded portion at distance \( x \) from \( A \).

Weight on this strip = \( w \times \delta x = 1600 \delta x \)

Fixed end moments \( M_A \) and \( M_B \) due to a concentrated load at distances \( a \) and \( b \) from \( A \) and \( B \) respectively are given by

\[
M_A = \frac{-Wab^3}{l^2}, \quad M_B = \frac{-Wab^3}{l^2}
\]

Due to elementary weight \( w \times \delta x \), fixed end moments \( \delta M_A \) and \( \delta M_B \) are given by

\[
\delta M_A = \frac{-w \times \delta x \times x(l-x)^3}{l^2}
\]

\[
\delta M_B = \frac{-w \times \delta x \times x^2(l-x)}{l^2}
\]
\[ M_A = - \int_0^L \frac{1600d_\alpha x \times (8-x)^2}{8^2} \]
\[ = - \int_0^4 \frac{1600}{64} (64x \times 16x^3 + x^5) \, dx \]
\[ = -25 \left[ \frac{64x^2}{2} - \frac{16x^3}{3} + \frac{x^4}{4} \right]_0^4 \]
\[ = -25 \left[ 32(16-4) - \frac{16}{3} (64-8) + \frac{1}{4} (256-16) \right] \]
\[ = -25 \left[ 384 - \frac{896}{3} + 60 \right] \]
\[ = -25 \times 436 = -3633.3 \text{ kgm} \]

\[ M_B = \int_0^L \frac{1600d_\alpha x^2}{8} \times (8-x) \]
\[ = -25 \int_0^4 (8x^3 - x^2) \, dx \]
\[ = -25 \left[ \frac{8x^4}{3} - \frac{x^4}{4} \right]_0^4 \]
\[ = -25\left[ \frac{1}{3} (64-8) - \frac{1}{4} (256-16) \right] \]
\[ = -25\left[ \frac{1}{3} \times 56 - \frac{1}{4} \times 240 \right] \]
\[ = -2233.3 \text{ kgm} \]

**Second Method.**

Simply supported B.M. diagram is shown in Fig. 2-9 (c)

\[ M_A + M_B = - \frac{2A}{l} = - \frac{2}{8} \left[ \frac{1}{2} \times 2 \times 4000 + \frac{1}{2} \times 4 \times 4800 \right] \]
\[ + \int_0^4 \left( 2000x - \frac{1600(x-2)^2}{2} \right) \, dx \]
\[ = -\frac{1}{4} \left[ 4000 + 9600 + \left\{ \frac{2000x^2}{2} - \frac{800(x-2)^3}{3} \right\} \right] \]
\[ = -\frac{1}{4} \left[ 13,600 + \left\{ 1000(16-4) - \frac{800}{3} (8-0) \right\} \right] \]
\[ = \frac{1}{4} \times 23,133.33 \]
\[ = 5783.33 \]

\[ M_A + 2M_B = - \frac{6A}{l} = - \frac{6}{8} \times 8 \left[ 4000 \times \frac{2}{3} \times 2 \right] \]
\[ + 9600 \left( 4 + \frac{4}{3} \right) + \int_0^4 \left( 2000x - \frac{1600(x-2)^2}{2} \right) \, dx \]
\[ = -\frac{3}{32} \left[ 16,000 + 9,600 \times 16 \right] \]
\[ + \int_0^4 \left( 2000x^3 - 800(x^3 - 4x^2 + 4x) \right) \, dx \]

... (1)
\[ M_A + 2M_B = -\frac{3}{32} \left[ 56,533.33 + \left( \frac{5200x^4}{3} - \frac{800x^4}{4} - \frac{3200x^2}{2} \right)_x \right] \]

\[ = -\frac{3}{32} \left[ 56,533.33 + \left( \frac{5200}{3} x (64 - 8) - 200(256 - 16) - 1600 \times 12 \right) \right] \]

\[ = -\frac{3}{32} \left[ 56,533.33 + 97,066.67 - 48,000 - 19,200 \right] \]

\[ = -\frac{3}{32} \times 86,400 = -8100 \]

\[ M_A + 2M_B = -8100 \]

\[ M_A + M_B = -5866.67 \]

\[ \therefore M_B = -2233.33 \text{ kg. m.} \]

\[ M_A = -3633.34 \text{ kg. m.} \]

Taking moments about \( A \).

\[ R_A \times 8 + 3633.34 - 2233.33 - 2 \times 1600 \times 3 = 0 \]

\[ R_A = 1200 - \frac{1400}{8} = 1200 - 175 = 1025 \text{ kg.} \]

\[ R_A = 1600 \times 2 - 1025 = 3200 - 1025 = 2175 \text{ kg.} \]

S.F. and B.M. diagrams are shown in Figs. 2:9 (d, e).

Ex. 2:4. Find the fixed end moments for the beam carrying uniformly varying load as shown in Fig. 2:10.

Solution. Consider a strip of width \( \delta x \) at distance \( x \) from support \( A \). Intensity of loading at this section is \( \frac{wx}{l} \).

\[ \text{Fig. 2:10} \]

Weight of elementary strip = \( \frac{wx}{l} \times \delta x \).

Fixed end moments \( \delta M_A \) and \( \delta M_B \) due to this elementary weight \( \frac{wx}{l} \times \delta x \) are given by

\[ \delta M_A = -\left( \frac{wx}{l} \right) \delta x \times \frac{x(l-x)^2}{2} \]
\[5M_x = - \left( \frac{wx}{l} \right) \times dx \times \frac{x(l-x)^2}{l^3}\]

\[M_A = \int_0^1 \frac{wx}{l} \times dx \times \frac{x(l-x)^2}{l^3}\]

\[= - \frac{w}{l} \left[ \frac{\int_0^1 (l^2 - 2lx + x^2) \, dx}{3} - \frac{\int_0^1 l^3}{4} + \frac{\int_0^1 x^5}{5} \right]_0^1\]

\[= - \frac{w}{l} \left[ \frac{l^3}{3} - \frac{l^3}{2} + \frac{l^3}{5} \right] - \frac{w l^2}{30}\]

\[M_B = \left( \frac{wx}{l} \right) \times dx \times \frac{x^2(l-x)^3}{l^5}\]

\[= - \frac{w}{l} \left[ \int_0^1 (lx^3 - x^4) \, dx \right]\]

\[= - \frac{w}{l} \left[ \frac{l^3}{4} - \frac{l^5}{5} \right]_0^1\]

\[= - \frac{w l^2}{20}\]

**Ex. 2.5** Find the fixed end moments and plot S.F. and B.M. diagrams for the fixed beam loaded as shown in Fig. 2.11 (a).

**Solution.** Simply supported B.M. diagram for the beam is shown in Fig. 2.11 (c). From this \(\frac{M}{I}\) diagram is plotted and is shown in Fig. 2.11 (d). Let \(M_A\) and \(M_B\) be the fixed end moments at two supports. B.M. diagram due to fixing moments is shown in Fig. 2.11 (e). \(\frac{M}{I}\) diagram due to fixing moments is shown in Fig. 2.11 (f).

\[\frac{A_s}{I} + \frac{A_i}{I} = 0\]

\[\therefore \left[ \frac{1}{2} \times \frac{4000}{I} \times 2 + \frac{1}{2} \times \frac{2000}{I} \times 4 \right] \]

\[= \left[ \frac{M_A}{I} + \frac{(2M_A + M_B)}{3I} \right] \times 2\]

\[= \frac{(2M_A + M_B)}{6I} + \frac{M_B}{3I} \times 4 = 0\]

\[\therefore \frac{8000}{I} + \frac{M_A}{I} + \frac{2M_A}{3I} + \frac{M_B}{3I} + \frac{2M_A}{3I} + M_B + \frac{M_B}{I} - 0\]
\[
\frac{1}{6} M_A + \frac{1}{6} M_B = -8000
\]

\[3000 \text{ kg.}\]

(a)

(b)

(c)

SIMPLY SUPPORTED B.M DIAGRAM.

(d)

4000

21

M I DIAGRAM

(e)

M_A

FAIRED END MOMENT DIAGRAM.

(f)

M_B

\[ \frac{M_A}{1} \quad \frac{M_B}{21} \]

\[\frac{2M_A + M_B}{61}\]

M I DIAGRAM

(g)

\[2100\text{ kg}\]

\[+ + +\]

SF DIAGRAM

(h)

22703

\[\text{ kgm}\]

\[+3000\]

\[B M \text{ DIAGRAM}\]

\[14M_A + M_B = -4800\]

\[\frac{A_B}{f} + \frac{A_E}{f} = 0\]
\[ \frac{1}{2} \times \frac{4000}{l} \times 2 \times \frac{2}{3} \times 2 + \frac{1}{2} \times \frac{2000}{l} \times 4 \times \left( 2 + \frac{1}{3} \times 4 \right) + \frac{M_A}{l} \times 2 \times 1 + \frac{1}{2} \times \frac{(M_B - M_A)}{3l} \times 2 \times \frac{2}{3} \times 2 + \frac{(2M_A + M_B)}{6l} \times 4 \times 4 + \left[ \frac{M_B}{2l} - \frac{2M_A + M_B}{6l} \right] \times \frac{1}{8} \times 4 \times (2 + \frac{1}{2} \times 4) \]

\[ \frac{16,000}{3} + 4000 \times \frac{10}{3} + 2M_A + \frac{4}{9} \left( M_B - M_A \right) \]

\[ + \frac{8}{3} \left( 2M_A + M_B \right) + \frac{28}{3} \left[ \frac{M_B}{2} - \frac{2M_A + M_B}{6} \right] = 0 \]

\[ 56,000 \times 3 + 18M_A - 4M_B - 4M_A + 43M_A + 24M_B + 28M_B - 28M_A = 0 \]

\[ 34M_A + 56M_B = -163,000 \]

\[ 0.6071M_A + M_B = -3000 \]

(2) — (1) gives 0.793 \[ M_A = -1800 \]

\[ M_B = -4,300 - 1\cdot4M_A \]

\[ = -4,300 + 3,175\cdot4 = -1621\cdot6 \text{ kg. m.} \]

Taking moments about \( A \),

\[ R_B \times 6 + 2270\cdot3 - 1621\cdot6 = 3000 \times 2 = 0 \]

\[ R_B = 1000 - \frac{648.7}{6} = 1000 - 108\cdot1 = 891\cdot9 \text{ kg.} \]

\[ R_A = 3000 - 891\cdot9 = 2108\cdot1 \text{ kg.} \]

S.F. and B.M. diagrams are shown in Fig. 2.11.

Ex. 2.6. Find the fixed end moments for the fixed beam with applied moment at distance \( a \) from one end.

Solution. Consider simply supported beam with applied moment \( M \). Reaction at each support will be \( M/l \). The B.M. diagram for the simply supported beam will be as shown in Fig. 2.12 (c).

\[ M_A + M_B = -\frac{2A}{l} \]

\[ \frac{2}{l} \left[ -\frac{1}{2} \times \frac{Ma}{l} \times a + \frac{1}{2} \times \frac{Mb}{l} \times b \right] \]

\[ -\frac{M}{l^2} (b^2 - a^2) \]

\[ M_A + 2M_B = \frac{-6A \cdot x}{l^3} \]

\[ \frac{6}{l^3} \left[ -\frac{1}{2} \times \frac{Ma}{l} \times a + \frac{2}{3} a + \frac{1}{2} \times \frac{Mb}{l} \times b \left( a + \frac{b}{3} \right) \right] \]

\[ 6 \left[ -\frac{Ma^3}{3l} + \frac{Mb^3}{2l} \times \frac{(3a + b)}{3} \right] \]
ANALYSIS OF STRUCTURES

\[
M = -\frac{6}{l^2} \times \frac{M}{ul} \left[ -2a^2 + 3ab^2 + b^2 \right]
\]
\[
\frac{M}{l^2} (a+b)(b^2 + 2ab - 2a^2)
\]

(a) 

(b) 

SIMPLY SUPPORTED B.M. DIAGRAM

(c) 

(d) 

\[ M_A + 2M_B = -\frac{M}{l^2} (b^3 + 2ab - 2a^2) \]

But 

\[ M_A + M_B = -\frac{M}{l^2} (b^3 - a^2) \]

\[ M_B = -\frac{M}{l^2} \left[ b^3 + 2ab - 2a^2 - b^3 + a^2 \right] \]

\[ = -\frac{M}{l^2} (2ab - a^2) \]

\[ = -\frac{Ma}{l^2} (2b - a) \]

\[ M_A = -\frac{M}{l^2} (b^3 - a^2) + \frac{Ma}{l^2} (2b - a) \]

\[ = -\frac{M}{l^2} (b^3 - a^2 - 2ab + a^2) \]

\[ = -\frac{Mb}{l^2} (b - 2a) \]

\[ = \frac{Mb}{l^2} (2a - b) \]
Second Method.

Applied moment is equal to two equal and opposite applied loads \( M/2d \) at distance \( 2d \) apart as shown in Fig. 2.12 (c).

Due to a single load applied at distance \( a \) from one end and \( b \) from other end, fixed end moments are \(-\frac{Wab^3}{l^3}\) and \(-\frac{Wa^3b}{l^3}\)

Due to loads shown in Fig. 2.12 (c).

\[
M_A = -\frac{M}{2d^2}[(a+d)(b^3-2bd+d^3)-(a-d)(b^3+2bd+d^3)]
\]

\[
= -\frac{M}{2dl^2} [a^3-2abd+ad^3+b^3d-2b^3d^3-a^3b-\]

\[ad^3+b^3d+2bd^3+\]

\[
= -\frac{M}{2d^2l} [2bd^3-4abd+d^3]
\]

\[
= -\frac{M}{l^3} [b^3-2ab+a^3].
\]

In the limit as \( d \) tends to zero,

\[
M_A = -\frac{M}{l^3} (b^3-2ab) = \frac{Mb(2a-b)}{l^3}
\]

\[
M_B = -\frac{M}{2d} \left[\frac{(a+d)(a+d)^3}{l^3} + \frac{M}{2d} \frac{(a-d)^3(b+d)}{l^3}\right]
\]

\[
= -\frac{M}{2dl^2} [(a-d)(a^3+2ad+d^3)-(a^3-2ad+d^3)(b+d)]
\]

\[
= -\frac{M}{2dl^2} [a^3b+2abd+bd^3-a^3d-2ad^3-d^3-a^3b+2abd-
\]

\[bd^3-a^3d+2ad^3-d^3]\]

\[
= -\frac{M}{l^3} [2abd-a^3-d^3]
\]

In the limit as \( d \) tends to zero

\[
M_B = -\frac{M}{l^3} a(2b-a).
\]

2.3. Effect of sinking of support.

Let the support \( B \) sink by \( \Delta \) with respect to \( A \) and take position \( B' \).
As the slopes at $A$ and $B'$ will be zero, the total area of $\frac{M}{EI}$ diagram will be zero. Thus the fixing moments at the two supports will be equal and opposite. Let $M_A$ and $M_B$ be the fixing moments at two supports.

$$M_A = -M_B \ldots (1)$$

The intercept on the vertical by the two tangents drawn at $A$ and $B$ will be $\Delta$.

$$\left( \frac{M_A \times \frac{1}{2} \times \frac{2}{3} \cdot l + M_B \times \frac{l}{2} \times \frac{l}{3}}{EI} \right) = -\Delta$$

$$\frac{l_A l_A^2}{3} + \frac{M_B l^2}{6} = EI \Delta$$

$$2M_A + M_B = -\frac{6EI \Delta}{l^2}$$

$$M_A = -\frac{6EI \Delta}{l^2}$$

$$M_B = \frac{6EI \Delta}{l^2}$$

2.4. Effect of rotation at a support.

Let rotation at support $B$ be $\theta_B$ anticlockwise. The angle between the two tangents at $A$ and $B$ will be $\theta_B$. Let $M_A$ and $M_B$ be the fixing moments at two supports.

$$\frac{A}{EI} = \theta_B$$

$$M_A \times \frac{l}{2} + \frac{M_B \times l}{2}$$

$$\frac{EI}{l}$$

$$\therefore M_A + M_B = -\frac{2EI\theta_B}{l} \ldots (1)$$

The intercept on the vertical at $B$ due to tangents at $A$ and $B$ will be zero.
\[
M_A \times \frac{l}{2} \times \frac{2}{3} l + M_B \times \frac{l}{2} \times \frac{1}{3} \frac{EI}{l}
\]

\[
2M_A + M_B = 0 \quad \text{...(2)}
\]

(2) - (1) gives

\[
M_A = -\frac{2EI\theta_B}{l}
\]

\[
M_B = +\frac{4EI\theta_B}{l}
\]

Ex. 2.7. A beam AB span 4 m. fixed at A and B carries a uniformly distributed load of 1500 kg/m. The support B sinks by 1 cm. Find the fixed end moments and draw the bending moment diagram of the beam. \( E = 2.0 \times 10^6 \) kg/cm². \( I = 8000 \) cm⁴.

Solution. Due to uniformly distributed load

\[
M_A = M_B = -\frac{wl^2}{12} = -\frac{1500 \times 4 \times 4}{12}
\]

\[
M_A = M_B = -2000 \text{ kg. m.}
\]

Due to settlement of support B

\[
M_A = -M_B = -\frac{6EI \Delta}{l}
\]

\[
= -\frac{6 \times 2 \times 10^4 \times 8000 \times 1}{(400)^2} \times \frac{l}{100} \text{ kg. m.}
\]

\[
= -6000 \text{ kg. m.}
\]

\[\text{Fig. 2.15}\]

Total moment will be

\[
M_A = -2000 - 6000 = -8000 \text{ kg. m.}
\]

\[
M_B = -2000 + 6000 = +4000 \text{ kg. m.}
\]
2:5. Slope and Deflection at a point, by moment area method.

Let \( D \) be the point, at distance \( x \) from support \( B \), where slope and deflection is required.

**Slope \( \theta_D \)**

The tangents drawn at support \( B \) and the point \( D \) on the deflected form intersect at angle \( \theta_D \) which is also slope at \( D \), as slope at \( B \) is zero. The angle measured from tangent at left point \( D \) to the tangent at \( B \) is measured clockwise and, therefore, is caused by \(-ve\) B.M.

\[ \therefore \theta_D = -\left( \frac{A_z}{EI} \right) \]

where \( A_z \) is total of bending moment area between \( B \) and \( D \).

**Deflection \( y_D \)**

As the tangent at support \( B \) is horizontal, the intercept on the vertical at \( D \) by the two tangents drawn at \( B \) and \( D \) will be the deflection at \( D \). As the intercept is below the tangent drawn at \( B \), the total moment of the B.M. diagram between \( B \) and \( D \) about vertical at \( D \) will be \(-ve\).

\[ y_D = \frac{A_z z_a}{EI} \]

where \( z_a \) is the C.G. of total B.M. diagram area between \( B \) and \( D \).

**Ex. 2:8.** Find slope and deflection at quarter span for centrally loaded beam shown in Fig. 2:17. Find also the deflection at the centre \( C \).

**By Moment Area Method**

**Solution.** The simply supported and fixed end moment diagrams for the beam are shown in Fig. 2:17,

\[ \theta_D = -\frac{A_z}{EI} \]

\[ = -\frac{1}{EI} \left\{ \frac{WL}{8} \times \frac{l}{4} - \frac{WL}{8} \times \frac{l}{4} \right\} \]

\[ = -\frac{1}{EI} \left[ \frac{-WL}{64} \right] \]

\[ = \frac{WL}{64EI} \]

\[ y_D = -\frac{A_z z_a}{EI} = -\frac{1}{EI} \left[ \frac{WL}{8} \times \frac{l}{4} \times \frac{1}{3} \times \frac{l}{4} - \frac{WL}{8} \times \frac{l}{4} \times \frac{l}{8} \right] \]
Central deflection

\[ y = \frac{1}{EI} \left[ \frac{Wl^3}{4} \times \frac{l}{4} \times \frac{1}{3} \times \frac{l}{2} - \frac{Wl^3}{8} \times \frac{l}{2} \times \frac{1}{4} \right] \]

\[ = -\frac{1}{EI} \left[ \frac{Wl^3}{96} - \frac{Wl^3}{64} \right] \]

\[ = \frac{Wl^3}{192EI}. \]

II. By Macaulay's Method

B.M. at distance \( x \) from \( A \) in \( CB \) is given by

\[ M_x = -\frac{Wl}{8} + \frac{W}{2} x - W \left( x - \frac{l}{2} \right) \]

\[ \therefore EI \frac{d^2y}{dx^2} = -M_x = \frac{Wl}{8} - \frac{W}{2} x + W \left( x - \frac{l}{2} \right) \]

\[ EI \frac{dy}{dx} = \frac{Wl}{8} x - \frac{W}{2} x^2 + \frac{W(x-l/2)^2}{2} + C_1 \]

At \( x = 0 \), \( \frac{dy}{dx} = 0 \quad C_1 = 0. \]

\[ EI y = \frac{Wl}{8} \times \frac{x^3}{2} - \frac{W}{4} \times \frac{x^3}{3} + \frac{W}{2} \frac{(x-l/2)^3}{3} + C_2. \]
At 
\[ EI \frac{dy}{dx} \frac{Wl}{8} x - \frac{Wx^3}{4} + \frac{W(x - l/2)^3}{6} \]

\[ Ely = \frac{Wl}{16} x^2 - \frac{Wx^3}{12} + \frac{W(x - l/2)^3}{6} \]

At 
\[ x = l/4, \]
\[ dy \frac{dx}{dx} = \frac{1}{EI} \left[ \frac{Wl}{8} \times \frac{l}{4} - \frac{W}{4} \times \left( \frac{l}{4} \right)^3 \right] \]
\[ = \frac{1}{EI} \left[ \frac{Wl^2}{32} - \frac{Wl^3}{64} \right] \]
\[ = \frac{Wl^2}{64EI} \]

\[ y_{l/4} = \frac{1}{EI} \left[ \frac{Wl}{16} \times \left( \frac{l}{4} \right)^2 - \frac{W}{12} \times \left( \frac{l}{4} \right)^3 \right] \]
\[ = \frac{1}{EI} \left[ \frac{Wl^2}{16 \times 16} - \frac{Wl^3}{64 \times 12} \right] \]
\[ = \frac{Wl^2}{364EI} \]

At 
\[ x = l/2 \]
\[ y_{l/2} = \frac{1}{EI} \left[ \frac{Wl}{16} \times \left( \frac{l}{2} \right)^2 - \frac{W}{12} \times \left( \frac{l}{2} \right)^3 \right] \]
\[ = \frac{1}{EI} \left[ \frac{Wl^2}{64} - \frac{Wl^3}{96} \right] \]
\[ = \frac{Wl^2}{1024EI} \]

Ex. 2.9. Find the central deflection for fixed beam of problem 2.5 given that \( EI = 5 \times 10^6 \) kg. cm².

Solution. \( \frac{M}{I} \) diagrams as simply supported beam and due to fixed end moments are shown in Fig. 2.18.

\[ y_c = -\frac{A\bar{x}}{E} \]

where \( A \) is the area of \( \frac{M}{I} \) diagram up to \( C \) and \( \bar{x} \) is the distance of C.G. of \( \frac{M}{I} \) diagram from the centre C.
\[ y_e = -\left( \frac{3}{4} \times 2000 \times \frac{3}{2} \times \frac{1}{3} \times 3 - \frac{810.8}{l} \times 3 \times 1.5 - \frac{3}{4} \times \frac{216.25}{l} \times \frac{3}{2} \times \frac{1}{3} \times 3 \right) \]

\[ = -\left( \frac{2250}{l} - \frac{3648.6}{l} - \frac{243.28}{l} \right) \]

\[ = \frac{1631.88}{l} \]

\[ = \frac{1641.88}{5 \times 10^3} \times (100)^3 \text{ cm.} \]

\[ = 0.32838 \text{ cm.} \]

**Ex. 2.10.** Find by Macaulay's method the central deflection of a fixed beam loaded with uniformly distributed load throughout the span.

**Solution.** B.M. at distance \( x \) from support is given by

\[ M_e = -\frac{wl^3}{12} + \frac{wl}{2} \times \frac{wx^2}{2} \]

\[ \frac{Eld^2Y}{dx^2} = -M_e \]

\[ \frac{Eld^2y}{dx^2} = \frac{wl^3}{12} - \frac{wl}{2} \times \frac{x^3}{6} + \frac{wx^2}{2} \]

\[ \frac{Eldy}{dx} = \frac{wl^3}{12} - \frac{wl}{2} \times \frac{x^2}{2} + \frac{wx^3}{6} + C_1 \]

At \( x = 0 \), \( \frac{dy}{dx} = 0 \). \( C_1 = 0 \)

\[ Ely = \frac{wl^3}{24} \times x^3 - \frac{wl^3}{12} + \frac{wx^4}{24} + C_2 \]

At \( x = 0 \), \( y = 0 \). \( C_2 = 0 \).

\[ Ely = \frac{wl^3}{24} - \frac{wl^3}{12} + \frac{wx^4}{24} \]

At \( x = l/2 \),

\[ Ely = \left[ \frac{wl^3}{24} \times \left( \frac{l}{2} \right)^3 - \frac{wl}{12} \times \left( \frac{l}{2} \right)^3 + \frac{wx^4}{24} \times \left( \frac{l}{2} \right)^4 \right] \]

\[ \frac{1}{EI} \left[ \frac{wl^6}{96} - \frac{wl^6}{96} + \frac{wl^6}{24 \times 16} \right] \]

\[ y_e = \frac{wl^6}{384EI} \]

**Ex. 2.11.** A beam of span 1 meter carries a uniformly distributed load of \( w \) kg/m. The ends are so constrained that the end slope is 0, the restraining couple at the supports is 0. Prove...
that the magnitude of restraining couple at each end is \( \frac{kw^3}{12(kl + 2EI)} \)
and that the magnitude of central deflection is \( \frac{wl^4(kl + 10EI)}{384 EI(kl + 2EI)} \).

Find the value of \( k \) which gives least value of numerical B.M.

\[ w/UNIT\ LENGTH \]

\[ k \theta \]

\[ \frac{wl}{2} \]

\[ x - \frac{wx^3}{2} \]

\[ M_x = -k \theta + \frac{wl}{2} x - \frac{wx^3}{2}. \]

**Solution.** Fixing moment at each end is \( k \theta \). B.M. at section \( x \) distance from support is

\[ EI \frac{dy}{dx} = -M_x \]

\[ k \theta \]

\[ \frac{wl}{2} x^{\frac{3}{2}} - \frac{wx^3}{2} + C_1 \]

At \( x = 0 \), \( \frac{dy}{dx} = C_1 = EI \theta \)

\[ EI \frac{dy}{dx} = k \theta \times x - \frac{wl}{4} x^3 + \frac{wx^3}{6} + EI \theta \]

\[ EI \frac{dy}{dx} = \frac{k \theta \times x^3}{2} - \frac{wl}{4} x^3 + \frac{wx^4}{24} + EI \theta \times x + C_a \]

At \( x = 0 \), \( y = 0 \)

\[ C_a = 0 \]

\[ EI y = \frac{k \theta \times x^3}{2} - \frac{wl}{12} x^3 + \frac{wx^4}{24} + EI \theta \times x \]

At \( x = l \), \( y = 0 \)

\[ 0 = \frac{k \theta \times l^3}{2} - \frac{wl^3}{12} + \frac{wl^4}{24} + EI \theta \times l \]

\[ \frac{wl^4}{12} - \frac{wl^4}{24} = \theta \left( \frac{kl + 2EI}{2} \right) \]

\[ \theta = \frac{wl^3}{12(kl + 2EI)} \]

**Fixing couple** = \( k \theta = \frac{kw^3}{12(kl + 2EI)} \)

At \( x = l/2 \)

\[ EI \frac{dy}{dx} = \frac{k \theta \times (l/8)}{12} - \frac{wl}{8} x^3 + \frac{wl}{24} \times \frac{l^3}{16} + EI \theta \times \frac{l}{8} \]
\[ EIy_e = \frac{6l}{8} (4EI + kl) - \frac{wl^4}{96} + \frac{wl^4}{384}. \]

Substituting the value of \( \theta \)

\[ EIy_e = \frac{wl^4}{12(kl + 2EI)} \times \frac{l}{8} (4EI + kl) - \frac{3wl^4}{384} \]

\[ = \frac{wl^4}{96} (kl + 4EI) - \frac{3wl^4}{384} \]

\[ = \frac{wl^4}{384} \frac{4(kl + 4EI) - 3(kl + 2EI)}{(kl + 2EI)} \]

\[ = \frac{wl^4}{384} \frac{kl + 10EI}{kl + 2EI} \]

\[ y_e = \frac{wl^4}{384EI} \frac{(kl + 10EI)}{(kl + 2EI)}. \]

For least value of B.M., support moment should be equal to B.M. at centre.

\[ \text{Support moment} = \frac{kwl^4}{12(kl + 2EI)} \]

\[ \text{B.M. at centre} = \frac{wl^4}{8} - \frac{kwl^4}{12(kl + 2EI)} \]

\[ \frac{wl^4}{8} - \frac{kwl^4}{12(kl + 2EI)} = \frac{wl^4}{2kl + 2EI} - \frac{kwl^4}{12(kl + 2EI)} = \frac{wl^4}{4kl} = \frac{3(kl + 2EI)}{6EI} \]

\[ kl = \frac{6EI}{l}. \]

**Ex. 2:12.** In the fixed beam of Problem 2:5, if the support \( A \) rotates by 0.002 radian clockwise, find the fixed end moments. \( EI = 1 \times 10^{10} \text{ kg cm}^2. \)

**Solution.** Let \( M_A \) and \( M_B \) be the moments at supports \( A \) and \( B \) respectively. As the support \( A \) rotates in clockwise direction, the moments \( M_A \) and \( M_B \) will act in clockwise direction. \( \frac{M}{EI} \) diagrams for \( M_A \) and \( M_B \) are shown in Fig. 2:21.

As the slope at \( A \) is \( \theta_A \) and that at \( B \) is zero, the angle between tangents at \( A \) and \( B \) will be \( \theta_A \). Therefore, the area of \( M/EI \) diagram between \( A \) and \( B \) will be \( \theta_A \).

The deflection of point \( A \) with respect to tangent at \( B \) is zero, therefore the moment of \( \frac{M}{EI} \) diagram about \( A \) will be zero.
Area of $\frac{M}{EI}$ diagram

\[
\frac{M_A + 2M_A}{3EI} \times 2 + \frac{M_A}{3EI} \times 4 \times \frac{2}{2}
\]

\[
\frac{M_B}{2EI} + \frac{M_B}{6EI} \times 4 - \frac{M_B}{3EI} \times 2 \times \frac{2}{2}
\]

Fig. 2.21

\[
\frac{5}{3} \frac{M_A}{EI} + 2 \frac{M_A}{3EI} - \frac{4}{3} \frac{M_B}{EI} - \frac{M_B}{3EI}
\]

\[
= \frac{7M_A}{3EI} - \frac{5M_B}{3EI} \quad \theta_A = 0.002
\]

\[7M_A - 5M_B = 0.006EI\]  \hspace{1cm} (1)

Moment of $\frac{M}{EI}$ diagram about $A = 0$

\[
\frac{M_A}{EI} \times \frac{2}{2} \times \frac{2}{2} + \frac{2}{3} \frac{M_A}{EI} \times 2 \times \frac{2}{3}
\]

\[
\frac{M_A}{3EI} \times \left(2 \times \frac{4}{3}\right)
\]

\[
\frac{M_B}{3EI} \times \frac{2}{2} \times \frac{4}{3} - \frac{M_B}{6EI} \times \frac{4}{2} \times \left(2 + \frac{4}{3}\right)
\]

\[
- \frac{M_B}{2EI} \times 2 \times \left(2 + \frac{8}{3}\right) = 1
\]
\[
\frac{2M_A}{EI} + \frac{8M_A}{9EI} + \frac{20M_A}{9EI} - \frac{4M_B}{9EI} - \frac{10M_B}{9EI} - \frac{14M_B}{3EI} = 0.
\]

\[
\therefore \quad \frac{34M_A}{9EI} - \frac{56M_B}{9EI} = 0.
\]

\[
\therefore \quad M_A = \frac{28}{17} M_B
\]

Substituting in (1)

\[
7 \times \frac{28}{17} M_B - 5M_B = 0.006EI
\]

\[
\therefore \quad M_B = \frac{14}{111} \times 0.006EI
\]

\[
= \frac{17}{111} \times 0.006 \times 1 \times 10^{10} \times 10^4 \text{ kg. m.}
\]

\[
= 918.9 \text{ kg. m.}
\]

\[
\therefore \quad M_A = \frac{28}{17} \times 918.9
\]

\[
= 1513.5 \text{ kg. m.}
\]

**Ex. 2.13.** Find the fixing moments at \( A \) and \( B \) and draw the B.M. diagram for the beam shown in Fig. 2.22. There is a moment of 6 t.m. applied at \( C \).

(A.M.I.E. May 1967)

**Sol.** Let \( M_A \) and \( M_B \) be the fixed end moments at \( A \) and \( B \) respectively.

(i) Total area of \( \frac{M}{EI} \) diagram = 0

\[
\therefore \quad - \frac{1}{2} \times \frac{2l}{3} \times \frac{1}{2EI} + \frac{1}{2} \times \frac{l}{3} \times \frac{2}{EI} + \frac{M_A}{3EI} + \frac{M_A}{24EI} \times \frac{2l}{3} + \frac{M_A}{3EI} \times \frac{l}{2} \times \frac{2}{3} + \frac{M_B}{12EI} \times \frac{2l}{3} + \frac{M_B}{6EI} \times \frac{l}{2} \times \frac{2}{3} + \frac{2M_B}{3EI} + \frac{M_B}{EI} \times \frac{l}{3} = 0
\]

\[
\therefore \quad - \frac{l}{6EI} + \frac{l}{3EI} + \frac{M_A l}{18EI} + \frac{M_A l}{18EI} + \frac{M_B l}{36EI} + \frac{5M_B l}{18EI} = 0
\]

\[
\therefore \quad \frac{l}{6EI} + M_A l + \frac{11M_B l}{36EI} = 0.
\]

\[
\therefore \quad 6 + 4M_A + 11M_B = 0.
\]

\[
\therefore \quad M_A = -1.5 - \frac{11}{4} M_B.
\]
ANALYSIS OF STRUCTURES

SIMPLY SUPPORTED BEAM

(c) SIMPLY SUPPORTED B.M. DIAGRAM

(e) B.M. DIAGRAM

(f) M/EI DIAGRAM

(g) B.M. DIAGRAM

(h) M/EI DIAGRAM

Fig. 2.22
(ii) Total moment of \(\frac{M}{EI}\) diagram about support \(B = 0\).

\[
\begin{align*}
\therefore - \frac{l}{6EI} \left( \frac{2l}{3} \times \frac{1}{3} + \frac{l}{3} \right) &+ \frac{M_A}{3EI} \times \frac{2l}{3} \times \frac{2}{3} \\
&+ \frac{24EI}{2l} \times \frac{2l}{2} \times \frac{2l}{3} \times \left( \frac{1}{3} + \frac{2l}{3} \times \frac{3}{3} \right) + \frac{M_A}{3EI} \times \frac{l}{2} \times \frac{2l}{3} \\
&+ \frac{12EI}{M_B} \times \frac{2l}{2} \times \frac{l}{3} \times \left( \frac{1}{3} + \frac{1}{3} \times \frac{2l}{3} \right) \\
&\quad + \frac{2M_B}{3EI} \times \frac{2l}{2} \times \left( \frac{2l}{3} \times \frac{l}{3} \right) + \frac{M_B}{3EI} \times \frac{l}{2} \times \frac{l}{3} \times \frac{l}{3} = 0
\end{align*}
\]

\[
\begin{align*}
\therefore - \frac{5l^2}{6 \times 9EI} + \frac{2l^2}{27EI} + \frac{7M_A l^2}{24 \times 9EI} + \frac{5M_A l^2}{72 \times 9EI} + \frac{M_A l^2}{9 \times 9EI} \\
&+ \frac{5M_B l^2}{36 \times 9EI} + \frac{2M_B l^2}{9 \times 9EI} + \frac{M_B}{6 \times 9EI} = 0
\end{align*}
\]

\[
\begin{align*}
\therefore -60 + 48 + 21M_A + 5M_A + 8M_A + 10M_B + 16M_B + 12M_B = 0
\end{align*}
\]

\[
\begin{align*}
\therefore -12 + 34M_A + 38M_B = 0
\end{align*}
\]

\[
\begin{align*}
\therefore M_A = \frac{6}{17} - \frac{19}{17} M_B
\end{align*}
\]

\[
\begin{align*}
\therefore -1.5 - \frac{11}{4} M_B = \frac{6}{17} - \frac{19}{17} M_B
\end{align*}
\]

\[
\begin{align*}
\therefore -1.5 \times 17 - \frac{11 \times 17}{4} M_B = 6 - 19M_B
\end{align*}
\]

\[
\begin{align*}
\therefore -31.5 = 27.75 M_B
\end{align*}
\]

\[
\begin{align*}
M_B = -\frac{31.5}{27.75} = -1.135 \text{ t. m.}
\end{align*}
\]

\[
\begin{align*}
M_A = -1.5 - \frac{11}{4} \times (-1.135)
\end{align*}
\]

\[
\begin{align*}
= -1.5 + 3.121 = +1.621 \text{ t. m.}
\end{align*}
\]

\[4.216\]

\[1.621\]

\[1.784\]

Final B.M. diagram.

Fig. 223
Problems

1. Find the fixed end moments for the beam shown in Fig. 2.24.

Fig. 2.24

[Ans. \(M_A = -5000\) kg.m.; \(M_B = -3000\) kg.m.]

2. Find the fixed end moments and plot the B.M. and S.F. diagrams for the beam shown in Fig. 2.25.

Fig. 2.25

[Ans. \(M_A = 9,600\) kg.m.; \(M_B = -22,600\) kg.m. \(R_A = 8,020\) kg.; \(R_B = -13,020\) kg.]

3. Find the fixed end moments and plot the S.F. and B.M. diagrams for the beam shown in Fig. 2.26.

Fig. 2.26

[Ans. \(M_A = -2,200\) kg.m.; \(M_B = -1000\) kg.m. \(R_A = 1,950\) kg.; \(R_B = 450\) kg.]

4. Find the fixed end moments for the beam shown in Fig. 2.27.

Fig. 2.27

[Ans. \(M_A = -21,375\) kg.m.; \(M_B = -19,125\) kg.m.]
5. A fixed beam carries load by a rigid bracket as shown in Fig. 2.28. Find the fixed end moments and plot the B.M. diagram.

Fig. 2.28

\[ \text{Ans. } M_A = -11,250 \text{ kg. m. ; } \]
\[ M_B = -15,300 \text{ kg. m. } \]

6. Find the fixed end moments for the beam having varying moments of inertia and loaded as shown in Fig. 2.29.

Fig. 2.29

\[ \text{Ans. } M_A = -9,610 \text{ kg. m. ; } \]
\[ M_B = -15,300 \text{ kg. m. } \]

7. Find the fixed end moments and central deflection for the beam shown in Fig. 2.30.

Fig. 2.30

\[ \text{Ans. } M_A = M_B = -\frac{1}{48} Wl. \]
\[ y_c = \frac{11 Wl^3}{3072EI} \]
8. Find the fixed end moments for the beam loaded as shown in Fig. 2.31.

\[ M_A = M_B = -12,500 \text{ kg m.} \]

Fig. 2.31

9. A beam of span \( l \) carries a central load \( W \). It is so constrained at the ends that when the end slope is \( \theta \) the restraining couple at the supports is \( kl \). Prove that the magnitude of restraining couple at each end is \( \frac{kw^2}{8(kl + 2EI)} \) and that the magnitude of the central deflection is \( \frac{WL^3}{192EI} \left( \frac{kl + 8EI}{kl + 2EI} \right) \).

Find the value of \( k \) which gives least value of numerical B.M. [Ans. \( k = 0 \).]

10. For the fixed beam in Fig. 2.29, determine the fixed end moments if the support \( A \) rotates 0.05 radian clockwise.

\[ EI = 6 \times 10^6 \text{ kg cm}^2. \]

11. The left support of the fixed beam shown in Fig. 2.31 rotates clockwise by 0.02 radian and settles by 1 cm. Find the fixed end moments and plot the B.M. diagram.

\[ EI = 1 \times 10^{10} \text{ kg cm}^2. \]

12. A timber beam 12 cm. wide, 20 cm. deep and 4 m. long is loaded with uniformly distributed load. The beam is fixed at both ends and during loading right support settles by an amount equal to \( wL^4/24EI \) where \( w \) is load per metre run of the beam, \( L \) is span of the beam and \( EI \) is flexural rigidity of the beam. Determine the permissible load on the beam if maximum allowable fibre stress is 100 kg/cm². [Ans. \( w = 150 \text{ kg/m.} \)]

13. A beam built in at both ends has uniform flexural rigidity \( EI \) throughout its length. \( P \). It carries a single point load \( P \) which is placed at a distance ‘a’ from the left hand. Calculate from the first principles, the fixed end moments (F.E.M.s) developed at the two ends.

Hence calculate the F.E.M.s. developed in it when carrying a distributed load whose intensity varies linearly from zero at the left end to \( \frac{2W}{l} \) at the right end, \( W \) being the total load carried. (A.M.I.E. May 1971)

14. A beam \( AB \) of span \( L \) has a uniform section throughout and is fixed at ends \( A \) and \( B \). It is subjected to a clockwise couple \( M \) at distance \( KL \) from \( A \). Draw the bending moment and shear force diagrams for the beam. (A.M.I.E. May 1970)
CONTINUOUS BEAMS

3.1. A beam which is supported on more than two supports is called a continuous beam. Such beams, when loaded, deflect in the form of a curve such that at the intermediate supports the slope of the elastic curve for the two spans will be the same. At the intermediate supports there will be bending moment. If the end supports are simply supported the bending moment there will be zero. When the end is fixed there will be fixed end moment and the slope at fixed end will be zero. The moment of inertia of beam in different spans may be same or may be different.

3.2. Analysis of Continuous Beams.

(i) Beams with varying moment of inertia for different spans.

Consider two consecutive spans $AB$ and $BC$ of lengths $l_1$ and $l_2$ of a continuous beam. Let $I_1$ and $I_2$ be the moments of inertia of spans $AB$ and $BC$ respectively.

Let $M_A$, $M_B$ and $M_C$ be the support moments at supports $A$, $B$ and $C$ respectively. Let $A_1$ and $A_2$ be the areas of B.M. diagrams for given loading, considering $AB$ and $BC$ as simply supported. Let $x_1$ and $x_2$ be the distances of centres of gravity of areas $A_1$ and $A_2$ from $A$ and $C$ respectively.
Draw $DBE$ tangent to the elastic curve at middle support $B$ cutting the verticals at $A$ and $C$ at $D$ and $E$ respectively.

Let $AD = Z_1$ and $CE = Z_2$. As shown in Fig. 3-1, $Z_1$ is above the elastic curve, and, therefore, will be $-ve$.

$$Z_1 = \left( A_1 \bar{x}_1 + M_A \times \frac{l_1}{2} + M_B \times \frac{l_1}{2} \times \frac{2l_1}{3} \right)$$

$$Z_2 = \left( A_2 \bar{x}_2 + M_C \times \frac{l_2}{2} + M_B \times \frac{l_2}{2} \times \frac{2l_2}{3} \right)$$

$$\theta_{11} = \frac{Z_1}{l_1} = \frac{Z_2}{l_2}$$

$$A_1 \bar{x}_1 + \frac{M_A l_1^2}{6} + \frac{2M_B l_1^2}{6}$$

$$E I_{11}$$

$$\therefore \quad A_2 \bar{x}_2 + M_C \times \frac{l_2^2}{6} + M_B \times \frac{2l_2^2}{6}$$

$$E I_{22}$$

$$\therefore \quad \frac{M_A l_1}{6l_1} + \frac{2M_B}{6} \left( \frac{l_1}{I_1} + \frac{l_2}{I_2} \right) + \frac{M_C l_2}{6l_2} = - \frac{A_1 \bar{x}_1}{I_{11}} - \frac{A_2 \bar{x}_2}{I_{22}}$$

$$\therefore \quad M_A \left( \frac{l_1}{I_1} \right) + 2M_B \left( \frac{l_1}{I_1} + \frac{l_2}{I_2} \right) + M_C \left( \frac{l_2}{I_2} \right)$$

$$= -6 \left[ \frac{A_1 \bar{x}_1}{I_{11}} + \frac{A_2 \bar{x}_2}{I_{22}} \right] \quad \cdots(3.1)$$

This is known as Clapeyron's theorem of three moments.

(i) **Beams with same moment of inertia for different spans.**

In this case $I_1 = I_2$ and Clapeyron's theorem will be

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = -6 \left[ \frac{A_1 \bar{x}_1}{l_1} + \frac{A_2 \bar{x}_2}{l_2} \right] \quad \cdots(3.2)$$

(iii) **Continuous beam with end support fixed.**

Consider span $AB$ with end $A$ as fixed and $B$ continuous.

Let $A$ be the area of B.M. diagram considering $AB$ as simply supported. Let $\bar{x}$ be the distance of centre of gravity of the B.M. area from $B$.

Let $M_A$ and $M_B$ be the moments at supports $A$ and $B$ respectively.

Fig. 3-2
The tangent drawn at A to the elastic curve will pass through B, therefore, the moment of the area of B.M. diagram about B will be zero.

\[ \therefore \quad Ax + \frac{M_A}{2} \times \frac{l}{2} \times \frac{2l}{3} + \frac{M_B}{2} \times \frac{l}{2} \times \frac{l}{3} = 0 \]

\[ 2M_A \frac{l^3}{6} + \frac{M_B}{6} l^3 = -Ax \]

\[ 2M_A + M_B = \frac{6Ax}{l^2} \]

This result can be obtained by applying Clapeyron's theorem by taking spans of zero length on the fixed end side.

\[ 2M_A (0 + l) + M_B \times l = \frac{6Ax}{l} \]

\[ 2M_A + M_B = \frac{6Ax}{l^2}. \]

(iv) Continuous beam with overhang on one side.

Applying theorem of three moments to span AR and BC.

\[ M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 \]

\[ = -6 \left( \frac{A_1 x_1}{l_1} + \frac{A_2 x_2}{l_2} \right). \]

Here \( M_C \) = Bending moment as for cantilever CD.

3.3. Reactions at the Supports.

Reaction at support B

\[ R_B = F_{BA} + F_{BC} \]

**Fig. 3.5**
Taking moments about $A$.

\[ F_{BA} \times l_1 + M_A - M_B + W_1 x_1 = 0 \]

\[ \therefore \quad F_{BA} = \frac{W_1 x_1}{l_1} + \frac{(M_B - M_A)}{l_1} \]

Taking moments about $C$,

\[ F_{BC} \times l_2 + M_C - M_B - W_2 x_2 = 0 \]

\[ \therefore \quad F_{BC} = \frac{W_2 x_2}{l_2} + \frac{(M_B - M_C)}{l_2} \]

\[ \therefore \quad R_R = \left( \frac{W_1 x_1}{l_1} + \frac{W_2 x_2}{l_2} \right) + \left( \frac{M_B}{l_1} + \frac{M_A}{l_2} \right) + \left( \frac{M_B - M_C}{l_2} \right) \]

\[ F_{AB} = W_1 - F_{BA} \]

\[ F_{CH} = W_2 - F_{BC} \]

**Ex. 3.1.** A continuous beam $ABC$ is simply supported at $A$ and $C$. Spans $AB$ and $BC$ are of lengths $l_1$ and $l_2$ respectively. The beam is loaded with $W_1$ and $W_2$ acting at the centre of $AB$ and $BC$ respectively. Find the support moment $M_B$.

**Solution.** Applying theorem of three moments to spans $AB$ and $BC$.

\[ M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = -6 \left[ \frac{A_1 x_1}{l_1} + \frac{A_2 x_2}{l_2} \right] \]

As the ends $A$ and $C$ are simply supported $M_A = 0$ and $M_C = 0$.

**Fig. 3.6.**

Simply supported B.M. diagram for spans $AB$ and $BC$ is shown in Fig. 3.6.

\[ = \frac{W_1 l_1}{2} \times \frac{l_1}{2} = \frac{W_1 l_1^2}{8} \left( x_1 = \frac{l_1}{2} \right) \]

\[ A_2 = \frac{W_2 l_2}{4} \times \frac{l_2}{2} = \frac{W_2 l_2^2}{8} \left( x_2 = \frac{l_2}{2} \right) \]

\[ \therefore \quad 2M_B (l_1 + l_2) = -6 \left[ \frac{W_1 l_1^2}{8} \times \frac{l_1}{2} + \frac{W_2 l_2^2}{8} \times \frac{l_2}{2} \right] \]

\[ \therefore \quad M_B = -\frac{3}{8} \left( \frac{W_1 l_1^2 + W_2 l_2^2}{l_1 + l_2} \right) \]
CONTINUOUS BEAMS

(ii) \( l_1 = l_2 = l, \ W_1 = W_2 = W \)

\[
M_2 = -\frac{16}{16} \times \frac{2WL^3}{2l} = -\frac{3}{16} WL.
\]

Reactions.
Consider beam \( AB \).
Taking moments about \( B \),

\[
R_A l + \frac{3}{16} Wl - W \frac{l}{2} = 0
\]

\[
R_A = \frac{W}{2} - \frac{3}{16} W = \frac{5}{16} W.
\]

From symmetry \( R_C = \frac{5}{16} W \).

\[
R_B = 2W - 2 \times \frac{5W}{16} = \frac{11}{8} W.
\]

S.F. diagram is shown in Fig. 3.8.

The B.M. diagram is drawn by superimposing the B.M. diagram considering spans as simply supported over the \( - \)vo B.M. diagram due to support moments as shown in Fig. 3.8.

Let the point where the B.M. is zero, i.e. point of contraflexure be at a distance \( x_0 \) from \( A \).

\[
R_A \times x_0 - W\left( x_0 - \frac{l}{2} \right) = 0
\]
\[
\frac{5}{16} W x_0 - \frac{W l}{2} = 0
\]
\[
\frac{11}{16} x_0 = \frac{l}{2} = 8 \frac{W}{11} l.
\]

**Maximum Positive B.M.**

Maximum +ve B.M. will occur where S.F. is zero, i.e. under the load.

Maximum +ve B.M. = \( R_A \times \frac{l}{2} \)

\[
= \frac{5}{16} W \times \frac{l}{2} = \frac{5}{32} W l.
\]

**Ex. 32.** A continuous beam ABCD with three equal spans AB, BC and CD is loaded with uniformly distributed load \( w \) per unit length throughout the span. Find the support moments and draw S.F. and B.M. diagrams.

**Solution.** Simply supported B.M. diagrams for spans AB, BC and CD are shown in Fig. 3.9 (b).

![Diagram](image_url)
From symmetry $M_B = M_C$

Ends $A$ and $D$ are simply supported, 

$$M. \quad M_D = 0.$$

Applying theorem of three moments to spans $AB$ and $BC$

$$M_A \times l + 2M_B \times (l+l) + MC \times l$$

$$= 6 \left( \frac{A_1 x_1}{l_1} + \frac{A_2 x_2}{l_2} \right)$$

$$A_1 = A_2 = \frac{2}{3} \times \frac{wl^2}{8} \times l = \frac{wl^3}{12}$$

$$A = 0 + 2M_B \times 2l + M_B \times l$$

$$= -6 \left( \frac{wl^3}{12} \times \frac{l}{2} + \frac{wl^3}{12} \times \frac{l}{2} \right)$$

$$\therefore 5M_B l = -6 \left( \frac{wl^3}{12} \right)$$

$$\therefore M_B = -\frac{wl^2}{10}$$

Reactions

Cut the beam at $B$, taking moments about $B$

$$R_A \times l + \frac{wl^2}{10} - wl \times \frac{l}{2} = 0$$

$$R_A = \frac{wl}{2} - \frac{wl}{10} = -\frac{wl}{5}$$

$$R_D = R_A = \frac{2}{5}wl$$

$$R_B = R_C : 3wl - 2 \times \frac{3}{5}wl = \frac{11}{10}wl$$

B.M. and S.F. diagrams are shown in Fig. 3.10 (d) and (e).

**Maximum +ve B.M.**

**Span AB**

Maximum +ve B.M. occurs at $\frac{1}{3}l$ from $A$ where shear force is zero.

$$\text{Max. +ve B.M.} = R_A \times \frac{2}{3}l - w \times \frac{3}{8}l \times \frac{2}{3}l \times \frac{1}{6}$$

$$= \frac{3}{8}wl \times \frac{3}{8}l - w \times \frac{3}{8}l \times \frac{2}{3}l \times \frac{1}{6}$$

$$= \frac{3}{8}wl^2$$

**Span BC**

Maximum +ve B.M. = $\frac{wl^2}{8} - \frac{wl^2}{10} = \frac{wl^2}{40}$
Points of contraflexure

Span AB
Let it be at \( x_0 \) from \( A \).
\[
M_x = R_A \times x_0 - \frac{wx_0^2}{2}
\]
\[
x_0 \left[ \frac{3}{8}wl - \frac{w}{2} \times x_0 \right] = 0
\]
\[
x_0 = \frac{4}{3}l
\]

Span BC
Let it be at \( x_1 \) from \( B \).
\[
M_x = \frac{wl}{2} \times x_1 - \frac{wl^2}{10} - \frac{wx_1^2}{2} = 0
\]
\[
x_1^2 - x_1l + \frac{l^2}{5} = 0
\]
\[
\left( x_1 - \frac{l}{2} \right)^2 = \frac{l^2}{4} - \frac{l^2}{5} = \frac{l^2}{20}
\]
\[
\therefore \quad x_1 = \pm \sqrt{\frac{l^2}{20} + \frac{l}{2}}
\]
\[
= \frac{l}{2} \left[ 1 \pm \sqrt{\frac{1}{5}} \right]
\]
\[
= \frac{l}{2} \left[ 1 \pm 0.4472 \right]
\]
\[
\therefore \quad x_1 = 0.7236l \text{ or } 0.2764l.
\]

**Ex. 3.3.** Continuous beam ABCDE is loaded as shown in Fig. 3.11 (a). Draw S.F. and B.M. diagrams for the beam.

**Solution.** Simply supported B.M. diagrams for spans BC and CD are shown in Fig. 3.11 (b).

\[
M_B = -2000 \times 1 = -2000 \text{ kg. m.}
\]
\[
M_D = -1000 \times 2 = -2000 \text{ kg. m.}
\]
\[
A_1 = \frac{2}{3} \times 4000 \times 4 = \frac{32,000}{3}
\]
\[
= 2 \int_0^3 \left( 6000x - \frac{4000}{3} x \times \frac{x}{2} \times \frac{x}{3} \right)
\]
\[
= 2 \int_0^3 \left( 6000x - \frac{2000}{9} x^3 \right)dx
\]
\[
= 2 \left[ 6000 \times \frac{x^2}{2} - \frac{2000}{9} \times \frac{x^4}{4} \right]_0^3
\]
\[
= 2 \left[ 3000 \times 9 - \frac{2000}{9} \times \frac{81}{4} \right]
\]
\[
= 54,000 - 9,000
\]
\[
= 45,000
\]
Applying theorem of three moments to spans $BC$ and $CD$

$M_B \times 4 + 2M_C(4+6) + M_D \times 6$

\[-6 \left\{ \frac{32000 \times 2}{3} + \frac{45000 \times 3}{6} \right\} \]

![Diagram with labeled forces and reactions](image)


SIMPLY SUPPORTED B.M. DIAGRAM

\[\begin{align*}
2000 & \text{kg} \\
2000 & \text{kg/m} \\
-4000 & \text{kg/m} \\
1000 & \text{kN} \\
R_B & = 4662.5 \text{ kg} \\
R_C & = 12229.2 \text{ kg} \\
R_D & = 6108.3 \text{ kg} \\
R_E & = 6395.5 \text{ kg} \\
\end{align*}\]

\[\begin{align*}
\text{S.F. DIAGRAM} \\
2000 & \text{ kg} \\
6891.7 & \text{ kg} \\
1000 & \text{ kN} \\
2000 & \text{ kN} \\
5337.5 & \text{ kN} \\
5108.3 & \text{ kN} \\
\end{align*}\]

\[\begin{align*}
\text{B.M DIAGRAM} \\
1000 & \text{ kNm} \\
4000 & \text{ kNm} \\
7350 & \text{ kNm} \\
12000 & \text{ kNm} \\
2000 & \text{ kNm} \\
\end{align*}\]

Fig. 3.11

\[-4 \times 2000 + 20M_C - 6 \times 2000 = -6 \left\{ \frac{16000}{3} + \frac{45000}{2} \right\} \]

\[-20,000 + 20M_C = -167,000 \]

$20M_C = -147,000$

$M_C = -7350 \text{ kg} \cdot \text{m}$

**Reactions**

Consider left portion of beam, taking moments about $C$

$R_B \times 4 + 7350 - 2000 \times 5 - 2000 \times 4 \times 2 = 0$

$4R_B = 10,000 + 16,000 - 7350$

$= 18,650$

$R_B = 4662.5 \text{ kg}$.
Consider right portion of beam, taking moments about C

\[ R_D \times 6 + 7350 - 1000 \times 8 - \frac{1}{3} \times 4000 \times 6 \times 3 = 0 \]

\[ 6R_D = 8000 + 36,000 - 7350 = 36,650 \]

\[ R_D = 6108.3 \text{ kg.} \]

\[ R_C = 2000 + 1000 + 2000 \times 4 + \frac{1}{3} \times 4000 \times 6 - 6108.3 - 4662.5 \]

\[ = 23,000 - 10,770.8 \]

\[ = 12,229.2 \text{ kg.} \]

S.F. and B.M. diagrams are shown in Fig. 3.11 (c) and (d).

**Ex. 3.4.** A continuous beam of constant moment of inertia has three supports and two overhanging ends as shown in Fig. 3.12. The overhangs are adjusted such that the reactions on the three supports due to uniformly distributed load covering the entire beam are equal. Find the value of \( x \) in terms of \( l \).

**Solution.**

\[ M_B = -\frac{wx^2}{2} \]

\[ M_L = -\frac{wx^2}{2} \]

Applying theorem of three moments to spans BC and CD.

\[ M_B \times l + 2M_C \times (l + l) + M_D \times l \]

\[ = - \left( \frac{2}{3} \times \frac{wl^2}{8} \times l \times \frac{l}{2} + \frac{2}{3} \times \frac{wl^2}{8} \times l \times \frac{l}{2} \right) \]

\[ = -\frac{wl^3}{4} \times l = -\frac{wl^3}{4} \]

\[ \therefore M_C = -\frac{wl^3}{8} \]

Total load on the beam = \( w(2x + 2l) \)

\[ = 2w(x + l) \]

All reactions are to be equal.

\[ \therefore R_B = R_C = R_D = \frac{2wx + 2l}{3} \]

Fig. 3.13

Cut the beam at C and consider the left portion of the beam

\[ R_B \times l - M_C - \frac{w(l+x)^2}{6} = 0. \]
CONTINUOUS BEAMS

\[ R_B \frac{w(l+x)^2}{2l} + \frac{M_C}{l} = \frac{w(l+x)^2}{2l} + \left( -\frac{wl^2}{8} + \frac{wx^2}{4} \right) \]

But \( R_B \) is equal to \( \frac{2w(x+l)}{a} \)

\[ \therefore \quad \frac{2}{3} w(x+l) = \frac{w(l+x)^2}{2l} + \left( -\frac{wl^2}{8} + \frac{wx^2}{4} \right) \]

\[ 8 \times \frac{2}{3} (x+l) \times l = 4(l+x)^2 - l^2 + 2x^2 \]

\[ 16l^3 + 16lx = 12(l^2 - 2lx + x^2) - 3l^2 + 6x^2 \]

\[ 16l^2 + 16lx = 12l^2 + 24lx + 12x^2 - 3l^2 + 6x^2 \]

\[ \therefore \quad 18x^2 + 8lx - 7l^2 = 0 \]

\[ x = \frac{-8l + \sqrt{64l^2 + 4 \times 18 \times 7l^2}}{2 \times 18} \]

\[ -8l + 2l\sqrt{16 + 12l^2} \]

\[ 36 \]

\[ \left( -4 + \sqrt{144} \right) \]

\[ = -4 + 11.916 \]

\[ = -7.916 \]

\[ \frac{l}{18} \]

\[ = 0.4397 l. \]

Ex. 3.5. A continuous beam ABC of constant moment of inertia carries a central load of 10,000 kg. in span AB and a central clockwise moment of 30,000 kgm. in span BC. Span AB = 10 m. and span BC = 15 m. Find the support moment and plot S.F. and B.M. diagrams for the beam.

Solution. Simply supported B.M. diagrams for spans AB and BC are shown in Fig. 3.14 (b). As the ends B and C are simply supported \( M_A = M_C = 0 \).

Applying theorem of three moments to spans AB and BC.

\[ M_A \times l_1 + 2M_B(l_1 + l_2) + M_C \times l_2 = -6 \left[ \frac{A_1 x_1}{l_1} + \frac{A_2 x_2}{l_2} \right] \]

\[ 0 + 2M_B(10 + 15) + 0 = -6 \left[ \frac{1 \times 25,000 \times 10 \times 5}{10} + \frac{1 \times 15,000 \times 15^2 \times 5 - \frac{1}{15} \times 15,000 \times 15^2 \times (7.5 + 2.5)}{15} \right] \]

\[ \therefore \quad 50M_B = -6 \left[ \frac{125,000}{2} + \frac{75,000}{4} - \frac{150,000}{15} \right] \]

\[ \therefore \quad 50M_B = -6 \times \frac{175,000}{4} \]

\[ \therefore \quad M_B = -5250 \text{ kg. m.} \]
Reactions.
Consider equilibrium of span \( AB \). Taking moments about \( B \),
\[
R_A \times 10 + 5250 - 10,000 \times 5 = 0
\]
\[\Rightarrow R_A = 5000 - 525 = 4,475 \text{ kg.}\]
Consider portion \( BC \) of beam. Taking moments about \( B \),
\[
R_C \times 15 + 5250 - 30,000 = 0
\]
\[\Rightarrow R_C = 2000 - 350 = 1,650 \text{ kg.}\]
\[\Rightarrow R_B = 10,000 - 4,475 - 1,650 = 3,875 \text{ kg.}\]
The S.F. and B.M. diagrams are shown in Fig. 3.14.
Ex. 3-8. Find the fixed end moments and plot B.M. diagram the continuous beam shown in Fig. 3-15.

Solution. Simply supported B.M. diagrams for spans AB and CD are shown in Fig. 15. Add zero span to the left of A.

Applying theorem three moments to the span and span AB.

\[ 0 \times 0 + 2M_A \left( 0 + \frac{8}{2I} \right) + M_B \times \frac{8}{2I} = -6 \]

\[ + \frac{20,000 \times \frac{4}{3} \times 8 \times 4}{2I \times 8} \]

\[ 16M_A + 8M_B = -240,000 \]

\[ 2M_A + M_B = -30,000 \]

...(1)

Applying theorem of three moments to spans AB and BC.

\[ M_A \times \frac{8}{2I} + 2M_B \left( \frac{8}{2I} + \frac{8}{I} \right) + M_C \times \frac{8}{I} \]

\[ = -6 \left[ \frac{20,000 \times \frac{4}{3} \times 8 \times 4}{2I \times 8} + 0 \right] \]

\[ 4M_A + 24M_B + 8M_C = -120,000 \]

\[ 2M_A + 12M_B + 4M_C = -60,000 \]

...(2)

Applying theorem of three moments to spans BC and CD.

\[ M_B \times \frac{8}{I} + 2M_C \left( \frac{8}{I} + \frac{8}{2I} \right) + M_D \times \frac{8}{2I} \]

\[ = -6 \left[ 0 + \frac{8 \times 40,000 \times 8 \times 4}{2I \times 8} \right] \]

\[ M_D = -5000 \times 4 \times 2 = -40,000 \text{ kg. m.} \]

\[ 8M_B + 24M_C - 4 \times 40,000 = -320,000 \]

\[ 8M_B + 24M_C = -160,000 \]

\[ M_B + 3M_C = -20,000 \]

...(3)

Equations (1), (2) and (3) can be solved for values of \( M_A, M_B \) and \( M_C \).

(2)−(1) gives

\[ 11M_B + 4M_C = -30,000 \]

...(4)

Multiplying (3) by 11

\[ 11M_B + 33M_C = -220,000 \]

...(5)
(5) - (4) gives

\[ 29 \quad M_c = -190,000 \]
\[ M_c = -6551.7 \text{ kg m.} \]
\[ M_b = -20,000 - 3M_c \]
\[ = -20,000 + 19,655.7 \]
\[ M_b = -344.3 \text{ kg m.} \]

\[ \therefore \]
\[ 2M_a = 30,000 - M_b \]
\[ = -30,000 + 344.3 \]
\[ = 29,655.7 \]
\[ M_a = -14,827.85 \text{ kg m.} \]

B.M. diagram is shown in Fig. 3.15(c).

**Ex. 3.7.** Beam AEBCD shown in Fig. 3.16 (a) is supported at A, B, C, and D and hinged at E. Plot the B.M. diagram for the beam.

![Beam Diagram](image)

**Solution.** Consider free body diagram of portion AE (Fig. 3.16 (b)).
\[ R_B = R_A = \frac{4000 \times 3}{2} = 6000 \text{ kg.} \]

On beam \( EBCD \) reaction at hinge will be \( R_B = 6000 \text{ kg.} \) downwards at \( E \).

Simply supported B.M. diagrams for spans \( BC \) and \( CD \) are shown in Fig. 3.16 (d).

Applying theorem of three moments to spans \( BC \) and \( CD \).

\[ M_B \times 5 + 2M_C(5+6) + M_D \times 6 = -6 \left[ \frac{A_1 x_1}{l_1} + \frac{A_2 x_2}{l_2} \right] \]

\[ M_D = 0 \text{ as the end } D \text{ is simply supported.} \]

\[ M_B = -6000 \times 3 - 4000 \times 3 \times \frac{3}{4} \]

\[ = -18,000 - 18,000 \]

\[ = -36,000 \text{ kg. m.} \]

\[ \therefore -5 \times 36,000 + 2M_C \times 11 + 0 \]

\[ = -6 \left[ \frac{12,500 \times \frac{3}{4} \times \frac{1}{4}}{5} + \frac{\frac{3}{4} \times 18,000 \times 6 \times 3}{6} \right] \]

\[ \therefore -180,000 + 22M_C \]

\[ = -6 \left[ \frac{62,500}{4} + 36,000 \right] \]

\[ \therefore -180,000 + 22M_C = -93,750 - 216,000 \]

\[ \therefore 22M_C = -129,750 \]

\[ \therefore M_C = -5,897 \text{ kg. m.} \]

B.M. diagram is shown in Fig. 3.16 (e).

3.4. Effect of Sinking of Supports.

Let the level of middle support \( B \) be \( \delta_1 \) below support \( A \) and \( \delta_2 \) below support \( C \). Draw tangents at \( B \) to the elastic curve cutting the verticals at \( A \) and \( C \) at \( D \) and \( E \) respectively.

Let the spans \( AB \) and \( BC \) be of lengths \( l_1 \) and \( l_2 \) and moment of inertia \( I_1 \) and \( I_2 \) respectively.

\[ Z_1 = \frac{A_1 x_1 + M_A \times \frac{l_1}{2} \times \frac{l_1}{3} + M_B \times \frac{l_1}{2} \times \frac{2l_2}{3}}{EI_1} \]

\[ Z_2 = \frac{(A_2 x_2 + M_C \times \frac{l_2}{2} \times \frac{l_2}{3} + M_B \times \frac{l_2}{2} \times \frac{2l_2}{3})}{EI_2} \]

\[ \frac{Z_1 - \delta_1}{l_1}, \quad \frac{Z_2 - \delta_2}{l_2} (\theta_B \text{ being small}) \]

\[ \frac{Z_1}{l_1}, \quad \frac{Z_2}{l_2} + \frac{\delta_1}{l_1} + \frac{\delta_2}{l_2} \]
\[ A_1 \bar{x}_1 + \frac{W_A \times l_1^2}{6} + \frac{2M_B \times l_1^2}{6} \]

\[ \Rightarrow \]

\[ E I_1 l_1 \]

\[ = - \left( A_2 \bar{x}_2 + \frac{M_C \times l_2^3}{6} + \frac{2M_B \times l_2^2}{6} \right) \]

\[ E I_2 l_2 \]

\[ \frac{\delta_1}{l_1} + \frac{\delta_2}{l_2} \]

SIMPLY SUPPORTED B.M. DIAGRAM

\[ \bar{x}_1 \]

\[ \bar{x}_2 \]

A_1

A_2

\[ M_A \frac{l_1}{6I_1} + \frac{2M_B l_1}{6I_1} + \frac{M_C l_2}{6I_2} + \frac{2M_B l_2}{6I_2} \]

\[ \Rightarrow \]

\[ = - \frac{A_1 \bar{x}_1}{I_1 l_1} - \frac{A_2 \bar{x}_2}{I_2 l_2} + E \left( \frac{\delta_1}{l_1} + \frac{\delta_2}{l_2} \right) \]

\[ M_A \left( \frac{l_1}{l_1} \right) + 2M_B \left( \frac{l_1}{l_1} + \frac{l_2}{l_2} \right) + M_C \left( \frac{l_2}{l_2} \right) \]

\[ = -6 \left[ \frac{A_1 \bar{x}_1}{I_1 l_1} + \frac{A_2 \bar{x}_2}{I_2 l_2} \right] + 6E \left[ \frac{\delta_1}{l_1} + \frac{\delta_2}{l_2} \right] \]

In case \( l_1 = l_2 \)

\[ M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 \]

\[ = -6 \left[ \frac{A_1 \bar{x}_1}{l_1} + \frac{A_2 \bar{x}_2}{l_2} \right] + 6EI \left[ \frac{\delta_1}{l_1} + \frac{\delta_2}{l_2} \right] \] \( \ldots (3.3) \)

**Ex. 3.8.** A beam ABCD 20 metres long is continuous over 3 spans AB, BC and CD of 6 m., 8 m., and 6 m. respectively. There is a uniformly distributed load of 1000 kg/m. on span AB, a central load of 10,000 kg. on span BC and a load of 5000 kg. on span CD at 2 m.
from D. The ends A and D are fixed. During loading support B sinks by 1 cm. Find the fixed end moments and draw B.M. diagram for the beam.

\[ I_{BC} = 2I_{AB} = 2I_{CD} = 30,000 \text{ cm}^4; \quad E = 2.0 \times 10^6 \text{ kg/cm}^2. \]

**Solution.** Simply supported B.M. diagrams for spans AB, BC and CD are shown in Fig. 3.18.

Applying theorem of three moments to imaginary zero span and span AB.

\[
0 + 2M_A \left( 0 + \frac{6}{I} \right) + M_B \times \frac{6}{I} = -6 \left[ 0 + \frac{2}{3} \times \frac{4500 \times 6 \times 3}{6I} \right] + \frac{6 \times E \times (-1500)}{6}
\]

\[
\therefore 12M_A + 6M_B = -6 \times 9000 - \frac{6 \times 2 \times 10^8 \times 100^2}{6 \times 100} \times \frac{15,000}{(100)^4}
\]

\[ 2M_A + M_B = -9000 - 5000 = -14,000 \text{ kg. m.} \quad \ldots(1) \]

![Simply supported B.M. diagram](image)

**SIMPLY SUPPORTED B.M. DIAGRAM**

![B.M. diagram](image)

**Fig. 3.18**

Applying theorem of three moments to spans AB and BC

\[
M_A \times \frac{6}{I} + 2M_B \left( \frac{6}{I} + \frac{8}{2I} \right) + M_C \times \frac{8}{2I} = -6 \left[ \frac{2}{3} \times \frac{4500 \times 6 \times 8}{6I} + \frac{1}{2} \times \frac{20,000 \times 8 \times 4}{8 \times 2I} \right] + 6E \left( \frac{1}{100} \times \frac{1}{6} + \frac{1}{100} \times \frac{1}{8} \right)
\]
\[
\begin{align*}
6M_A + 20M_B + 4M_C & = -6[9,000 + 20,000] + 6EI\left(\frac{1}{600} + \frac{1}{800}\right) \\
& = 174,000 + 6 \times 2 \times 10^8 \times 100^2 \times \frac{15,000}{(100)^4} \times \frac{14}{4800} \\
& = -174,000 + 52,500 \\
\therefore \ 3M_A + 10M_B + 2M_C & = -60,750 \text{ kg. m.} \quad \ldots(2)
\end{align*}
\]

Applying theorem of three moments to spans BC and CD.
\[
M_B \times \frac{8}{2I} + 2M_C\left(\frac{8}{2I} + \frac{6}{I}\right) + M_D \times \frac{6}{I} \\
= -6\left[\frac{1}{2} \times \frac{20,000 \times 8 \times 4}{8 \times 2I} - \frac{1}{100} \times \frac{8000 \times 6 \times \frac{3}{8}}{I \times 6}\right] + 6EI\left(-\frac{1}{100} \times \frac{1}{8}\right) \\
= -6\left(\frac{20,000}{3}\right) - 6 \times 2 \times 10^8 \times (100)^2 \times \frac{15,000}{(100)^4} \times \frac{1}{800} \\
\therefore \ 4M_B + 20M_C + 6M_D & = -184,000 - 22,500 = -206,500 \\
\therefore \ 2M_B + 10M_C + 3M_D & = -103,250 \text{ kg. m.} \quad \ldots(3)
\]

Applying theorem of three moments to span CD and zero span.
\[
M_C \times \frac{6}{I} + 2M_D\left(\frac{6}{I} + 0\right) + 0 = -6\left[\frac{1}{I} \times \frac{8000 \times 6}{6I} \times \frac{10}{3}\right] \\
\therefore \ 6M_C + 12M_D & = -80,000 \\
\therefore \ M_C - 2M_D & = -13,333.3 \text{ kg. m.} \quad \ldots(4)
\]

Equations (1), (2), (3) and (4) can be solved for \(M_A, M_B, M_C\) and \(M_D\).

\[
\begin{align*}
2M_A + M_B & = -14,000 \quad \ldots(1) \\
3M_A + 10M_B + 2M_C & = -60,750 \quad \ldots(2) \\
2M_B + 10M_C + 3M_D & = -103,250 \quad \ldots(3) \\
M_C + 2M_D & = 13,333.3 \quad \ldots(4)
\end{align*}
\]

Multiplying Eqn. (3) by 2,
\[
4M_B + 20M_C + 6M_D = -206,500 \quad \ldots(5)
\]
CONTINUOUS BEAMS

Multiplying Eqn. (4) by 3
\[ 3M_C + 6M_D = -40,000 \] ...(0)

(5) - (6) gives \[ 4M_B + 17M_C = -166,600 \]

\[ \therefore \quad M_B + 4.25M_C = -41,625 \] ...(7)

Multiplying Eqn. (1) by 3
\[ 6M_A + 3M_B = -42,000 \] ...(8)

Multiplying Eqn. (2) by 2,
\[ 6M_A + 20M_B + 4M_C = -121,500 \] ...(9)

(9) - (8) gives \[ 17M_B + 4M_C = -79,500 \]

\[ \therefore \quad M_B + 0.235M_C = -46.6 \] ...(10)

(7) - (10) gives \[ 4.015M_C = -36,949 \]

\[ \therefore \quad M_C = -9200 \text{ kg m.} \]

\[ \therefore \quad M_B = -41,625 - 4.25M_C 
= -41,625 - 4.25 \times 9200 
= -2525 \text{ kg m.} \]

\[ 2M_A + M_B = -14,000 \]

\[ 2M_A = -14,000 + 2525 \]

\[ \therefore \quad M_A = -5737.5 \text{ kg m.} \]

\[ M_C + 2M_D = -13,333.3 \]

\[ 2M_D = -13,333.3 + 9200 = -4133.3 \]

\[ \therefore \quad M_D = -2066.7 \text{ kg m.} \]

B.M. diagram is shown in Fig. 3.18.

Ex. 3.9. A continuous beam ABCD 18 m. long is loaded as shown in Fig. 3.19. During loading support B sinks by 1 cm. Find support moments and plot S.F. and B.M. diagrams for the beam. \( I = 8,000 \text{ cm}^4 \quad E = 2 \times 10^8 \text{ kg/cm}^2 \).

Solution. Simply supported B.M. diagrams for spans AB, BC and CD are shown in Fig. 3.19 (b).

Support moments \( M_A \) and \( M_D \) will be zero.

Applying theorem of three moments to spans AB and BC.

\[ 0 + 2M_B \left( \frac{4}{I} + \frac{8}{2I} \right) + M_C \times \frac{8}{2I} \]

\[ = -6 \left[ \frac{\frac{1}{3} \times 3000 \times 4 \times \frac{2}{3}}{4I} + \frac{\frac{1}{3} \times 8000 \times 8 \times 4}{8 \times 2I} \right] + 6EI \left( \frac{1}{100} \times \frac{1}{4} + \frac{1}{100} \times \frac{1}{3} \right) \]

\[ 16M_B + 4M_C = -6 \left[ \frac{7,000}{2} + \frac{32,000}{3} \right] + 6EI \left( \frac{1}{400} + \frac{1}{800} \right) \]
\[ 16M_B + 4M_C = -85,000 + 6 \times 2 \times 10^4 \times (100)^3 \times \frac{8000}{(100)^4} \times \frac{3}{800} \]
\[ = -85,000 + 36,000 = -49,000 \]
\[ 4M_B + M_C = -12,250 \text{ kg. m.} \]

Applying theorem of three moments to span \( BC \) and \( CD \).

\[ M_B \times \frac{8}{2I} + 2M_C \left( \frac{8}{2I} + \frac{6}{1.5I} \right) + 0 = -6 \left[ \frac{2}{3} \times \frac{8000 \times 8 \times 4}{8 \times 2I} + \frac{1}{2} \times \frac{6000 \times 6 \times 3}{6 \times 1.5I} \right] + 6EI \left[ \left( -\frac{1}{100} \times \frac{1}{8} \right) + 0 \right] \]

\[ 4M_B + 16M_C = -6 \left[ \frac{32,000}{3} + 6000 \right] - 6EI \times \frac{1}{800} \]

\[ = -100,000 - 6 \times 2 \times 10^4 \times (100)^2 \times \frac{8000}{(100)^4} \times \frac{1}{800} \]

\[ = -112,000 \text{ kg. m.} \]

**Fig. 3.19**
(2)—(1) gives
\[ 15M_C = -99,750 \]
\[ M_C = -6,650 \text{ kg m.} \]
\[ 4M_B = -12,250 + 6,650 = -5600 \]
\[ \therefore M_B = -1400 \text{ kg m.} \]

**Reactions**

Consider span \( AB \), taking moments about \( B \)

\[ R_A \times 4 + 1400 - 4000 \times 1 = 0 \]

\[ R_A = 1000 - 350 = 650 \text{ kg.} \]

Consider spans \( AB \) and \( BC \).

\[ R_A \times 12 + R_B \times 8 + M_C - 1000 \times 8 \times 4 - 4000 \times 9 = 0 \]

\[ 8R_B = 32,000 + 36,000 - 6,650 - 12 \times 650 \]

\[ = 53,550 \]

\[ \therefore R_B = 8893.8 \text{ kg.} \]

Consider span \( CD \), taking moments about \( C \),

\[ R_D \times 6 + M_C = 4000 \times 3 \]

\[ R_C = 4000 + 4000 + 8000 - 650 - 8893.8 - 891.7 \]

\[ = 5564.5 \text{ kg.} \]

B.M. and S.F. diagrams are shown in Fig. 3.19 (c) and (d).

**Ex. 3.10.** A straight elastic beam \( ABCD \) of uniform section rests on four similar floats at \( A, B, C \) and \( D \) which are spaced \( t \) metres apart. The buoyancy of each float is such that every addition of one tonne of load increases its immersion by \( 'd' \). Initially all the floats are equally immersed. If two loads, each of \( W \) tonnes, are placed on the girder at \( B \) and \( C \), show that the proportion of the load carried by vertical float is

\[ \frac{W \left(1 + \frac{6EI_d}{5l^3}\right)}{1 + \frac{12EI_d}{5l^3}} ; \text{ where } EI \text{ is flexural rigidity of the beam.} \]
Solution. Let \( R \) be the reaction at \( A \). From symmetry reaction \( R_D \) will also be \( R \).

\[ R_B = R_O = -R. \]

Immersion of floats at \( A \) and \( B \) will be \( R_d \) and that of \( B \) and \( C \) will be \( (W-R)d \).

Deflection of \( B \) relative to \( A \) will be \( (W-R)d - R \times d = (W-2R)d \).

The deflection of \( B \) relative to \( C \) will be zero.

Ends \( A \) and \( C \) are simply supported so \( M_A = M_D = 0 \).

As the beam is symmetrical and loading is symmetrical \( M_B = M_C \).

Applying theorem of three moments to spans \( AB \) and \( BC \).

\[
M_A \times l_1 + 2M_B(l_1 + l_2) + M_O \times l_2
\]

\[= -6 \left[ \frac{A_1 \delta_1}{l_1} + \frac{A_2 \delta_2}{l_2} \right] + 6EI \left( \frac{\delta_1}{l_1} + \frac{\delta_2}{l_2} \right)
\]

\[0 + 2M_B(l_1 + l) + M_B \times l = 0 + 6EI \left( \frac{(W-2R)d}{l} \right)+ 0\]

\[\therefore 5M_B l = \frac{6EI(l)(W-2R)}{l}\]

\[\therefore M_B = \frac{6EI(l)(W-2R)}{5l^2}.
\]

Taking moments about \( B \),

\[R \times l - M_B = 0\]

\[\therefore R \times l - \frac{6EI(l)(W-2R)}{5l^2} = 0\]

\[\therefore R = \frac{6EI(l)(W-2R)}{5l^2} \times 2R = \frac{6EI(l)(W)}{5l^2}\]

\[R \times l \left(1 + \frac{12EI(l)}{5l^2}\right) = \frac{6EI(l)(W)}{5l^2}\]

\[\therefore R = \frac{6EI(l)(W)}{5l^2} \times \frac{5l^2}{1 + \frac{12EI(l)}{5l^2}}\]

\[\therefore R_B = W - R = W - W \times \frac{6EI(l)}{5l^2} \times \frac{5l^2}{1 + \frac{12EI(l)}{5l^2}}\]
\[
W \left( 1 + \frac{6EI \delta}{5l^3} \right) \quad \frac{12EI \delta}{5l^3}
\]

\[\therefore \text{Portion of load carried by central float is:} \quad W \left( 1 + \frac{6EI \delta}{5l^3} \right) \quad \frac{12EI \delta}{5l^3}\]

**Ex. 3.11.** A steel strip of cross-section 3 cm. \(\times\) 2 mm. is placed through a row of five rigid pegs as shown in Fig. 3.22. The pegs have a diameter of 1.2 cm. and are spaced at equal intervals of 50 cm. with their centres on a straight line. Neglecting the weight of the strip, find the thrust on each of the pegs. \(E = 2 \times 10^6\) kg/cm².

**Solution.** Support moments \(M_A\) and \(M_E\) will be zero. Support moments \(M_B\) and \(M_D\) will be equal as the strip is symmetrical.

Consider spans \(AB\) and \(BC\). Support \(B\) sinks by 1.2 cm. relative to supports \(A\) and \(C\).

Applying theorem of three moments to spans \(AB\) and \(BC\).

\[
M_A \times l_1 + 2M_B \times (l_1 + l_2) + M_C \times l_2
\]

\[
= -6 \left( \frac{A_1 \delta_1}{l_1} + \frac{A_2 \delta_2}{l_2} \right) + 6EI \left( \frac{\delta_1}{l_1} + \frac{\delta_2}{l_2} \right)
\]

\[
0 + 2M_B(50+50) + M_C \times 50 = -0 + 6EI \left( \frac{1.2}{50} + \frac{1.2}{50} \right)
\]

\[
0 + 2M_B(100) + M_C \times 100 = -0 + 6EI \left( \frac{1.2}{50} - \frac{1.2}{50} \right)
\]

\[
\therefore \quad 200M_B + 50M_C = 6 \times 2 \times 10^6 \times \frac{1}{12} \times 3 \times \left( \frac{2}{10} \right)^3 \times \frac{2.4}{50}
\]

\[
= 1152.
\]

\[
\therefore \quad 4M_B + M_C = 23.04 \text{ kg. cm.} \quad \ldots (1)
\]

Applying theorem of three moments to spans \(BC\) and \(CD\).

\[
M_B \times 50 + 2M_C (50+50) + M_D \times 50 = 0 + 6EI \left( \frac{-1.2}{50} - \frac{1.2}{50} \right)
\]

From symmetry \(M_B = M_D\)

\[
\therefore \quad M_B \times 100 + M_C \times 200 = -6 \times 2 \times 10^6 \times \frac{1}{12} \times 3 \times \left( \frac{2}{10} \right)^3 \times \frac{2.4}{50}
\]
\[ 100M_B + 200M_C = -1152 \]

\[ \frac{M_B}{2} + M_C = -5.76 \text{ kg cm.} \quad \ldots(2) \]

(1) – (2) gives \[ 3.5M_B = 28.8 \]

\[ M_B = 8.23 \text{ kg cm.} \]

\[ M_C = -5.76 - \frac{M_B}{2} = -5.76 - 4.12 \]

\[ = -9.88 \text{ kg cm.} \]

Consider beam \( AB \).

Taking moments about \( B \).

\[ R_A \times 50 = 8.2 \times 0 \]

\[ \therefore R_A = 0.164 \text{ kg.} \]

\[ R_B = 0.164 \text{ kg.} \]

Consider spans \( AB \) and \( BC \).

Taking moments about \( C \).

\[ R_A \times 100 + R_B \times 50 = 9.88 \]

\[ R_A = -2R_A - 0.1976 \]

\[ = -0.1976 - 0.328 = -0.5256 \text{ kg.} \]

\[ R_D = -0.5256 \text{ kg.} \]

\[ R_C = 2 \times 0.5256 - 2 \times 0.164 \text{ (upwards, Fig. 3.23)} \]

\[ = 1.0512 - 0.328 = 0.7232 \text{ kg.} \]

**Ex. 3.12.** Explain the theorem of three moments in the most general form. Hence or otherwise analyse the continuous beam shown in Fig. 3.24 and draw the bending moment and shear force diagrams. Relevant values of moment of inertia are shown against the members.

(A.M.I.E. Nov. 1967)
Solution.

\[ M_A \left( \frac{l_1}{I_1} \right) + 2M_B \left( \frac{l_1}{I_1} + \frac{l_2}{I_2} \right) + M_C \left( \frac{l_2}{I_2} \right) = -6 \left[ \frac{A_x \bar{x}_1}{I_1} + \frac{A_x \bar{x}_2}{I_2} \right] \]

Take ABC

\[ M_A \left( \frac{3}{2I} \right) + 2M_B \left( \frac{3}{2I} + \frac{4\cdot5}{3I} \right) + M_C \left( \frac{4\cdot5}{3I} \right) = -6 \left( \frac{\frac{2}{3} \times 2\cdot531 \times 4\cdot5 \times \frac{4\cdot5}{2}}{3I \times 4\cdot5} \right) \]

\[ M_A = -3 \times 0\cdot915 = -2\cdot745 \]

\[ -1\cdot5 \times 2\cdot745 + 6M_B + 1\cdot5M_C = -6 \times 1\cdot266 \]

\[ 4M_B + M_C = -4 \times 1\cdot266 + 2\cdot745 = -2\cdot0 \ldots \text{(i)} \]

![B.M. Diagram.](image)

Take BCD

\[ M_B \left( \frac{4\cdot5}{3I} \right) + 2M_C \left( \frac{4\cdot5}{3I} + \frac{3}{2I} \right) + M_D \left( \frac{3}{2I} \right) = \frac{2}{3} \times 2\cdot531 \times 4\cdot5 \times \frac{4\cdot5}{2} - \frac{2\cdot896 \times 3}{3I \times 4\cdot5} \times \left( \frac{1\cdot22 + 3}{3} \right) \]

\[ M_D = 0 \]

\[ 1\cdot5M_B + 6M_C = -6 \left[ 1\cdot266 + 1\cdot018 \right] \]

\[ M_B + 4M_C = -4 \times 2\cdot284 = -9\cdot136 \ldots \text{(ii)} \]

\[ 4 \times \text{(i)} \]

\[ 16M_B + 4M_C = -9\cdot276 \ldots \text{(iii)} \]

\[ 15M_B = -0\cdot140 \]

\[ M_B = -0\cdot0093 \text{ t.m.} \]

\[ M_C = -2\cdot319 + 4 \times 0\cdot0093 = -2\cdot319 + 0\cdot0372 \]

\[ = -2\cdot2818 \text{ t.m.} \]
Reactions.

\[ R_A \times 3 = 3(915 + 3) - 0.0093 \]
\[ R_A = 3 \times 915 - 0.0031 \]
\[ = 3 \times 9119 \text{ t.} \]

\[ R_B = F_{AB} + F_{BC} = \frac{-0.0093 - 2.745}{3} + \frac{0.0093 + 1 \times 4.5^2}{2} - 2.2818 \]
\[ = -0.0093 + 1.7339 \]
\[ = 0.8220 \]

\[ R_C = F_{CB} + F_{CD} = (4.5 - 1.7339) + \frac{(4 \times 1.22 + 2.2818)}{3} \]
\[ = 2.7661 + 2.3873 \]
\[ = 5.1534 \text{ t.} \]

\[ R_D = F_{DC} = 4 - 2.3873 \]
\[ = 1.6127 \text{ t.} \]

(a)

(b)

(c)

S.F. Diagram.

Fig. 3 26
Ex. 3·13. Analyse the continuous beam shown in Fig. 3·27. It carries a uniformly distributed load of 6·6 t/m, on AB and BC. The support B sinks by 6·35 mm. below A and C. EI is constant, $E = 2000$ t/m², $I = 33,200$ cm⁴.

(A.M.I.E. May 1968)

![Fig. 3·27](image)

Solution. If support B is $\delta_1$ below A and $\delta_2$ below C, general formula will be,

$$M_A \left( \frac{I_1}{l_1} \right) + 2M_B \left( \frac{l_1}{I_1} + \frac{l_2}{I_2} \right) + MC \left( \frac{l_2}{I_2} \right) = -6 \left[ \frac{A_1 x_1}{I_1 l_1} + \frac{A_2 x_2}{I_2 l_2} \right] + 6EI \left[ \frac{\delta_1}{l_1} + \frac{\delta_2}{l_2} \right]$$

As $I_1 = I_2 = I$,

$$M_A l_1 + 2M_B (l_1 + l_2) + MC l_2$$

$$= -6 \left[ \frac{A_1 x_1}{l_1} + \frac{A_2 x_2}{l_2} \right] + 6EI \left[ \frac{\delta_1}{l_1} + \frac{\delta_2}{l_2} \right]$$

![Fig. 3·28](image)

ABC

$$EI = \frac{2000 \times 33,200}{(100)^3} = 6,640 \text{ t.m}^2.$$  

$$\frac{\delta_1}{l_1} = \frac{6·35 \times 1000}{3·6} = 1·764 \times 10^{-3}.$$  

$$\frac{\delta_2}{l_2} = \frac{6·35 \times 1000}{4·8} = 1·323 \times 10^{-3}.$$  

$$3·6 M_A + 2M_B (3·6 + 4·8) + MC \times 4·8$$

$$-6 \left( \frac{2}{3} \times 10·690 \times 3·6 \times \frac{3·6}{2} \times \frac{2}{3} \times 19'000 \times 4·8 \times \frac{4·8}{2} \right)$$

$$+ 6 \times 6,640 \left[ 1·764 \times 10^{-3} + 1·323 \times 10^{-3} \right]$$
\[ M_A = 0 \]
\[ 16.8 \; M_B + 4.8 \; M_C = -6[12.83 + 30.41] + 39.84 \times 3.087 \]
\[ = -259.44 + 123.0 = -136.44 \]
\[ 14M_B + 4M_C = -113.7 \]
\[ 7M_B + 2M_C = -56.85 \]
\[ \begin{array}{c}
\text{B} \quad \text{O} \quad \text{C} \quad \text{R} \\
\hline
4.8
\end{array} \]
\[ \text{Fig. 3.29} \]

Take end span \( CR = 0 \)
\[ \frac{\delta_8}{l_2} = -1.323 \times 10^{-3} \]
\[ \frac{\delta_9}{l_3} = 0. \]

\[ M_B \; 4.8 + 2M_C (4.8 + 0) = -6 \left( \frac{2}{3} \times 19.00 \times 4.8 \times \frac{4.8}{2} \right) \]
\[ + 6 \times 6,640\left[ -1.323 \times 10^{-3} \right] \]
\[ \therefore 4.8 \; M_B + 9.6 \; M_C = -6 \times 30.41 - 39.84 \times 1.323 \]
\[ = -182.46 - 52.71 = -235.17 \]
\[ \therefore M_B + 2M_C = -49.01 \]

Equation (i) - (ii), \( 6M_B = -7.84 \)
\[ M_R = -1.307 \text{ t.m.} \]
\[ M_C = -49.01 + 1.307 = -23.852 \text{ t.m.} \]

Final moments are shown in Fig. 3.30

**Problems**

1. A continuous beam \( ABCD \), 40 m. long is continuous over 3 spans, \( AB = 16 \text{ m.}, \ BC = 12 \text{ m.} \) and \( CD = 12 \text{ m.} \) The supports being at the same level. There is a uniformly distributed load of 1.5 tonnes/m. over \( BC \), on \( AB \) there is a point load of 15 tonnes at 4 m. from \( A \) and on \( CD \), there is a point load of
20 tonnes at 9 m. from D. Calculate the moments and reactions at the supports.

\[ \text{Ans. } M_B = -21.75 \text{ t.m.}, \quad M_C = -27.76 \text{ t.m.}, \]
\[ R_A = 9.89 \text{ t.}, \quad R_B = 13.61 \text{ t.}, \quad R_C = 26.81 \text{ t.}, \quad R_D = 2.69 \text{ t.} \]

\( \frown \) A continuous beam \( ABC \), 11 m. long is continuous over 2 spans, \( AB = 6 \text{ m.} \) and \( BC = 5 \text{ m.} \). The supports are at the same level. \( AB \) is loaded with a uniformly distributed load of 2000 kg/m and \( BC \) with a concentrated load of 5,000 kg at 3 m. from \( C \). Find the support moments and reactions.

\[ \text{Ans. } M_B = -7091 \text{ kg.m.}, \quad R_A = 4818 \text{ kg.}, \quad R_B = 11,600 \text{ kg.}, \quad R_C = 582 \text{ kg.} \]

\( \frown \) A continuous beam of constant moment of inertia is loaded as shown in Fig. 3.31. Find the support moments and reactions.

![Fig. 3.31](image)

\[ \text{Ans. } M_A = -875 \text{ kg.m.}, \]
\[ M_B = +1750 \text{ kg.m.}, \quad R_A = 323.1 \text{ kg.} \]
\[ R_B = -2236.1 \text{ kg.}, \quad R_C = 3958 \text{ kg.} \]

\( \frown \) A continuous beam \( ABDEG \) shown in Fig. 3.32 is supported at \( A, \) \( B, \) \( D, \) \( E \) and \( G \), and hinged at centres of \( AB \) and \( FG \) as shown in Fig. 3.32. The beam is loaded with uniformly distributed load of 1000 kg/m. Plot the B.M. diagram for the beam.

\[ \text{Ans. } M_A = M_G = 0, \quad M_E = -4000 \text{ kg.m.}, \]
\[ M_D = -6000 \text{ kg.m.} \]

\( \frown \) A continuous beam \( ABCD \), 28 m. long, is continuous over 3 spans of 10 m, 10 m. and 8 m. There is a uniformly distributed load of 3 tonnes/m over each of 10 m. span and a load of 6 tonnes/m over the 8 m. span. The ends are freely supported and during load support \( B \) sinks by 1 cm. Find the fixed end moments and draw B.M. and S.F. diagrams for the beam.

\[ I = 2 \times 10^8 \text{ kg/cm}^2; \quad I = 30,000 \text{ cm}^4. \]

\[ \text{Ans. } M_B = 28,800 \text{ kg.m.}, \quad M_C = 35,700 \text{ kg.m.}, \quad R_A = 12,320 \text{ kg.}, \]
\[ R_B = 39,785 \text{ kg.}, \quad R_C = 44,363 \text{ kg.}, \quad R_D = 19,532 \text{ kg.} \]
6. A straight elastic beam of uniform cross-section and length 18 m. carries a uniformly distributed load of 1500 kg/m. throughout its length and is supported on four similar elastic supports, which are spaced 6 m. apart. The supports are such that they compress 1 cm. for each 1,000 kg. of load upon them. Find the reaction on central supports. \( E = 2 \times 10^6 \) kg/cm.²; \( I = 20,000 \) cm.⁴.

[Ans. 8930 kg.]

7. A continuous beam \( ABCD \), 20 m. long is continuous over 3 spans \( AB = 8 \) m., \( BC = 4 \) m. and \( CD = 8 \) m. Moment of inertia of \( AB \) is \( 2I \), that of \( BC \) is \( I \) and that of \( CD \) is \( 2I \). There is a uniformly distributed load of 1500 kg/m. over spans \( AB \) and \( BC \). On \( CD \) there is a central load of 4000 kg. The ends are fixed and during load support \( B \) sinks by 1 cm. Find the fixed end moments. \( I = 16,000 \) cm.⁴; \( E = 2 \times 10^6 \) kg/cm.².

[Ans. \( M_A = -16,533 \) kg. m., \( M_B = +3066 \) kg. m.,
\[ M_C = -7733 \) kg. m. \( M_D = -2133.5 \) kg. m.]

8. A continuous beam \( ABCD \) is loaded as shown in Fig. 33. Determine the support moments and plot B.M. diagram for the beam.

[Ans. \( M_B = -6993 \) kg. m., \( M_C = -8887 \) kg. m.]

9. A continuous beam \( ABCDE \), 14 m. long is continuous over four spans. \( AB = 3 \) m., \( BC = 4 \) m., \( CD = 4 \) m. and \( DE = 3 \) m. Moment of inertia of \( AB \) is \( I \), that of \( BC \) is \( 2I \), that of \( CD \) is \( I \) and that of \( DE \) is \( I \). Span \( AB \) is loaded with a load of 1800 kg. at 1 m., from \( A \), \( BC \) is loaded with a uniformly distributed load of 1500 kg/m. \( CD \) is loaded with central load of 4000 kg. and \( DE \) is loaded with a uniformly distributed load of 800 kg/m. The ends are simply supported and during load support \( C \) sinks by 1 cm. Determine the support moments and plot S.F. and B.M. diagrams for the beam. \( I = 8,000 \) cm.⁴; \( E = 2 \times 10^6 \) kg/cm.².

[Ans. \( M_B = -5203 \) kg. m., \( M_C = +5615 \) kg. m.,
\[ M_D = -5263 \) kg. m., \( R_A = -534 \) kg., \( R_B = 8038 \) kg.,
\[ R_C = -424 \) kg., \( R_D = 7,674 \) kg., \( R_E = -554 \) kg.]

10. A beam \( ABC \) of constant flexural rigidity \( EI \) and length \( 2L \) carries a uniformly distributed load of \( w/\)unit length and simply supported on rigid supports at ends \( A \) and \( C \) and on a spring at centre \( B \). When the load is applied to the beam the spring is found to deflect by \( L/k \). Find the deflection constant of spring in terms of \( EI, L, w \) and \( k \).

\[ \text{Ans. } \frac{u}{w} = \frac{6EI}{k} \]
11. A uniform continuous girder $ABC$ rests upon three similar floating supports, situated at each end and at the middle point $B$. The buoyancy of each float is such that every additional tonne of load increases its immersion by $3\ell$. Initially all the floats are equally immersed. If a load $W$ tonnes be placed on the girder at $B$, show that the proportion of the load carried by the central float is

$$W(1 + \frac{3EI\ell}{a^3})$$

$$1 + \frac{9EI\ell}{a^3}$$

where $2a$ is the length of the girder.

12. A steel strip of uniform cross-section is laced through a row of five rigid pegs. The pegs are of same diameter and are spaced at equal intervals, with their centres on a straight line. Neglecting the weight of the strip, show that the reactions on the pegs are in the ratio of $5 : 16 : 22 : 16 : 5$.

13. A straight uniform beam $ABCD$ of length $4l$ is freely supported at its ends $A$ and $D$, and at two intermediate supports $B$ and $C$, distance $l$ from either end. The supports at $A$ and $D$ are rigid but those at $B$ and $C$ are such that they deflect by an amount $\lambda$ for each unit of load which is placed upon them. The beam carries a uniformly distributed load $w$/unit length along its entire length.

Show that the reactions at supports are

$$\frac{wl}{8} \left[ \frac{7l^3 + 48EI\lambda}{4l^3 + 4EI\lambda} \right]$$

and

$$\frac{3wl}{8} \left[ \frac{19l^3}{4l^3 + 3EI\lambda} \right].$$

14. A beam of constant flexural rigidity $B$ and length $2L$ carries a uniformly distributed load $w$/unit length and is supported on three springs, one at its centre and one at each end. When the load is applied to the beam the end springs are found to deflect $L/100$ and the centre spring to deflect $3L/100$. Find the deflection constants of the springs in terms of $B$, $L$ and $w$.

$$\left[ \text{Ans. Centre } 41.63w - \frac{4B}{L^3}, \text{ outer } 37.5w + \right.$$ 

15. A uniform beam of length $2l$ is supported at three points $A$, $B$ and $C$ distance $l$ apart. The supports are at the same level when beam carries no load. When the beam is loaded the reactions at each of the points of support is $1/k$ times the deflection at that point.

Prove that the reaction at $B$ due to two equal loads $W$, applied at distances $a$ and $c$ from $A$ and $C$ respectively, is

$$\frac{12kEI + 3l^3(a + c) - a^3 - c^3}{18kEI + 2l^3} W$$

where $I$ is the moment of inertia of the cross-section of the beam.
16. A beam of length $2a$ and flexural rigidity $EI$ carries a uniformly distributed load $w/\text{unit length}$ and rests on 3 supports, one at each end and one in the middle. Assuming that the beam was straight before loading, show that, for the greatest $+ve$ B.M. to be as small as possible, the central support must be

$$w\left(8\sqrt{2} - 11\right) a^4 \frac{2a}{24EI}$$

lower than the end support which are at the same level.

17. A beam $ABCD$ 9.15 m. long, is simply supported at $A$, $B$ and $C$ such that span $AB$ is 3.05 m., span $BC$ is 4.58 m. and the overhang $CD$ is 1.52 m. It carries a uniformly distributed load of 1790 kg/m, in span $AB$ and a point load of 908 kg. at the free end $D$. The moment of inertia of the beam in span $AB$ is $I$ and that in span $BC$ is $2I$. Draw the B.M. and S.F. diagrams for the beam. (A.M.I.E., Nov. 1968)

18. A beam $ABCD$ is 9 m. long and is simply supported at $A$, $B$ and $C$. Span $AB$ is 3 m., $BC$ is 4.5 m., and overhang $CD$ is 1.5 m. Moment of inertia of beam in span $AB$ is $I$ and that in span $BC$, $2I$. It is loaded with a uniformly distributed load of 2000 kg/m, in span $AB$ and with a point load of 1000 kg. at the free end $D$. Draw the bending moment and shear force diagrams for the beam. (A.M.I.E., May 1969)

19. A beam of length $L$ is supported at ends $A$ and $C$ and at its centre $B$. It is loaded with a uniform load $W$ per unit length over its entire length. The $EI$ for the beam is the same throughout. The three supports are at the same level initially, but under load, the central support sinks by 's' relative to other supports. Draw the B.M. and S.F. diagrams for the beam and point out the values of maximum B.M. in the beam. (A.M.I.E. Nov., 1969)

20. A beam $ABCD$ has a built-in support at $A$ and roller support at $B$, $C$ and $D$, $DE$ being an overhang. $AB = 7$ m., $BC = 5$ m., $CD = 4$ m. and $DE = 1.5$ m. The values of $I$, the moment of inertia of the section, which is uniform over each of these lengths are $3I'$, $2I'$, $I'$ and $I'$ respectively. The beam carries a point load of 10 t at a point 3 m. from $A$, a U.D. load of 4.5 t/m, over whole of $BC$, a concentrated load of 9 t on $CD$, 1.5 m. from $C$ and another point load of 3 t at $E$, the tip of the overhang.

Determine :-

(i) The moments developed over each of supports $A$, $B$, $C$ and $D$.

(ii) Draw the B.M. diagram for the beam stating values at salient points. (A.M.I.E., Nov. 1970)
ELASTIC THEOREMS AND ENERGY PRINCIPLES

4.1. Elastic materials which follow Hooke's law, obey certain principles. These are useful in the analysis of structures.

The general principles are
(i) Principle of superposition.
(ii) Reciprocal deflection theorem or Clark Maxwell's theorem.
(iii) Betti's theorem.
(iv) Castigliano's theorem.
(v) Müller Breslau's Principle.

4.2. Principle of Superposition.

For structures subjected to multiple loads, the total effect as for S.F., B.M., and deflections concerned will be same as sum of effects due to individual loads taken separately. This principle holds good only when deflections are linear functions of the applied loads.

Consider beam $AB$ subjected to loads $W_1$ and $W_2$ as shown in Fig. 4.1.

Taking moments about $B$,

$$R_A = \frac{W_1a + W_2b}{l}$$

$$R_B = \frac{(W_1 + W_2) - \frac{W_1a + W_2b}{l}}{l} + \frac{W_1(l-a)}{l} + \frac{W_2(l-b)}{l}$$

Fig. 4.1
S.F. and B.M. at a section distance $x$ from $B$ are

$$F_X = W_1 - R_B$$
$$= \frac{W_1}{l} - \frac{W_2(l-b)}{l}$$

$$M_X = R_B x - W_1(x-a)$$
$$= \frac{W_1}{l} (l-a) + \frac{W_2}{l} (l-b) x - W_1 (x-a)$$
$$= \frac{W_1}{l} a(l-x) + \frac{W_2}{l} b lx$$

Now consider loads $W_1$ and $W_2$ applied separately to beam $AB$ as shown in Fig. 4.1 (b) and (c).

Due to load $W_1$ on beam $AB$,

$$R_{B1} = W_1 \left(\frac{l-a}{l}\right)$$

$$F_{X1} = -\frac{W_1(l-a)}{l} + W_1$$

$$M_{X1} = \frac{W_1}{l} (l-a)x - W_1(x-a)$$
$$= \frac{W_1}{l} a(l-x)$$

Due to load $W_2$ on beam $AB$,

$$R_{B2} = \frac{W_2(l-b)}{l}$$

$$F_{X2} = -\frac{W_2}{l} (l-b)$$

$$M_{X2} = \frac{W_2}{l} (l-b)x$$

$$R_{B1} + R_{B2} = \frac{W_1}{l} (l-a) + \frac{W_2}{l} (l-b) = R_B$$

$$F_{X1} + F_{X2} = \frac{W_1}{l} a - \frac{W_2}{l} (l-b) = F_X$$

$$M_{X1} + M_{X2} = \frac{W_1}{l} a(l-x) + \frac{W_2}{l} (l-b)x = M_X$$

Thus it is seen that the total effect of several loads is equal to the sum of the effects of each load taken separately.

It is already shown that when original beam is loaded with bending moment diagram, the S.F. and B.M. at any point in the conjugate beam give slope and deflection respectively in the original beam. As the S.F. and B.M. at any point is the sum of effects
of all the loads taken separately, the deflection at any point will be the sum of deflections produced by individual loads taken separately.

**Ex. 4.1.** Find the reactions in the continuous beam loaded as shown in Fig. 4.2 (a).

**Solution.** Let $R_B$ be the reaction at $B$. Consider the beam $AB$ as loaded downward with $w$/unit length and with upward load $R_B$ at $B$. The two loadings are shown separately in Fig. 4.2 (b) and (c).

Downward deflection of $B$ due to downward load

$$
\delta_{B1} = \frac{5}{384} \frac{w(2l)^4}{EI} \\
= \frac{5wl^4}{24EI}
$$

Upward deflection of $B$ due to $R_B$,

$$
\delta_{B2} = \frac{R_B(2l)^3}{48EI} = \frac{R_Bl^3}{6EI}
$$

As there is no sinking of support $B$,

$$
\delta_{B1} = \delta_{B2} \\
= \frac{5wl^4}{24EI} \cdot \frac{R_Bl^3}{6EI}
$$

$$
R_B = \frac{w}{4} \cdot \frac{wl}{3}
$$

$$
R_A = R_C = \frac{1}{2} \left(2wl - \frac{5}{4}wl\right)
$$

**Ex. 4.2.** A fixed beam, span 4 m. is loaded with uniformly distributed load of 1.5 t/m throughout the span. During loading left support rotates clockwise by 0.001 radian, determine the fixed end moments and draw B.M. diagram $EI = 1 \times 10^{10}$ kg cm$^2$. 

Solution. The fixed end moments in the beam can be calculated by superposing fixed end moments due to rotation of support on fixed end moments due to loading only.

Due to uniformly distributed load, fixed end moments will be

$$M_{A_1} = M_{B_1} = -\frac{wl^2}{12}$$

$$= -\frac{1500 \times 4^2}{12}$$

$$= -2000 \text{ kg. m}$$

Due to loading

$$M_{A_2} = \frac{4EI \theta}{l}$$

$$= \frac{4 \times 1 \times 1 \times 10}{(100)^2} \times \frac{0.001}{4}$$

$$= 1000 \text{ kg. m}$$

$$M_{B_2} = -\frac{2EI \theta}{l}$$

$$= -\frac{2 \times 1 \times 10}{(100)^2} \times \frac{0.001}{4}$$

$$= -500 \text{ kg. m}$$

Total moments are

$$M_A = M_{A_1} + M_{A_2}$$

$$= -2000 + 1000$$

$$= -1000 \text{ kg. m}$$

$$M_B = M_{B_1} + M_{B_2}$$

$$= +2000 - 500$$

$$= -2500 \text{ kg. m}$$

4.3. Reciprocal Deflection Theorem

Consider a body subjected to load $W$ at $A$. Let $\delta_1$ be the deflection produced by load $W$ in direction 1 at $B$. Now if the load at $A$ is removed and applied at $B$ in direction 1, the deflection at $A$, in the direction in which the load $W$ applied at $A$, will also be $\delta_1$.

This is known as Clark Maxwell's theorem of reciprocal deflection. Simple proof of the above theorem for case of a cantilever is given below.

Consider a cantilever $AB$, loaded with load $W$ at free end $B$. The deflection at $C$, distance $a$ from $A$ is equal to moment of $\frac{M}{EI}$ diagram between $A$ and $C$ about $C$. 


Now consider same cantilever loaded with load $W$ at $C$. The deflection at $B$ will be equal to

\[
\delta_C = \frac{1}{EI} \left[ \frac{W(l-a)}{2} +Wa \times \frac{a}{2} \times \frac{2}{3} a \right] \\
\frac{Wa^2}{2EI} \left[ l-a+\frac{2}{3} a \right] \\
\frac{wa^2}{2EI} \left( l-\frac{a}{3} \right) \quad \ldots \quad (1)
\]

Thus

\[
\delta_B = \frac{Wa \times \frac{a}{2}}{EI} \left( l-\frac{a}{3} \right)
\]

\[
= \frac{Wa^2}{2EI} \left( l-\frac{a}{3} \right)
\]

**Ex. 4:3.** A beam $AB$ of length $l$ m. long is simply supported on rigid supports at its ends and the centre rests across the free end of a cantilever $BD$ of length $l/4$. The flexural rigidity of the beam is twice that of cantilever. Obtain the influence line for central reaction when a concentrated unit load rolls over the beam.

**Solution.** When load moves on beam $AB$, cantilever $CD$ will offer upward reaction $R_C$ at $C$ to the beam. The load acting on the cantilever will be downward reaction $R_C$. For any position of load, downward deflection of beam $AB$ at $C$ will be equal to downward deflection of cantilever $CD$ at $C$. 
Consider beam $AB$ without central support at $C$. Let the unit load act at distance $x$ from $A$. By reciprocal theorem deflection at $C$ when the load is at $X$ will be same as deflection at $X$ when load is at $C$.

Consider Fig. 4.8 (b) when the unit load acts at $C$. As the load is acting symmetrically about centre line, tangent at $C$ will be horizontal.

Displacement $X$ above the tangent at $C$ will be equal to moment of B.M. diagram between $X$ and $C$ about $X$

$$= \frac{1}{EI} \left[ \frac{x}{2} \left( \frac{l}{2} - x \right) \left( \frac{l}{2} - x \right) \right]$$

$$+ \frac{1}{2} \left( \frac{l}{2} - x \right) \left( \frac{l}{4} - \frac{x}{2} \right) \times \frac{2}{3} \left( \frac{l}{2} - x \right)$$

$$= \frac{1}{EI} \left[ \frac{x}{4} \left( \frac{l}{2} - x \right) \right]$$

$$= \frac{(l-x)^2}{4EI} \left[ \frac{x}{3} + \frac{l}{3} \right]$$

$$= \frac{(l-2x)^2(l+x)}{48EI}$$

Actual deflection at $X$ will be central deflection at $C$ minus displacement of $X$ above tangent at $C$,

$$\delta_x = \frac{l^3}{48EI} - \frac{(l-2x)^2(l+x)}{48EI}$$

This will be deflection of $C$ when unit load is at $X$. Let the reaction at $C$ due to restraint provided by cantilever be $R_C$ when the unit load is at $X$. The upward deflection of $C$ due to reaction $R_C$ will be

$$\frac{R_C l^3}{48EI}$$

Net downward deflection of point $C$ is

$$\frac{l^3}{48EI} - \frac{(l-2x)^2(l+x)}{48EI} = \frac{R_C l^3}{48EI} \quad \ldots(1)$$

This deflection will be equal to downward deflection of cantilever.
Downward deflection of cantilever

\[ \frac{R_{C}(l/4)^3}{3 \times \frac{EI}{2}} = \frac{R_{C}l^3}{96EI} \]

...\(2\)

Equating (1) and (2)

\[ \frac{l^3}{48EI} - \frac{(l-2x)^3(l+x)}{48EI} - \frac{R_{C}l^3}{48EI} = \frac{R_{C}l^3}{96EI} \]

\[ \therefore \frac{l^3-(l-2x)^3(l+x)-R_{C}l^3}{2} \]

\[ \therefore \frac{3}{2} R_{C}l^3 = l^3-l^3+3l^2x-4x^3 \]

\[ \therefore R_{C} = \frac{2}{3}\left(3l^2x-4x^3\right) = \frac{2x}{3}\left(3l^2-4x^3\right) \]

This represents equation of \(R_{C}\). When the value of \(x\) is zero i.e. load is at \(A\), \(R_{C}\) is zero. Maximum value of \(R_{C}\) is \(2/3\) and occurs when load is at \(C\). The influence line is shown in Fig. 4.8 (d).

**Ex. 4.4.** A uniform beam shown in Fig. 4.9 is supported at 3 points \(A\), \(B\) and \(C\), which are at the same level when the beam carries no load. When the beam is loaded the reaction at each of the points of supports is \(1/k\) times the deflection at that point. Prove that the reaction at \(B\) due to two equal loads \(W\), applied at the points indicated is

\[ \frac{12kEI-3L^3(a+c)-a^2-c^2}{18kEI+2L^3} \]

**Solution.** Consider beam \(AC\) without central support \(B\).

Let the load \(W\) act at \(B\). By reciprocal theorem deflection at \(B\) when load is at \(X\) will be same as deflection at \(X\) when load is at \(B\).

Consider beam with central support \(B\) removed and downward load \(W\) acting at \(B\).

As the load is acting symmetrically about centre line, tangent at \(B\) will be horizontal. Displacement of \(X\) above tangent at \(B\) will be equal to moment of B.M. diagram between \(X\) and \(B\) about \(X\).

![Fig. 4.9](image-url)
ANALYSIS OF STRUCTURES

\[
\delta_x = \frac{W(2L)^3}{48EI} - \frac{W(L-x)^3(x+2L)}{12EI}
\]

\[
= \frac{W}{12EI} [3L^2x - x^3]
\]

...(1)

In the continuous beam \(ABC\), let \(R_A, R_B\) and \(R_C\) be reactions at \(A\), \(B\), and \(C\) respectively

\(R_A + R_B + R_C = 2W\)

\(R_A + R_C = 2W - R_B\)

Reaction at each support is \(1/k\) times the deflection at each support.

\[\delta_A = kR_A, \delta_B = kR_B \text{ and } \delta_C = kR_C\]

where \(\delta_A, \delta_B\) and \(\delta_C\) are settlements of supports \(A, B\) and \(C\) respectively

\[\delta_A + \delta_C = kR_A + kR_C = k(2W - R_B)\]

Deflection at \(B\) due to \(W\) at \(D\) is equal to deflection at \(D\) due to \(W\) at \(B\).

\[\therefore \text{ Due to } W \text{ at } D \text{ downward deflection of } B \text{ will be } \frac{W}{12EI} (3L^2a - a^3) \text{ [substituting } x=a \text{ in (1).]}\]

Similarly downward deflection of \(B\) due to \(W\) at \(E\) will be \(\frac{W}{12EI} (3L^3c - c^3)\).

Upward deflection of \(B\) due to upward reaction

\[R_B = \frac{R_B (2L)^3}{48EI} = \frac{R_B L^3}{6EI}\]

Due to settlement \(\delta_A\) of \(A\) and \(\delta_C\) of \(C\), the downward deflection of \(B\) will be \(\frac{\delta_A + \delta_C}{a} \).
Net downward deflection of \( R_B \) is

\[
\delta_B = \frac{W}{12EI} \left( 3L^2a - a^3 \right) + \frac{W}{12EI} \left( 3L^2c - c^3 \right) - \frac{R_B \cdot L^3}{6EI} + \frac{1}{2} (\delta_A + \delta_C)
\]

\[
kR_B = \frac{W}{12EI} \left[ 3L^2(a+c) - a^3 - c^3 \right] - \frac{R_B L^3}{6EI} + \frac{k}{2} (2W - R_B)
\]

\[
\frac{3}{2} kR_B + \frac{R_B L^3}{6EI} = \frac{W}{12EI} [3L^2(a+c) - a^3 - c^3] + kW
\]

\[
= \frac{W}{12EI} [3L^2(a+c) - a^3 - c^3 + 12kEI]
\]

\[
R_B = \frac{W}{12EI} \frac{[12kEI + 3L^2(a+c) - a^3 - c^3]}{18kEI + 2L^3}
\]

4.4. Betti’s Theorem

This law states that in an elastic structure with unyielding supports and at constant temperature, the work done on a given structure by a system of 1st loading on the corresponding displacements of the 2nd loading is equal to the work done by 2nd loading on the displacement of the 1st loading.

Let \( P_1 \) and \( P_2 \) be the first loading and \( P_3 \) and \( P_4 \) be the second loading acting at points 1, 2, 3, and 4 respectively. Let \( \delta_1', \delta_2', \delta_3', \delta_4' \) be the deflections at points 1, 2, 3 and 4 in the directions of the respective loads due to first loading and \( \delta_1'', \delta_2'', \delta_3'', \delta_4'' \) be the deflections due to the second loading in direction of respective loads.

If the forces \( P_1, P_2, P_3 \) and \( P_4 \) be applied gradually and simultaneously at points 1, 2, 3 and 4 respectively, the deflections at the four points will be \( (\delta_1' + \delta_1''), (\delta_2' + \delta_2''), (\delta_3' + \delta_3'') \) and \( (\delta_4' + \delta_4'') \).

The work done when only \( P_1 \) and \( P_2 \) are applied,

\[
U_1 = \frac{1}{2} P_1 \delta_1' + \frac{1}{2} P_2 \delta_2'
\]

The work done on the structure when \( P_3 \) and \( P_4 \) are gradually applied, while the forces \( P_1 \) and \( P_2 \) are still there,

\[
U_2 = \frac{1}{2} P_3 \delta_3' + \frac{1}{2} P_4 \delta_4' + P_1 \delta_1'' + P_2 \delta_2''
\]
The work done when \( P_1, P_2, P_3 \) and \( P_4 \) are gradually applied will be equal to
\[
U = \frac{1}{2} P_1 (\delta_1' + \delta_1'') + \frac{1}{2} P_2 (\delta_2' + \delta_2'') + \frac{1}{2} P_3 (\delta_3' + \delta_3'') + \frac{1}{2} P_4 (\delta_4' + \delta_4'')
\]

As the work done remains the same when the loads are applied simultaneously or in any other order,
\[
U = U_1 + U_2
\]
\[
\therefore \frac{1}{2} P_1 (\delta_1' + \delta_1'') + \frac{1}{2} P_2 (\delta_2' + \delta_2'') + \frac{1}{2} P_3 (\delta_3' + \delta_3'') + \frac{1}{2} P_4 (\delta_4' + \delta_4'')
\]
\[
= \frac{1}{2} P_1 \delta_1' + \frac{1}{2} P_2 \delta_2' + \frac{1}{2} P_3 \delta_3'' + \frac{1}{2} P_4 \delta_4'' + P_1 \delta_1'' + P_2 \delta_2''
\]
\[
\therefore \frac{1}{2} P_3 \delta_3' + \frac{1}{2} P_4 \delta_4' = \frac{1}{2} P_1 \delta_1'' + \frac{1}{2} P_2 \delta_2''
\]
\[
\therefore \frac{1}{2} P_1 \delta_1'' + \frac{1}{2} P_2 \delta_2'' = P_3 \delta_3' + P_4 \delta_4'
\]

Thus work done by first loading on the deflections caused by the second loading is equal to the work done by second loading on the deflections caused by the first loading.

The above relation holds good for generalised forces and deflections i.e. for moments and angular deflections in the directions of the moments as well as for forces and linear deflections in the direction of forces.

4.5. Castigliano's Theorems.

First Theorem.

The first theorem of Castigliano states that the partial derivative of the total strain energy in any structure with respect to applied force or moment gives the displacement or rotation respectively at the point of application of the force or moment in the direction of applied force or moment.

Consider a body subjected to forces \( P_1, P_2 \) and \( P_3 \) as shown in Fig. 4.11. Let the displacements be \( \Delta_1, \Delta_2 \) and \( \Delta_3 \) in the directions of \( P_1, P_2 \) and \( P_3 \) respectively. Strain energy stored will be equal to
\[
U = \frac{P_1 \Delta_1}{2} + \frac{P_2 \Delta_2}{2} + \frac{P_3 \Delta_3}{2}
\]
\[
\therefore 2U = P_1 \Delta_1 + P_2 \Delta_2 + P_3 \Delta_3 \tag{1}
\]

Now let the force \( P_1 \) be increased by an amount \( \delta P_1 \). This increment of force will cause additional displacements in directions of \( P_1, P_2 \) and \( P_3 \). These displacements in direction of \( \Delta_1, \Delta_2 \) and \( \Delta_3 \) will be
\[
\delta \Delta_1 = \frac{\partial \Delta_1}{\partial P_1} \delta P_1, \; \delta \Delta_2 = \frac{\partial \Delta_2}{\partial P_1} \times \delta P_1
\]
and
\[
\delta \Delta_3 = \frac{\partial \Delta_3}{\partial P_1} \delta P_1
\]