The extra energy stored will be
\[ \frac{\partial U}{\partial P_1} \times \delta P_1 = P_1 \delta \Delta_1 + P_2 \delta \Delta_2 + P_3 \delta \Delta_3 \]
\[ P_1 \times \frac{\partial \Delta_1}{\partial P_1} \times \delta P_1 + P_2 \times \frac{\partial \Delta_2}{\partial P_1} \times \delta P_1 + P_3 \times \frac{\partial \Delta_3}{\partial P_1} \times \delta P_1 \]
\[ \frac{\partial U}{\partial P_1} = P_1 \frac{\partial \Delta_1}{\partial P_1} + P_2 \frac{\partial \Delta_2}{\partial P_1} + P_3 \frac{\partial \Delta_3}{\partial P_1} \]

Differentiating equation (1) with respect to \( P_1 \),
\[ 2 \times \frac{\partial U}{\partial P_1} = \Delta_1 + P_1 \frac{\partial \Delta_1}{\partial P_1} + P_2 \frac{\partial \Delta_2}{\partial P_1} + P_3 \frac{\partial \Delta_3}{\partial P_1} \]

Subtracting (2) from (3)
\[ \frac{\partial U}{\partial P_1} = \Delta_1 \]

The above theorem can also be proved as follows:
The energy stored when \( P_1, P_2, \) and \( P_3 \) are applied gradually, is
\[ U = \frac{P_1 \Delta_1}{2} + \frac{P_2 \Delta_2}{2} + \frac{P_3 \Delta_3}{2} \]

Now when force \( P_1 \) is increased by \( \delta P_1 \), let the increase in displacements in directions of \( P_1, P_2, \) and \( P_3 \) be \( \delta \Delta_1, \delta \Delta_2, \) and \( \delta \Delta_3 \), respectively. The additional energy stored, neglecting terms like \( \delta P_1 \times \delta \Delta_1 \) will be
\[ \delta U = P_1 \delta \Delta_1 + P_2 \delta \Delta_2 + P_3 \delta \Delta_3 \]

Total energy stored will be
\[ U + \delta U = \frac{P_1 \Delta_1}{2} + \frac{P_2 \Delta_2}{2} + \frac{P_3 \Delta_3}{2} + P_1 \delta \Delta_1 + P_2 \delta \Delta_2 + P_3 \delta \Delta_3 \]

Now if the forces \( (P_1 + \delta P_1) \), \( P_2 \text{ and } P_3 \) are applied gradually, then by principle of superposition the deflections and strain energy stored should be the same as in first case.
\[ (U + \delta U) = \frac{(P_1 + \delta P_1)(\Delta_1 + \delta \Delta_1)}{2} + \frac{P_2(\Delta_2 + \delta \Delta_2)}{2} + \frac{P_3(\Delta_3 + \delta \Delta_3)}{2} \]

Expanding and neglecting terms like \( \delta P_1 \times \delta \Delta_1 \)
\[ (U + \delta U) = \frac{P_1 \Delta_1}{2} + \frac{P_1 \delta \Delta_1}{2} + \frac{\delta P_1 \Delta_1}{2} + \frac{P_2 \Delta_2}{2} + \frac{P_2 \delta \Delta_2}{2} + \frac{P_3 \delta \Delta_3}{2} \]

Equating (4) and (5)
\[ \frac{P_1 \delta \Delta_1}{2} + \frac{P_2 \delta \Delta_2}{2} + \frac{P_3 \delta \Delta_3}{2} = \frac{\delta P_1 \Delta_1}{2} \]
\[ \therefore \quad P_1 \delta \Delta_1 + P_2 \delta \Delta_2 + P_3 \delta \Delta_3 = \delta P_1 \Delta_1 \]
\[ \therefore \quad \frac{\delta U}{\delta P_1} = \Delta_1. \]
Thus the partial derivative of strain energy with respect to
$P_1$ gives displacement in the direction of $P_1$.

**Second Theorem of Castigliano**

The second theorem of Castigliano states that the work done
by external forces in a structure will be minimum. This theorem
is very useful in analysis of indeterminate structures.

Let $W$ be the work done by external forces on structure, $U$
be the strain energy stored in the structure and $W_1$ be the work
done by the reactive forces.

\[
\text{Strain energy } U = W + W_1
\]

\[
\therefore
W = U - W_1.
\]

By Castigliano's second theorem $W$ should be minimum. Thus
the partial derivative of the work done with respect to external
forces will be zero.

In case the supports are unyielding the work done by re-
active forces will be zero. Strain energy stored will be equal to
the work done by external forces and will be the minimum. Thus
the partial derivative of strain energy with respect to redundant
reaction will be zero.

The application of these theorems have been shown in
Chapters 5 and 7.

**Ex. 4:5.** Analyse the frame shown in Fig. 4:12 for clockwise
rotational yield of 0 at $A$, vertical yield of $\delta_3$ at $D$ and horizontal
yield of $\delta_1$, towards $A$, at $D$. $EI$ is constant for the frame.

**Solution.** Let $V$ and $H$ be vertical reaction and horizontal
reaction respectively at $D$ in the directions shown in Fig. 4:12 (b)
Let $M_A$ be moment at $A$.

\[H_A = H\]

Taking moments about $A$

\[M_A - V \times l = 0\]

\[\therefore M_A = Vl\]

Total strain energy for the frame will consist of strain energy
for the portions $AB$, $BC$ and $CD$. The bending moment operations
and origin of reference for three portions are shown in Table 4:1.

Strain energy

\[
= \int \frac{M^2}{2EI} dx
\]
Strain energy

\[ U = \frac{1}{2EI} \int_0^l (Hy)^3 dy + \frac{1}{2EI} \int_0^l (Hl-Vx)^3 dx + \frac{1}{2EI} \int_0^l (Vl-Hy)^3 dy \]

<table>
<thead>
<tr>
<th>Portion</th>
<th>Origin at</th>
<th>Moment</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>DO</td>
<td>D</td>
<td>Hx(y)</td>
<td>0(-l)</td>
</tr>
<tr>
<td>CB</td>
<td>C</td>
<td>Hx(l) - Vx(a)</td>
<td>0(-l)</td>
</tr>
<tr>
<td>AB</td>
<td>A</td>
<td>(M_A - Hx\y \equiv Vx\l - Hx\y)</td>
<td>0(-l)</td>
</tr>
</tbody>
</table>

\[ \therefore U = \frac{1}{2EI} H^2 \left[ \left( \frac{y^3}{3} \right)_0^l \right] + \frac{1}{2EI} \int_0^l (H^2l^3 + V^2x^3 - 2HlVx) dx + \frac{1}{2EI} \int_0^l (V^2l^3 + H^2y^3 - 2HlVy) dy \]

\[ = \frac{1}{2EI} \times \frac{H^2l^3}{3} + \frac{1}{2EI} \left[ \frac{H^2l^3 x}{3} - HlVx^2 \right]_0^l \]

\[ + \frac{1}{2EI} \left[ \frac{V^2l^3 y + H^2y^2}{3} - HlVy^3 \right]_0^l \]

\[ = \frac{1}{2EI} \left[ \frac{H^2l^3}{3} + H^2l^3 + \frac{V^2l^3}{3} - HlV^3 + \frac{H^2l^3}{3} - HlV^3 \right] \]

\[ = \frac{1}{2EI} \left[ \frac{5}{3} H^2l^3 + \frac{4}{3} V^2l^3 - 2HlV^3 \right] \]

Work done by supports \( W_1 = Hx\delta_1 - Vx\delta_2 + M_A \times \theta \)

\[ = H\delta_1 - V\delta_2 + Vl\theta. \]

Work done by external load \( W = U - W_1 \)

\[ \therefore W = \frac{1}{2EI} \left( \frac{5H^2l^3}{3} + \frac{4}{3} V^2l^3 - 2HlV^3 \right) - H\delta_1 + V\delta_2 - Vl\theta \]

Partial derivative of work done with respect to \( H \) and \( V \) should be zero.

\[ \frac{\delta W}{\delta H} = \frac{1}{2EI} \times \frac{10Hl^3}{3} - \frac{2Vl^3}{2EI} - \delta_1 \]

\[ \frac{\delta W}{\delta V} = \frac{1}{2EI} \times \frac{8}{3} Vl^3 - \frac{1}{2EI} \times 2Hl^3 + \delta_1 - \theta = 0 \]

Equation (1) gives

\[ \frac{5Hl^3}{3} \frac{Vl^3}{2EI} + \delta_1 \]

\[ \therefore H = \frac{3}{5} V + \frac{3EI\delta_1}{5l^3} \]
Substituting in (2)
\[
\frac{4}{3EI} Vl^3 - \left( \frac{3}{5} \frac{Vl^3}{EI} + \frac{3}{5} \delta_1 \right) + \delta_2 - l\theta = 0
\]
\[
\therefore V \left( \frac{4}{3EI} l^3 - \frac{3l^3}{5EI} \right) = l\theta - \delta_2 + \frac{3}{5} \delta_1
\]
\[
\therefore \frac{11Vl^3}{15EI} = l\theta - \delta_2 + \frac{3}{5} \delta_1
\]
\[
V = \frac{15EI}{11l^5} [l\theta - \delta_2 + \frac{3}{5} \delta_1]
\]
\[
= \frac{3EI}{11l^5} [5l\theta - 5\delta_2 + 3\delta_1]
\]
\[
H = \frac{9EI}{11l^5} [l\theta - \delta_2 - \frac{3}{5} \delta_1] + \frac{3EI\delta_1}{5l^3}
\]
\[
= \frac{3EI}{55l^5} [15l\theta - 15\delta_2 + 20\delta_1]
\]
\[
= \frac{3EI}{11l^5} (3l\theta - 3\delta_2 + 4\delta_1).
\]


This states that the influence line for any stress function of a structure, such as S.F. and B.M. or any reactive force or moment, is given by the deflected curve of the structure obtained by imposing a unit distortion in the direction of the stress function.

To draw influence line for reaction \( R_B \), a displacement equal to unity is given in the direction of \( R_B \) as shown in Fig. 4.13.

By Müller-Breslau Principle,
\( R_B \times 1 - 1 \times y = 0 \).

\[
R_B = y, \text{ thus ordinate of deflected form will be the influence line for } R_B.
\]

To draw influence line for \( M_A \) a rotation equal to unity is given in the direction of \( M_A \) shown in Fig.

By Müller-Breslau Principle
\( M_A \times 1 - 1 \times x = 0 \).

\[
M_A = v.
\]
Similarly to draw influence line for $R_A$, a displacement equal to unity is given in the direction of $R_A$ as shown in Fig. 4.15. The deflected form has zero slope at $A$, so that work done by $M_A$ is zero.

$$R_A \times 1 - 1 \times y = 0$$

$$R_A = y$$

![Fig. 4.15](image)

To draw influence line for shear at intermediate point $C$, the beam is cut at $C$ and displacement equal to unity is given at $C$ in direction of shear force in such a way that the slopes of the two portions at $C$ should be same so that the net work done by $M_C$ is zero as shown in Fig. 4.16.

$$y_1 + y_2 = 1$$

$$F_C(y_1 + y_2) - 1 \times y = 0$$

$$F_C = y$$

![Fig. 4.16](image)

To draw influence line for $M_C$, the beam is cut at $C$ and a hinge is inserted and rotation is given equal to unity as shown in Fig. 4.17.

$$M_C \theta - 1 \times y = 0$$

$$\theta = 1$$

$$M_C = y.$$

![Fig. 4.17](image)


For an elastic material if the load increased gradually and corresponding deformations are measured and a graph plotted, in case of a linear elastic material the curve will be a straight line as shown in Fig. 4.18 (a), and for non-linear elastic material, the curve will be as shown in Fig. 4.18 (b).

The area below the curve is the work done is called strain energy and the area above the curve is known as complementary energy.
Linear elastic material
Fig. 4·18 (a)

\[ U = \text{strain energy} \quad \int P d\Delta \]

\[ C = \text{complementary energy} = \int \Delta dP. \]

For linear elastic material

\[ U = \int P d\Delta \quad \text{and} \quad \Delta = \frac{PL}{AE} \quad \therefore \quad P = \frac{\Delta AE}{L} \]

\[ = \int \frac{\Delta AE}{L} d\Delta \]

\[ = \frac{\Delta^2 AE}{2L} = \frac{PL}{AE} \]

\[ \therefore U = \int \frac{PL}{AE} dP \]

\[ = \frac{PL}{AE} \int \frac{L}{P} dP \]

\[ = \frac{P^2 L}{2AE} = \Delta. \]

4·8. First Theorem of Complementary Energy.

The partial derivative of complementary energy with respect to the given force gives the displacement in the direction of force.

Let \( P_1, P_2, \ldots, P_n \) be the forces and \( \Delta_1, \Delta_2, \ldots, \Delta_n \) be the deflections respectively.

Let the force \( P_n \) be increased by \( \delta P_n \). The change in complementary energy will be \( \frac{\partial C}{\partial P_n} \times \delta P_n \). The change in complementary energy will be due to change in force \( P_n \).

From Fig. 4·18, the change in complementary energy will be equal to \( \delta P_n \times \Delta_n \).

\[ \therefore \frac{\partial C}{\partial P_n} \delta P_n = \delta P_n \times \Delta_n \]

\[ \therefore \frac{\partial C}{\partial P_n} = \Delta_n. \]

This theorem is applicable to both linear and non-linear elastic structures.

Ex. 4·6. Find the vertical deflection for the structure consisting of 3 bars of same linear elastic material and having same area. They are symmetrically placed as shown in Fig. 4·20, under a load \( P \). Find the force taken by the central bar.
Solution. Let $kP$ be the force taken by the central bar. The force taken by inclined bars will be

$$\frac{(1-k)}{2} P \times \sec \theta.$$ 

Total complementary energy

Total strain energy

$$C = \left( \frac{kP}{2AE} \right)^2 \left[ \frac{(1-k)P \cos \theta}{2} \right]^2 \times L \sec \theta$$

$$= \frac{k^2LP^2}{2AE} + \left( 1 - \frac{k^2}{2} \right) \frac{LP sec^2 \theta}{4AE}$$

$$\frac{\partial C}{\partial P} = k^2 \frac{PL}{AE} + \left( 1 - \frac{k^2}{2} \right) \frac{LP sec^3 \theta}{2AE}$$

Now $\Delta = \frac{k^2PL}{AE}$

$$\therefore \frac{kPL}{AE} = k^2 \frac{PL}{AE} + \left( 1 - \frac{k^2}{2} \right) \frac{LP sec^3 \theta}{2AE}$$

$$\therefore \ k = k^2 + \left( 1 - \frac{k^2}{2} \right) sec^3 \theta$$

$$\frac{k - k^3}{2} \sec^3 \theta = \frac{k}{1-k} \sec^3 \theta$$

$$\frac{PL}{AE (2 \cos^3 \theta + 1)}$$

$$kP \left( \frac{P}{2 \cos^3 \theta + 1} \right)$$  the force in the central bar.

Ex. 4.7. Find the forces in the members shown in Fig. 4.21. The area of each member is $A$ sq. cm. The structure consists of members which follow elastic non-linear curve given by $e = p^2 \times 10^{-18}$ where $p$ is the stress in kq/cm. and $e$ is strain. Take $F = 10,000$ kg.

Solution. Let $P$ be the force in $OB$ the central bar.
The force in \( OA \) and \( OC \) will be
\[
\left( \frac{F - P}{2} \right) \sqrt{2} = \frac{F - P}{\sqrt{2}}
\]

Complementary energy in the central bar
\[
C_1 = \int_0^P \Delta_n \; dP_n
\]

Now
\[
P = pA
\]

\[
\therefore \quad dP = Adp
\]

\[
C_1 = \int_0^P eLAdp
\]

\[
= LA \int_0^P p^2 \times 10^{-12} \; dp
\]

\[
= LP \times 10^{-12} \left[ \frac{p^3}{3} \right]_0^P
\]

\[
= LA \times 10^{-12} \frac{p^3}{3} \times \left( \frac{P}{A} \right)^3
\]

\[
= \frac{L \times 10^{-12} \; p^3}{3A^2}
\]

Similarly, complementary energy in the side bars,
\[
C_2 = 2 \left[ \frac{L \sqrt{2} \times 10^{-12}}{3A^2} \left( \frac{F - P}{\sqrt{2}} \right)^3 \right]
\]

\[
= \frac{L \times 10^{-12}}{3A^2} - (F-P)^3
\]

\[
\therefore \quad \text{Total complementary energy } C = C_1 + C_2
\]

\[
C = \frac{L \times 10^{-12}}{3A^2} \left( p^3 - (F-P)^3 \right)
\]

\[
\frac{\partial C}{\partial F} \quad \delta = \frac{L \times 10^{-12}}{3A^2} \left[ 3(F - P)^2 \right]
\]

where \( \delta \) is the central deflection.

For the central bar,
\[
\delta = Le = LP^2 \times 10^{-13} = \frac{LP^2}{A^2} \times 10^{-13}
\]

\[
\frac{L \times 10^{-12}}{3A^2} \left[ 3(F - P)^2 \right] = \frac{LP^2}{A^2} \times 10^{-12}
\]

\[
(F-P)^2 = \frac{P^2}{F-P} \quad P
\]

\[
\therefore \quad P = F/2 = 5000 \; \text{kg.}
\]

The force in side bar
\[
\frac{F - P}{\sqrt{2}} \quad \frac{5000}{\sqrt{2}} \quad 2500 \sqrt{2} \; \text{kg.}
\]

The central deflection \( \delta \)
\[
= \frac{L}{A^2} \times 5000^2 \times 10^{-13}
\]

\[
\frac{25 \times 10^{-6} L}{A^2} = \frac{L}{40,000 \; A^2}
\]
DEFLECTION OF FRAMES AND OTHER STRUCTURES

5.1. When a structure is loaded it deflects. It is necessary to determine the deflection of a structure and restrict it within permissible limits. From aesthetic point of view the deflections are restricted, as excessive deflection gives feeling of unsoundness of structure. In case of buildings the deflection of beams is restricted so that the plaster of ceiling does not crack. In case of bridges too much deflection which is visible with naked eyes creates a feeling that the bridge is not safe. Also from point of view of vibrations the deflection of bridges is to be restricted. For determinate structures relative displacement of supports has no bearing on stresses so long the deflections are small.

The deflection of beams has been fully dealt with in Analysis of Structures Vol. I by the methods of moment area, Macaulay's method, conjugate beam method and unit load method. The deflection of frames and other structures is dealt with by strain energy method in this chapter.

5.2. Deflection by Strain Energy.

The strain energy in a member will be due to

\[(i) \text{ Bending moment} = \int \frac{M^2 ds}{2EI}\]

\[(ii) \text{ Axial force} = \int \frac{P^2 ds}{2EA}\]

\[(iii) \text{ Due to shear} = \int \int \frac{q_y^2}{2N} bdyds \quad \text{(see Vol. I)}\]

where, \(M\) is the moment at the section of which moment of inertia is \(I\)

\(P\) is axial force and \(A\) is area.

\(q_y\) is the shear stress at \(y\) distance from the neutral axis where breadth is \(b\).

Total strain energy in a member will be

\[U' = \int \frac{M^2 ds}{2EI} + \int \frac{P^2 ds}{2AE} + \int \int \frac{q_y^2}{2N} bdyds\]

Generally strain energy due to shear is neglected as it is small. For members subjected primarily to bending, the strain energy due to axial forces is small and is generally neglected.
In case of pin jointed frames, members are subjected to axial forces and hence strain energy due to axial forces only is to be considered.

Total strain energy in a structure will be the sum of strain energies in all members.

\[ U = \Sigma \left( \int \frac{M^2 ds}{2EI} + \int \frac{P^2 ds}{2AE} + \int \frac{q^2}{2N} bdyds \right) \]

Partial derivative of strain energy with respect to a load gives deflection in the direction of load and with respect to a moment gives rotation in the direction of moment.

In case deflection or rotation is desired at a point where no force or moment is acting a fictitious force or a moment is applied and partial derivative with respect to the fictitious force or moment is worked out, the deflection or rotation is obtained by putting fictitious force or moment to zero in the partial derivative.

5.3. Deflection and Slope of Beams and Frames.

1. Deflection.

Consider a cantilever AB as shown in Fig. 5.1. To find the deflection at free end B, fictitious load \( W \) is applied there. Bending moment at a section distance \( x \) from B is given by \( M = M_x - Wx \) where \( M_x \) is B.M. due to external loading.

Strain energy due to bending

\[ U = \int_0^l \frac{M^2 dx}{2EI} \]

\[ \frac{\partial U}{\partial W} = \int_0^l \frac{M}{EI} \times \frac{\partial M}{\partial W} dx \]

But

\[ \frac{\partial M}{\partial W} = -x \]

\[ \therefore \quad \frac{\partial U}{\partial W} = \int_0^l \frac{M_x - Wx}{EI} \times -x dx \]

when \( W = 0 \),

\[ \frac{\partial U}{\partial W} = \delta = \int_0^l - \frac{M_x}{EI} x dx \]

2. Slope

To find slope at B, a fictitious moment \( M_x \) is applied. Moment at a section distance \( x \) from B is given by

\[ M = M_x - M_x \]

\[ \frac{\partial M}{\partial M_x} = -1 \]

Fig. 5.2
Strain energy due to bending

\[ U = \frac{1}{2EI} \int M^2 ds \]

\[ \partial M = \left\{ \frac{M}{EI} \times \frac{\partial M}{\partial M_0} \right\} dx \]

\[ \int (M_x - M_0) \times \frac{\partial M}{\partial M_0} dx \]

when \( M_0 = 0 \)

\[ \frac{\partial M}{\partial M_0} = \left\{ \frac{-M_s}{EI} \right\} \]

In case of frames the deflections and slopes are found in a similar manner as for beams only strain energy for all the members is to be considered.

**Ex. 5·1.** Find the deflection at free end of the cantilever loaded with triangular load as shown in Fig. 5·3.

**Solution.** To find deflection at free end B, apply fictitious load at free end.

B.M. at a section distance \( x \) from free end is given by

\[ M_s = -\left[ Px + \frac{1}{2} \cdot \frac{wx}{l} \right] \times x \times \cdot \]

\[ \frac{\partial M}{\partial P} = -x \]

\[ \frac{\partial U}{\partial P} = \int_0^l -\left[ \frac{Px + \frac{wx^2}{6l}}{EI} \right] (-x) dx \]

Putting \( P = 0 \),

\[ \frac{\partial U}{\partial P} = \Delta_B = \frac{1}{EI} \int_0^l \frac{wx^4}{6l} dx \]

\[ = \frac{w}{6EL} \left[ \frac{x^5}{5} \right]_0^l \]

\[ = \frac{wl^4}{30EI} \]
Ex. 5.2. Find the deflection and slope at quarter span of simply supported beam of span \( l \) and loaded with uniformly distributed load of intensity \( w/\text{unit length} \) throughout the span.

**Solution.** Apply fictitious load \( P \) at quarter span.

Reactions will be

\[
R_A = \frac{wL}{2} + \frac{3}{4}P \\
R_B = \frac{wL}{2} + \frac{P}{4}
\]

For portion \( AC \), taking origin at \( A \),

\[
M = \left( \frac{wL}{2} + \frac{3}{4}P \right) x - \frac{wx^2}{2}
\]

For whole beam

\[
\frac{\partial M}{\partial P} = \frac{-x}{4}
\]

Fig. 5.4 (a), (b)

\[
\frac{\partial M}{\partial P} = \frac{-x}{4}
\]

For portion \( CB \), taking origin at \( B \),

\[
M = \left( \frac{wL}{2} + \frac{P}{4} \right) x - \frac{wx^2}{2}
\]

Putting \( P = 0 \)

\[
\frac{\partial U}{\partial P} = 0
\]

\[
\frac{\partial U}{\partial P} = \frac{1}{EI} \int_0^{l/4} \left[ \frac{3wL}{8} x^3 - \frac{3}{8} wx^3 \right] dx
\]

\[
+ \frac{1}{EI} \int_0^{3l/4} \left[ \frac{wL}{8} x^3 - \frac{wx^3}{8} \right] dx
\]

\[
= \frac{1}{EI} \left[ \frac{3wL}{8} \times \frac{L^4}{3} + \frac{3}{8} w \times \frac{L^4}{4}\right]_{0}^{l/4}
\]

\[
+ \frac{1}{EI} \left[ \frac{wL}{8} \times \frac{L^4}{3} - \frac{wL^4}{3} \times \frac{L^4}{3} \right]_{0}^{3l/4}
\]

\[
= \frac{1}{EI} \left[ \frac{3}{24} wL \times \frac{L^4}{64} - \frac{3w}{32} \times \frac{L^4}{256} \right]
\]

\[
+ \frac{1}{EI} \left[ \frac{wL}{24} \times \frac{27L^4}{64} - \frac{w}{8L^4} \times \frac{L^4}{256} \right]
\]
\[ \Delta = \frac{1}{EI} \left[ \frac{30 \, wL^4}{24 \times 64} - \frac{84 \, wL^4}{32 \times 256} \right] \frac{19 \, wL^4}{19 \, wL^4} = \frac{64 \times 32EI}{2048 \, EI} \]

**Slope at quarter span.**

To find slope at quarter span fictitious moment \( M_o \) is applied at quarter span.

Reactions will be

\[
\begin{align*}
R_A &= \frac{wL}{2} M_o \\
R_B &= \frac{wL}{2} + \frac{M_o}{l}
\end{align*}
\]

For portion \( AC \), taking origin at \( A \),

\[ M_o = \left( \frac{wL}{2} - \frac{M_o}{l} \right) x - \frac{wx^2}{2} \]

\[ \frac{\partial M_o}{\partial M_o} = -\frac{x}{l} \]

For portion, \( BC \) taking origin at \( B \),

\[ M_o = \left( \frac{wL}{2} + \frac{M_o}{l} \right) x - \frac{wx^2}{2} \]

\[ \frac{\partial M_o}{\partial M_o} = +\frac{x}{l} \]

\[
\frac{\partial U}{\partial M_o} = \frac{1}{EI} \left[ \frac{1}{4} \left( \frac{wL}{2} - \frac{M_o}{l} \right) x - \frac{wx^2}{2} \right] \left( -\frac{x}{l} \right) dx
\]

\[ + \frac{1}{EI} \left[ \frac{3}{4} \left( \frac{wL}{2} + \frac{M_o}{l} \right) x - \frac{wx^2}{2} \right] \left( \frac{x}{l} \right) dx \]

Putting \( M_o = 0 \)

\[
\theta = \frac{1}{EI} \left[ \frac{1}{4} \left( -\frac{wx^2}{2} + \frac{wx^3}{2l} \right) dx \right] + \frac{1}{EI} \left[ \frac{3}{4} \left( \frac{wx^3}{2} - \frac{wx^3}{2l} \right) dx \right]
\]

\[
\begin{align*}
\theta &= \frac{1}{2EI} \left[ -\frac{wx^3}{3} + \frac{wx^4}{4l} \right]_{0}^{l/4} + \frac{1}{2EI} \left[ \frac{wx^3}{3} - \frac{wx^4}{4l} \right]_{0}^{3l/4} \\
\theta &= \frac{1}{2EI} \left[ -\frac{wL^3}{3 \times 64} + \frac{wL^3}{4 \times 256} \right] + \frac{1}{2EI} \left[ \frac{27wL^3}{3 \times 64} - \frac{81wL^3}{4 \times 256} \right] \\
\theta &= \frac{1}{2EI} \left[ -\frac{26wL^3}{3 \times 64} + \frac{80wL^3}{4 \times 256} \right] \\
\theta &= \frac{37wL^3}{768 \, EI}
\end{align*}
\]

**Ex. 5.3.** A continuous member \( ABCD \) is bent in one plane and loaded in the same plane as shown in Fig. 5.5. It is rigidly fixed at \( D \) and moment \( M \) is applied at the free end. Find the vertical movement of \( A \). Flexural rigidity of member \( = EI \).

**Solution.** To find vertical deflection of \( A \), apply an imaginary load \( W \) at \( A \). Partial derivative of strain energy of member with respect to \( W \) will give deflection in direction of \( W \), when \( W = 0 \),
Free body diagrams of portions $AB$, $BC$ and $CD$ are shown in Fig. 5·6. Equation for B.M., values of $\frac{\partial M}{\partial W}$ and limits of integration for three portions are given in Table 5·1.

**Fig. 5·5**

**FREE BODY DIAGRAM**

**Table 5·1**

<table>
<thead>
<tr>
<th>Portion</th>
<th>$M$</th>
<th>$\frac{\partial M}{\partial W}$</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>$M+WLx$</td>
<td>$x$</td>
<td>$0-l$</td>
</tr>
<tr>
<td>$BC$</td>
<td>$M+WL$</td>
<td>$L$</td>
<td>$0-l$</td>
</tr>
<tr>
<td>$CD$</td>
<td>$(M+WL)-WLx$</td>
<td>$(l-x)$</td>
<td>$0-2L$</td>
</tr>
</tbody>
</table>
\[ \frac{\partial U}{\partial W} = \int_0^l \frac{(M+W \times x) x}{EI} \, dx + \int_0^l \frac{M+W \times l}{EI} \times dx \\
+ \int_0^{2l} \frac{M+Wl-Wx}{EI} (l-x) \, dx \]

Putting \( W = 0 \)

\[ \Delta = \frac{1}{EI} \left[ \int_0^l M x \, dx + \int_0^l \frac{Ml}{EI} \, dx \right] + \int_0^{2l} \frac{M}{EI} (l-x) \, dx \]

\[ = \frac{M}{EI} \left[ \frac{x^2}{2} \right]_0^l + \frac{Ml}{EI} l \left[ x \right]_0^l + \frac{M}{EI} \left[ lx - \frac{x^3}{2} \right]_0^{2l} \]

\[ = \frac{M}{EI} \times \frac{l^2}{2} + \frac{Ml^2}{EI} + \frac{M}{EI} \left[ \frac{2l^3 - 4l^2}{2} \right] \]

\[ \Delta = \frac{3}{2} \frac{Ml^2}{EI} \]

**Ex. 5.4.** Find the horizontal deflection of joint B and slope at C in the frame shown in Fig. 5.7 (a). Also find horizontal deflection at D.

**Solution.** Taking moments about \( A \),

\[ R_D \times l = P \times l \]
\[ R_D = P \]
\[ R_A = -P \]

Strain energy = \[ \int \frac{M^2}{2EI} \, dx \]

Deflection of \( B \)

\[ \Delta_B = \frac{\partial U}{\partial P} = \frac{1}{EI} \int M \frac{\partial M}{\partial P} \, ds \]

Equations for B.M. values of \( \frac{\partial M}{\partial P} \)

and limits of integration for various members are given in Table 5.2 (a).

\[ \frac{\partial U}{\partial P} = \Delta_B = \frac{1}{EI} \left[ \int_0^l \frac{Py \times y}{EI} \, dy \right] + \frac{1}{EI} \left[ \int_0^l \frac{Px \times x}{EI} \, dx \right] \]

\[ = \frac{P}{EI} \left[ \frac{y^3}{3} \right]_0^l + \frac{P}{EI} \left[ \frac{x^3}{3} \right]_0^l \]

\[ = \frac{P l^3}{3EI} + \frac{P l^3}{3EI} = \frac{2Pl^3}{3EI} \]

(b) To find slope at \( C \), moment \( M_o \) is applied at \( C \).

Slope at \( C \), \( \theta_C = \frac{\partial U}{\partial M_o} \)

\[ = \int \frac{M}{\frac{\partial M}{\partial M_o}} \, ds \]
### Table 5.2 (a)

<table>
<thead>
<tr>
<th>Member</th>
<th>Origin at</th>
<th>B.M.</th>
<th>$\frac{\partial M}{\partial P}$</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>$A$</td>
<td>$Pxy$</td>
<td>$y$</td>
<td>$0-1$</td>
</tr>
<tr>
<td>$BC$</td>
<td>$C$</td>
<td>$Pxx$</td>
<td>$x$</td>
<td>$0-1$</td>
</tr>
<tr>
<td>$CD$</td>
<td>$-$</td>
<td>$0$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

Equations for B.M., values of $\frac{\delta M}{\delta M_o}$ and limits of integration for various members are given in Table 5.2 (b).

### Table 5.2 (b)

<table>
<thead>
<tr>
<th>Member</th>
<th>Origin at</th>
<th>$M$</th>
<th>$\frac{\delta M}{\delta M_o}$</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>$A$</td>
<td>$Pxy$</td>
<td>$0$</td>
<td>$0-1$</td>
</tr>
<tr>
<td>$CD$</td>
<td>$-$</td>
<td>$0$</td>
<td>$-$</td>
<td>$0-1$</td>
</tr>
<tr>
<td>$CB$</td>
<td>$O$</td>
<td>$Px + \frac{M_o}{l}x - M_o$</td>
<td>$\frac{x}{l} - 1$</td>
<td>$0-1$</td>
</tr>
</tbody>
</table>

$$\frac{\partial U}{\partial M_o} = \frac{1}{EI} \int_0^l \left( Px + \frac{M_o}{l}x - M_o \right) \left( \frac{x}{l} - 1 \right) dx$$

Putting $M_o = 0$

$$\theta_c = \frac{1}{EI} \int_0^l Px \left( \frac{x}{l} - 1 \right) dx = \frac{P}{EI} \left[ \frac{x^3}{3} - \frac{x^2}{2} \right]_0 = \frac{P}{EI} \left[ \frac{l^3}{3} - \frac{l^2}{2} \right]$$

$$= -\frac{P l^2}{6EI}$$

Minus sign indicates slope will be anticlockwise.

(c) To find horizontal deflection of point $D$, a force $F$ is applied at $D$ as shown in Fig. 5.8. The horizontal reaction at $A$ will also be $F$.

$$\Delta D = \frac{\partial U}{\partial F} = \frac{1}{EI} \int M \frac{\partial M}{\partial F} ds$$

The equation for B.M., $\frac{\partial M}{\partial F}$ and limits of integration for various members are given in Table 5.2 (c).
TABLE 5.2 (c)

<table>
<thead>
<tr>
<th>Member</th>
<th>Origin at</th>
<th>( M )</th>
<th>( \frac{\partial M}{\partial F} )</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>A</td>
<td>( Py - Fy )</td>
<td>(-y)</td>
<td>0 - l</td>
</tr>
<tr>
<td>BC</td>
<td>C</td>
<td>( P \times x - Fl )</td>
<td>(-l)</td>
<td>0 - l</td>
</tr>
<tr>
<td>CD</td>
<td>D</td>
<td>( F \times y )</td>
<td>( y)</td>
<td>0 - l</td>
</tr>
</tbody>
</table>

\[
\frac{\partial U}{\partial F} = \int_0^l \left( \frac{Py - Fy}{EI} \right) (-y) dy + \int_0^l \left( \frac{Px - Fl}{EI} \right) (-l) dx + \int_0^l \frac{k \times y \times y}{EI} dy
\]

Putting \( F = 0 \)

\[
\Delta D = \frac{1}{EI} \int_0^l (-Py)^2 dy + \frac{1}{EI} \int_0^l -Px l dx
\]

\[
= -\frac{P}{EI} \left[ \frac{y^3}{3} \right]_0^l + \frac{Pl}{EI} \left[ \frac{x^2}{2} \right]_0^l
\]

\[
= \frac{Pl^3}{6EI} - \frac{2Pl^3}{3EI}
\]

\[
= -\frac{5}{6} \frac{Pl^3}{EI}
\]

Negative sign shows point D will deflect towards right.

Ex. 5.5. In Ex. 5.4, find the coefficient of friction \( \mu \) at the rollers if there is no horizontal displacement of the rollers.

**Solution.** Let \( F \) be horizontal reaction at D.

Taking moments about A,

\( R_D \times l = P \times l \)

\( R_D = P \)

\( F = \mu P \)

where \( \mu \) is co-efficient of friction.

Horizontal reaction at A will be \( (P-F) \).

Deflection at D is to be zero, therefore partial derivative of strain energy with respect to \( F \) should be zero.

\[
\frac{\partial U}{\partial F} = \int_0^l \left( \frac{Py - Fy}{EI} \right) (-y) dy + \int_0^l \frac{Fy \times y}{EI} dy + \int_0^l \left( \frac{Px - Fl}{EI} \right) (-l) dx
\]

Fig. 5.9
0 = \frac{1}{EI} \int_0^l \left[ -Py^2 + Fy^2 \right] dy + \frac{F}{EI} \int_0^l y^3 dy + \int_0^l \frac{(-Px + Fl^2)}{EI} dx \\
0 = \frac{1}{EI} \left[ -\frac{Py^3}{3} + \frac{Fy^3}{3} \right]_0^l + \frac{F}{EI} \left[ \frac{y^3}{3} \right]_0^l + \frac{1}{EI} \left[ -Pl \frac{x^2}{2} + Fl^2 x \right] \\
0 = \frac{1}{EI} \left[ -\frac{Pl^3}{3} + \frac{Fl^3}{3} \right] + \frac{Fl^3}{3EI} - \frac{Pl^3}{2EI} + \frac{Fl^3}{EI} \\
0 = \frac{1}{EI} \left[ \frac{5}{3} Fl^3 - \frac{5Pl^3}{6} \right] \\
\therefore \frac{5}{3} F = \frac{5P}{6} \\
\therefore F = \frac{P}{2} \\
But F = \mu P \\
\therefore \mu P = \frac{P}{2} \\
\mu = 0.5

**Ex. 5.6.** A thin semi-circular bracket AB of radius R is encastered at A and has at B a rigid arm BC of length R. The arm carries a vertical load W at C. Show that the vertical deflection at the load is \( \frac{\pi WR^3}{2EI} \) where EI is flexural rigidity of the bracket.

**Solution.** The bracket BC is rigid, hence EI for BC is infinite. The strain energy for BC will be zero.

Free body of strip AB is shown in Fig. 5.10 (b).

B.M. at any section X at angle \( \theta \) to centre of semi-circle is

\[ M = WR - WR(1 - \cos \theta) = WR \cos \theta \]

\[ \frac{\partial M}{\partial W} = R \cos \theta \]

\[ ds = Rd\theta \]

\[ \frac{\partial U}{\partial W} = \sum_{\Delta} \frac{1}{EI} \int_0^\pi M \frac{\partial M}{\partial W} ds \]
\[ \frac{1}{EI} \int_0^\pi WR \cos \theta \times R \cos \theta \times R \, d\theta \]

\[ \frac{WR^3}{EI} \int_0^\pi \cos^3 \theta \, d\theta \]

\[ = \frac{WR^3}{2EI} \left[ \frac{\pi (1 + \cos 2\theta)}{2} \right]_0^\pi \]

\[ = \frac{WR^3}{2EI} \times \pi \]

\[ = \frac{\pi WR^3}{2EI}. \]

**5.4. Deflection of Pin-jointed Frames.**

In case of pin-jointed frames, members carry only axial forces. The total strain energy of the system will be the sum of the strain energy due to direct forces in various members. The deflection of any joint in any direction will be given by partial derivatives of total strain energy with respect to the force acting at that joint in the direction in which deflection is desired. In case there is no external force acting at the joint in the direction in which deflection is desired, a fictitious load is applied in that direction and forces in all members are worked out. The total strain energy of the structure is found and partial derivative of strain energy with respect to fictitious load is found. Putting the fictitious load to zero, the partial derivative will give the desired deflection.

Let \( P \) be the force in any member due to external load. Let \( Q \) be the fictitious applied load at the joint where deflection is desired, acting in the direction of desired deflection. Let \( kQ \) be the force in the member due to fictitious load. Total force in the member will be

\[ F = P + kQ \]

Strain energy in the member \( = \frac{F^3l}{2AE} \), where \( A \) is the cross-sectional area of the member, \( l \) is its length and \( E \) Young’s modulus.

Total strain energy of structure

\[ U = \sum F^3l \cdot \sum \frac{(P + kQ)^3l}{2AE} \]

\[ \frac{\partial U}{\partial Q} = \sum \frac{2(P + kQ)^2}{6AE} k \]
Putting
\[ Q = 0 \]
\[ \Delta = \Sigma \frac{P_k l}{A E} \]

\[ \frac{P l}{A E} = \delta, \text{ the change in length of the member.} \]
\[ \Delta = \Sigma \delta k. \]

\( k \) is the force in a member due to unit load applied at the joint where deflection is desired, acting in the direction of desired deflection.

If it is required to find relative movement of two joints say \( B \) and \( G \), unit loads are applied at \( B \) and \( G \) in line with \( BG \) and forces in all members are worked out. The relative displacement will be \( \Sigma \frac{P_k l}{A E} \) where \( k_1 \) is the force in the member due to unit loads acting at \( B \) and \( G \) along line \( BG \).

If it is required to find relative rotation of joints \( E \) and \( F \) forces equal to \( \frac{1}{a} \) are applied at \( E \) and \( F \) as shown in Fig. 5.13 and forces in all members are calculated.

The relative rotation will be \( \frac{P_k l}{A E} \) where \( k_2 \) is the force in a member due to loads \( \frac{1}{a} \) acting at \( E \) and \( F \).

5.5. Effect of Temperature Change.

Due to change of temperature various members in a frame will elongate or shorten. This change in length of various members will cause deflection of various joints in the frame.

Let \( \delta_1 \) be the change in length of a member due to external loads and \( \delta_2 \) be the change in length due to temperature change. The deflection of a joint will be

\[ \Delta = \Sigma (\delta_1 + \delta_2) k, \]

where \( k \) is the force in the member due to unit load applied at the joint where deflection is required, the load being applied in the direction of required displacement.

\[ \delta_1 = \frac{P l}{A E} \]

\[ \delta_2 = l a t \]

where \( P \) is the force in a member,
\( l \) is the length of member,
\( A \) is the area of cross-section of member,
\( E \) is Young's modulus,
\( a \) is the co-efficient of linear expansion, and
\( t \) is the temperature rise.
Ex. 5.7. Find the horizontal and vertical deflections of joint $U_1$ of the frame shown in Fig. 5.14 (a) due to applied loading. The figures in parenthesis show the area of cross-section of the members. $E = 2 \times 10^6$ kg/cm².

**Solution.** Various members are denoted by 1, 2, 3... as shown in Fig. 5.14 (b)

\[ \tan \theta = \frac{1}{2} \]
\[ \sin \theta = \frac{1}{\sqrt{5}} \]
\[ \cos \theta = \frac{2}{\sqrt{5}} \]

**Forces in members due to loading.**

**Joint U₁**

Resolving forces at joint $U_1$,

$\Sigma V = 0$ gives

\[ P_4 = \frac{2000}{\sin \theta} = +2000\sqrt{5}$ kg.

$\Sigma H = 0$ gives

\[ P_1 = -P_4 \cos \theta = -2 \times 2000$ kg.
\[ = -4000$ kg.

**Joint U₂**

$\Sigma V = 0$ gives

\[ P_7 = 0 \]

$\Sigma H = 0$ gives

\[ P_2 = P_1 = -4000$ kg.

**Joint L₂**

Resolving forces at right angles to $U₁L₂$

\[ P_8 = 0 \text{ as } P_7 = 0 \]

Resolving forces along direction $L₂U₁$

\[ P_9 = P_4 = +2000\sqrt{5}$ kg.

**Joint U₃**

$\Sigma V = 0$ gives $P_3 = 0$ as $P_8 = 0$

$\Sigma H = 0$ gives $P_3 = P_2 = -4000$ kg.

**Joint L₃**

Resolving forces at right angles to $L₃U₁$

\[ P_{10} = 0 \text{ as } P_9 = 0 \]

Resolving forces along $L₃U₁$

\[ P_6 = P_5 = +2000\sqrt{5}$ kg.\]
To find vertical deflection of \( U_1 \), unit vertical load is applied at \( U_1 \). Let the forces in various members due to unit load be \( k_1, k_2, k_3, \ldots \) and so on. The values of \( k_1, k_2, \ldots \) can be obtained by diving forces in members due to applied loading by 2000.

\[
\begin{align*}
k_1 &= k_2 = k_3 = -2, \\
k_4 &= k_5 = k_6 = +\sqrt{5} \\
k_7 &= k_8 = k_9 = 0 \\
\Delta v &= \sum \frac{P_{kl}}{AE}
\end{align*}
\]

To find horizontal deflection of joint \( U_1 \), unit horizontal load is applied at \( U_1 \). Forces in various members due to this load will be

\[
\begin{align*}
k_1' &= -1, k_2' = -1, k_3' = -1 \\
k_4' &= k_5' = k_6' = k_7' = k_8' = k_9' = k_{10}' = 0
\end{align*}
\]

The values of length, area, \( P \), \( k \), \( k' \), \( \frac{P_{kl}}{AE} \) and \( \frac{P_{kl'}}{AE} \) for various members are tabulated in Table 5.3.

**Table 5.3**

<table>
<thead>
<tr>
<th>Member</th>
<th>Length (m)</th>
<th>Area (cm²)</th>
<th>Force (kg)</th>
<th>( k )</th>
<th>( k' )</th>
<th>( \frac{P_{kl}}{A} )</th>
<th>( \frac{P_{kl'}}{A} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>8</td>
<td>-4000</td>
<td>-2</td>
<td>-1</td>
<td>2000</td>
<td>1000</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>8</td>
<td>-4000</td>
<td>-2</td>
<td>-1</td>
<td>2000</td>
<td>1000</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>8</td>
<td>-4000</td>
<td>-2</td>
<td>-1</td>
<td>2000</td>
<td>1000</td>
</tr>
<tr>
<td>4</td>
<td>( \sqrt{5} )</td>
<td>12</td>
<td>+2000( \sqrt{5} )</td>
<td>+( \sqrt{5} )</td>
<td>0</td>
<td>\frac{2500}{3} ( \sqrt{5} )</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>( \sqrt{5} )</td>
<td>12</td>
<td>+2000( \sqrt{5} )</td>
<td>+( \sqrt{5} )</td>
<td>0</td>
<td>\frac{2500}{3} ( \sqrt{5} )</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>( \sqrt{5} )</td>
<td>12</td>
<td>+2000( \sqrt{5} )</td>
<td>+( \sqrt{5} )</td>
<td>0</td>
<td>\frac{2500}{3} ( \sqrt{5} )</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>[0]</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>( \sqrt{5} )</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>2( \sqrt{2} )</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ \Sigma = 6000 \quad | \Sigma = 3000 \quad +2500\sqrt{5} \]
\[
\Sigma \frac{Pkl}{A} = (6000 + 2500\sqrt{5}) \\
= 11,590 \\
\Delta v = \Sigma \frac{Pkl}{AE} \\
= \frac{11,590 \times 100}{2 \times 10^8} \text{ cm.} \\
= 0.5795 \text{ cm.} \\
\Delta h = \Sigma \frac{Pkl}{AE} \\
= \frac{3000 \times 100}{2 \times 10^8} \text{ cm.} \\
= 0.15 \text{ cm.}
\]

**Ex. 5.8.** The frame shown in Fig. 5.15 is hinged at A and on roller bearing at B. Under a vertical load \( W \) at C the tension members are stressed to 1000 kg/cm\(^2\) and compression members to 600 kg/cm\(^2\). Determine the horizontal movement of roller bearing. \( E = 2 \times 10^8 \text{ kg/cm}^2 \).

**Solution.** The nature of stresses due to applied loading in various members is shown in Fig. 5.15 (b).

\[
\Sigma V = 0 \text{ at } B \text{ gives } \\
P_A = 0 \\
\Sigma V = 0 \text{ at } A \text{ gives } \\
P_{10} = 0 \\
\Sigma V = 0 \text{ at } G \text{ gives } \\
P_7 = 0
\]

To find horizontal movement for roller B, apply unit horizontal force at B as shown in Fig. 5.15(c). Horizontal reaction \( H_A = 1 \). Taking moments about \( A \),

\[
V_B \times 6 = 1 \times 3 \\
\therefore \quad V_B = \frac{1}{3} \\
V_A = \frac{1}{3}
\]

Resolving forces at joint C

\[ k_1 = k_2 = 0 \]

Resolving forces at joint D

**Fig. 5.15**
Joint B
\[ \Sigma V = 0 \text{ gives} \]
\[ k_5 = \frac{1}{2} \times \frac{1}{\sin 45} = \frac{\sqrt{2}}{2} \]
\[ \Sigma H = 0 \text{ gives} \]
\[ k_6 = -(1 + P_3 \cos 45) = -\frac{3}{2} \]

Joint E
\[ \Sigma H = 0 \text{ gives} \]
\[ P_8 = P_6 = \frac{\sqrt{2}}{2} \]

**Table 5.4**

<table>
<thead>
<tr>
<th>Member</th>
<th>Length ( l )</th>
<th>( P/A )</th>
<th>( k )</th>
<th>( P\text{kt} / A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>-1000</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>-1000</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>( 3\sqrt{2} )</td>
<td>+600</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>( 3\sqrt{2} )</td>
<td>+600</td>
<td>+( \sqrt{2}/2 )</td>
<td>+1800</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>+600</td>
<td>-3/2</td>
<td>-2700</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>( 3\sqrt{2} )</td>
<td>-1000</td>
<td>+( \sqrt{2}/2 )</td>
<td>-3000</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>+610</td>
<td>-1/2</td>
<td>-900</td>
</tr>
<tr>
<td>10</td>
<td>( 3\sqrt{2} )</td>
<td>0</td>
<td>-( \sqrt{2} )</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>-1000</td>
<td>+1/2</td>
<td>-1500</td>
</tr>
</tbody>
</table>

\[ \Sigma = -6300 \]
\[ \Sigma H = 0, \text{ gives } P_7 = -(P_6 + P_8) \sin 45 \]
\[ = -\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) \frac{1}{\sqrt{2}} - 1 \]

**Joint G**

\[ \Sigma V = 0, \text{ gives } k_{10} = \frac{P_7}{\sin 45} = -\sqrt{2} \]
\[ \Sigma H = 0, \text{ gives } k_9 = -k_8 + k_{10} \sin 45 = -\frac{3}{4} + 1 \]
\[ = -\frac{1}{4} \]

**Joint A**

\[ \Sigma V = 0, \text{ gives } k_{11} = k_{10} \sin 45 - \frac{1}{4} \]
\[ = 1 - \frac{1}{4} = +\frac{3}{4} \]

The values of \( l \), \( P/A \), \( k \) and \( \frac{P_{kl}}{AE} \) are tabulated in Table 5·4.

\[ \Delta H = \Sigma \frac{P_{kl}}{AE} = 6300 \]
\[ = -2 \times 10^6 \times 100 \text{ cm.} \]
\[ = -0.315 \text{ cm.} \]

**Ex. 5·9.** In the truss shown in Fig. 5·16 (a), the top chord members have area of 9 cm\(^2\), verticals have area of 8 cm\(^2\) and bottom chord members and diagonals have area of 5 cm\(^2\). Find the vertical deflection of joint \( L_4 \) when load of 2000 kg. is applied at each of joints \( L_2 \) and \( L_3 \) as shown in Fig. 5·16. \( E = 2 \times 10^6 \) kg/cm\(^2\).

**Solution.** Numbering of various members as 1, 2, 3...........is shown in Fig. 5·16 (b). Reactions at supports are

\[ R_L = 2500 \text{ kg. and } R_R = 1500 \text{ kg.} \]

**Joint L_1**

\[ \Sigma H = 0, \text{ gives } P_4 = 0 \]
\[ \Sigma V = 0, \text{ gives } P_9 = +2500 \text{ kg.} \]

**Joint U_1**

\[ \Sigma V = 0, \text{ gives } P_{10} = -\frac{4/5}{\sin \theta} \]
\[ \therefore \]
\[ P_{10} = -\frac{2500}{4/5} = -3125 \text{ kg.} \]
\[ \Sigma H = 0, \text{ gives } P_5 = P_{10} \cos \theta \]
\[ \therefore \]
\[ P_5 = +3125 \times \frac{\sqrt{2}}{5} \]
\[ = +1875 \text{ kg.} \]

**Joint L_2**

\[ \Sigma V = 0 \]
\[ P_{11} + P_{10} \sin \theta + 2000 = 0 \]
\[ P_{11} = -2000 - (-3125) \times \frac{\sqrt{2}}{5} \]
\[ = -2000 + 2500 \]
\[ = +500 \]
\[ \Sigma H = 0 \text{ gives } P_2 = +P_{10} \cos \theta \]
\[ \therefore \]
\[ P_a = -3125 \times \frac{3}{5} = -1875 \text{ kg}. \]

![Diagram](image)

\[ R_a = 2500 \text{ kg} \quad \nabla 2000 \text{ kg} \quad \nabla 2000 \text{ kg} \]

\[ R_a = 1500 \text{ kg}. \]

**Joint U_2**

\[ \Sigma V = 0 \text{ gives} \]

\[ P_{11} + P_{12} \sin \theta = 0 \]

\[ P_{12} = -\frac{P_{11}}{\sin \theta} \]

\[ = -\frac{500}{\frac{4}{5}} = -625 \text{ kg}. \]

\[ \Sigma H = 0 \]

\[ P_6 = P_5 - P_{12} \cos \theta \]

\[ = 1875 + 625 \times \frac{3}{5} \]

\[ = 1875 + 375 \]

\[ P_6 = +2250 \text{ kg}. \]
Joint \( U_3 \)

\[ \Sigma V = 0, \text{ gives} \]
\[ P_{12} = 0 \]
\[ \Sigma H = 0, \text{ gives} \]
\[ P_7 = P_6 = +2250 \text{ kg.} \]

Joint \( L_3 \)

\[ \Sigma V = 0, \]
\[ P_{12} \sin \theta + P_{14} \sin \theta + 2000 = 0. \]

\[ \therefore \quad P_{14} = \frac{2000}{\sin \theta} - P_{12} \]
\[ = -2000 \times \frac{5}{4} + 625 \]
\[ = -2500 + 625 \]
\[ = -1875 \text{ kg.} \]

\[ \Sigma H = 0, \text{ gives} \]
\[ P_0 = -P_2 + P_{12} \cos \theta + P_{14} \cos \theta \]
\[ = 1875 - 625 \times \frac{3}{5} + 1875 \times \frac{3}{5} \]
\[ P_3 = -1125 \text{ kg.} \]

Joint \( L_5 \)

\[ \Sigma H = 0, \text{ gives} \]
\[ P_4 = 0 \]
\[ P_{17} = +1500 \text{ kg.} \]

Joint \( U_5 \)

\[ \Sigma V = 0, \text{ gives} \]
\[ P_{16} = -\frac{P_{17}}{\sin \theta} \]
\[ = -1500 \times \frac{5}{4} = -1875 \text{ kg.} \]

\[ \Sigma H = 0, \text{ gives} \]
\[ P_8 = -P_{16} \cos \theta \]
\[ = +1875 \times \frac{3}{5} = +1125 \text{ kg.} \]

Joint \( L_4 \)

\[ P_5 = P_{18} \cos \theta \]
\[ = -1875 \times \frac{5}{5} = -1125 \text{ kg.} \]
\[ P_{15} = -P_{16} \sin \theta \]
\[ = +1875 \times \frac{5}{5} = +1500 \text{ kg.} \]

To find vertical deflection of joint \( L_4 \), unit load is applied at \( L_4 \). The forces in various members, \( k_1, k_2, \ldots \) are worked out. These forces are given in Table 5.5. Table 5.5 also gives values of \( P_{k1} \) for various members.
<table>
<thead>
<tr>
<th>Member</th>
<th>l</th>
<th>A</th>
<th>P</th>
<th>k</th>
<th>$\frac{P_{kl}}{A}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
<td>-1875</td>
<td>-3/16</td>
<td>+3375/16</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>5</td>
<td>-1125</td>
<td>-9/16</td>
<td>+6075/6</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>9</td>
<td>+1875</td>
<td>+3/16</td>
<td>+1875/16</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>9</td>
<td>+2250</td>
<td>+3/8</td>
<td>+2250/8</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>9</td>
<td>+2250</td>
<td>+3/8</td>
<td>+2250/8</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>9</td>
<td>+1125</td>
<td>+9/16</td>
<td>+3375/16</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>8</td>
<td>+2500</td>
<td>+1/4</td>
<td>+2500/8</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>5</td>
<td>-3125</td>
<td>-5/16</td>
<td>+15,625/16</td>
</tr>
<tr>
<td>11</td>
<td>4</td>
<td>8</td>
<td>+500</td>
<td>+1/4</td>
<td>+500/8</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>5</td>
<td>-625</td>
<td>-5/16</td>
<td>+3125/16</td>
</tr>
<tr>
<td>13</td>
<td>4</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>5</td>
<td>5</td>
<td>-1875</td>
<td>+5/16</td>
<td>-9375/6</td>
</tr>
<tr>
<td>15</td>
<td>4</td>
<td>8</td>
<td>+1500</td>
<td>-1/4</td>
<td>-1500/8</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>5</td>
<td>-1875</td>
<td>-15/16</td>
<td>+23,125/16</td>
</tr>
<tr>
<td>17</td>
<td>4</td>
<td>8</td>
<td>+1500</td>
<td>+3/4</td>
<td>+4500/8</td>
</tr>
</tbody>
</table>

\[ \Delta = \sum \frac{P_{kl}}{AE} = \frac{4650}{2 \times 10^6} \times 100 \text{ cm.} = 0.2325 \text{ cm.} \]
Ex. 5.10. Find horizontal and vertical movement of joint G of the frame shown in Fig. 5.17 (a). All members have same cross-sectional area of 20 cm².

\[ E = 2 \times 10^6 \text{ kg/cm}^2. \]

**Solution.** The numbering of various members as 1, 2, 3 ..., is shown in Fig. 5.17 (b).

Reactions are \( R_A = R_B = 2500 \text{ kg}. \) As the frame is symmetrical and symmetrically loaded forces will be computed for half of the frame. Forces for other half will be similar to first half.

**Joint A**

\[ \Sigma H = 0, \text{ gives } \]
\[ P_2 = 0 \]
\[ \Sigma V = 0, \text{ gives } \]
\[ P_1 = +2500 \text{ kg}. \]

**Joint C**

\[ \Sigma V = 0, \text{ gives } \]
\[ P_3 = - \frac{P_1}{\cos 45^\circ} \]
\[ = -2500\sqrt{2} \text{ kg.} \]
\[ \Sigma H = 0, \text{ gives } \]
\[ P_6 = -P_3 \cos 45^\circ \]
\[ = +2500 \text{ kg}. \]

**Joint F**

\[ \Sigma V = 0, \text{ gives } \]
\[ P_7 \cos 45^\circ + P_3 \cos 45^\circ = 0 \]
\[ \therefore P_7 = -P_3 \]
\[ = +2500\sqrt{2} \text{ kg.} \]
\[ \Sigma H = 0, \text{ gives } \]
\[ P_{11} = P_3 \cos 45^\circ \]
\[ = -P_7 \cos 45^\circ \]
\[ = -2500 + 2500 \]
\[ = -5000 \text{ kg}. \]

Fig. 5.17
To find vertical deflection of $G$, unit vertical load is applied at $G$. The forces, due to this load in various members are denoted by $k_1, k_2, k_3$, and are shown in Table 5-6. Values of $Pkl$ for different members are also shown in the Table.

To find horizontal deflection of $G$, unit horizontal load is applied at $G$ as shown in Fig. 5-17 (d). The forces due to this load are denoted by $k_1', k_2', \ldots$ and are shown in Table 5-6. Values of $Pkl'$ for various members are also shown in the Table.

**TABLE 5-6**

<table>
<thead>
<tr>
<th>Member</th>
<th>Length $l$ m</th>
<th>$A$ cm$^2$</th>
<th>$P$</th>
<th>$k$</th>
<th>$'p$</th>
<th>$Pkl$</th>
<th>$Pkl'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>20</td>
<td>+2500</td>
<td>+1/4</td>
<td>+3/2</td>
<td>+3750</td>
<td>+22,500</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>20</td>
<td>+2500</td>
<td>+3/4</td>
<td>+1/2</td>
<td>+11,250</td>
<td>+7,500</td>
</tr>
<tr>
<td>3</td>
<td>$2\sqrt{2}$</td>
<td>10</td>
<td>$-2600\sqrt{2}$</td>
<td>$-\sqrt{2}/4$</td>
<td>$-3\sqrt{2}/2$</td>
<td>$+2500\sqrt{2}$</td>
<td>$+15000\sqrt{2}$</td>
</tr>
<tr>
<td>4</td>
<td>$2\sqrt{2}$</td>
<td>20</td>
<td>$-2500\sqrt{2}$</td>
<td>$-3\sqrt{2}/4$</td>
<td>$-\sqrt{2}/2$</td>
<td>$+7500\sqrt{2}$</td>
<td>$+5,000\sqrt{2}$</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>20</td>
<td>+2500</td>
<td>+1/4</td>
<td>+3/2</td>
<td>+2500</td>
<td>+15,000</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>20</td>
<td>+2500</td>
<td>+3/4</td>
<td>+1/2</td>
<td>+7500</td>
<td>+5,000</td>
</tr>
<tr>
<td>7</td>
<td>$2\sqrt{2}$</td>
<td>20</td>
<td>$+2500\sqrt{2}$</td>
<td>$+\sqrt{2}/4$</td>
<td>$-\sqrt{2}/2$</td>
<td>$+2500\sqrt{2}$</td>
<td>$-5000\sqrt{2}$</td>
</tr>
<tr>
<td>8</td>
<td>$2\sqrt{2}$</td>
<td>20</td>
<td>$+500\sqrt{2}$</td>
<td>$-\sqrt{2}/4$</td>
<td>$+\sqrt{2}/2$</td>
<td>$-2500\sqrt{2}$</td>
<td>$+5000\sqrt{2}$</td>
</tr>
<tr>
<td>9</td>
<td>$2\sqrt{5}$</td>
<td>20</td>
<td>0</td>
<td>0</td>
<td>$-\sqrt{5}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>$2\sqrt{5}$</td>
<td>20</td>
<td>0</td>
<td>0</td>
<td>$-\sqrt{5}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>4</td>
<td>20</td>
<td>$-5000$</td>
<td>$-1$</td>
<td>$-2$</td>
<td>$+10,000$</td>
<td>$+40,000$</td>
</tr>
</tbody>
</table>

$\sum Pkl = 33,600 + 10,000\sqrt{2} = 49,140$

$\sum Pkl' = 60,000 + 20,000\sqrt{2} = 115,290$
\[
\Delta v = \sum \frac{Pkl}{AE} = \frac{49.140}{20 \times 2 \times 10^3} \times 100 \text{ cm.}
\]
\[
= 0.1228 \text{ cm.}
\]
\[
\Delta h = \sum \frac{Pkl}{AE} = \frac{118.280}{20 \times 2 \times 10^3} \times 100 \text{ cm.}
\]
\[
= 0.2957 \text{ cm.}
\]

**Ex. 5.11.** The pin-jointed frame shown in Fig. 5.19 (a) is hinged to support at A and on roller bearing at B. If there is uniform rise of temperature in all members equal to 30° C above the temperature at which frame was erected, calculate the vertical movement of C. Coefficient of linear expansion = \(1.1 \times 10^{-5}/°C\).

**Solution.** Due to rise of temperature deformation of each member will be \(L \alpha t\) where \(L\) is length of member, \(\alpha\) coefficient of linear expansion and \(t\) is rise of temperature.

\[
\alpha t = 1.1 \times 10^{-5} \times 30 = 3.3 \times 10^{-4}
\]

To find vertical movement at C, unit vertical load is applied at C. Due to this load reactions will be

\[
R_A = R_B = \frac{1}{2}.
\]

As the frame is symmetrical and symmetrically loaded, forces in members will be same about centre line and hence forces for half the frame are calculated.

**Joint A**

\[
\Sigma H = 0, \text{ gives } k_5 = 0
\]

\[
\Sigma V = 0, \text{ gives } k_7 = +\frac{1}{2}
\]

**Joint D**

\[
\Sigma V = 0, \text{ gives }
\]

\[
k_3 = -\frac{k_7}{\sin \theta}
\]

\[
= -\frac{1}{2} \times \frac{5}{3} = -\frac{5}{6}
\]

\[
\Sigma H = 0, \text{ gives }
\]

\[
k_1 = -k_5 \cos \theta = +\frac{5}{6} \times \frac{4}{5} = +\frac{2}{3}
\]

**Joint E**

\[
\Sigma V = 0, \text{ gives } k_9 = 0
\]
### Table 5.7

<table>
<thead>
<tr>
<th>Member</th>
<th>$L$</th>
<th>$\Delta L = \text{Lat}$</th>
<th>$k$</th>
<th>$\Delta k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>$13.2 \times 10^{-4}$</td>
<td>$+2/3$</td>
<td>$+8.8 \times 10^{-4}$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>$13.2 \times 10^{-4}$</td>
<td>$+2/3$</td>
<td>$+8.8 \times 10^{-4}$</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>$16.5 \times 10^{-4}$</td>
<td>$-5/6$</td>
<td>$-13.75 \times 10^{-4}$</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>$16.5 \times 10^{-4}$</td>
<td>$-5/6$</td>
<td>$-13.75 \times 10^{-4}$</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>$16.5 \times 10^{-4}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>$16.5 \times 10^{-4}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>$19.8 \times 10^{-4}$</td>
<td>$+1/2$</td>
<td>$+9.9 \times 10^{-4}$</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>$19.8 \times 10^{-4}$</td>
<td>$+1/2$</td>
<td>$+9.9 \times 10^{-4}$</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>$9.9 \times 10^{-4}$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$$\delta_C = \Sigma \Delta k = 9.9 \times 10^{-4} \times 100 \text{ cm.} = 9.9 \times 10^{-2} \text{ cm.} = 0.099 \text{ cm.}$$

**Ex. 5.12.** In problem 5.9 find the rotation of the member $U_2 U_3$. Find also the relative displacement of joints $U_3$ and $L_2$.

**Solution.** Apply forces $\frac{1}{2}$ at $U_3$ and $U_2$ as shown in Fig. 5.20. The forces $k$ due to this load are given in Table 5.8.

![Fig. 5.20](image-url)
### Table 5.8

<table>
<thead>
<tr>
<th>Member</th>
<th>( l )</th>
<th>( A )</th>
<th>( P )</th>
<th>( k )</th>
<th>( \frac{Pkl}{A} )</th>
<th>( k' )</th>
<th>( \frac{P'k'l}{A} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
<td>(-1875)</td>
<td>(-\frac{1}{16})</td>
<td>(+\frac{1125}{16})</td>
<td>(+\frac{3}{5})</td>
<td>(-675)</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>5</td>
<td>(-125)</td>
<td>(+\frac{1}{16})</td>
<td>(-\frac{675}{16})</td>
<td>(0)</td>
<td>(0)</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>9</td>
<td>(+1875) (+\frac{1}{16})</td>
<td>(+\frac{625}{16})</td>
<td>(0)</td>
<td>(0)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>9</td>
<td>(+2250) (-\frac{1}{8})</td>
<td>(-\frac{1500}{16})</td>
<td>(+\frac{3}{5})</td>
<td>(+450)</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>9</td>
<td>(+2.50) (-\frac{1}{8})</td>
<td>(-\frac{1500}{16})</td>
<td>(0)</td>
<td>(0)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>9</td>
<td>(+1125) (-\frac{1}{16})</td>
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<td>(0)</td>
<td></td>
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<td>8</td>
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<td>(+\frac{2500}{24})</td>
<td>(0)</td>
<td>(0)</td>
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<td>(+\frac{15625}{48})</td>
<td>(0)</td>
<td>(0)</td>
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<td>4</td>
<td>8</td>
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<td>(+\frac{4}{5})</td>
<td>(+200)</td>
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<td>5</td>
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<td>(-\frac{3375}{16})</td>
<td>(-1)</td>
<td>(+675)</td>
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<td>8</td>
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<td>(-\frac{1}{3})</td>
<td>(0)</td>
<td>(+\frac{4}{3})</td>
<td>(0)</td>
</tr>
<tr>
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<td>5</td>
<td>5</td>
<td>(-1875) (+\frac{5}{48})</td>
<td>(-\frac{8375}{48})</td>
<td>(0)</td>
<td>(0)</td>
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<td>8</td>
<td>(+1500) (-\frac{1}{12})</td>
<td>(-\frac{1500}{24})</td>
<td>(0)</td>
<td>(0)</td>
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<tr>
<td>16</td>
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<td>5</td>
<td>(-1875) (+\frac{5}{48})</td>
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<td>(0)</td>
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<td>(-\frac{1500}{24})</td>
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<td>(0)</td>
<td></td>
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</table>

\[ \sum \quad \frac{6050}{16} \quad \sum \quad +650 \]
\[ \text{relative rotation} \quad -\frac{6050}{16 \times 2 \times 10^6} \text{ radians} \]

\[ = -189 \times 10^{-6} \text{ radians.} \]

The rotation will be clockwise.

Apply unit forces at \( U_3 \) and \( L_2 \) in the direction \( U_3L_2 \) as shown in Fig. 5.21. The forces due to this loading \( k' \) are given in Table 5.8.

![Fig. 5.21](image)

The displacement along \( U_3L_2 \) will be

\[ 650 \times 100 = 325 \times 10^{-4} \text{ cm.} \]

\[ = 0.0325 \text{ cm.} \]

**Ex. 5.13.** The pin-jointed frame shown in Fig. 5.22 (a) is hinged to supports at \( A \) and on roller bearing at \( B \). If there is a uniform rise of temperature in all the members equal to 20\(^\circ\)C above the temperature at which the frame was erected, calculate the horizontal movement of roller \( B \).

If \( B \) is also hinged calculate the horizontal thrust at \( A \) and \( B \).

Co-efficient of linear expansion of steel is \( 1.1 \times 10^{-5} \) per degree centigrade and \( E \) is \( 2 \times 10^8 \) kg./cm\(^2\). The areas of members are shown in parenthesis in Fig. 5.22 (a).

**Solution.** Horizontal movement at roller will be \( \sum \delta k \) where \( \delta \) is deformation of a member and \( k \) is force in the member due to unit load applied in the direction of desired deflection.

\[ \text{Lat} \]

\[ 1.1 \times 10^{-5} \times 20 \]

\[ = 2.2 \times 10^{-4} \]

Apply a unit horizontal load at \( B \) as shown in Fig. 5.22 (b).
Joint B

\[ \Sigma V = 0, \text{ gives } \]
\[ k_1 \sin 45 + k_3 \sin \theta = 0 \]
\[ k_1 = -k_3 \frac{\sin \theta}{\sin 45} \]
\[ \Sigma H = 0, \text{ gives } \]
\[ k_1 \cos 45 + k_3 \cos \theta + 1 = 0 \]
\[ k_1 = -5 \times \frac{4}{5} \times \sqrt{2} = -4 \sqrt{2} \]

Similarly \( k_4 = +5, \quad k_2 = -4 \sqrt{2} \)

Joint C

\[ k_5 = k_1 \sin 45 + k_3 \sin 45 = -4 \sqrt{2} \times \frac{1}{\sqrt{2}} = -4 \sqrt{2} \times \frac{1}{\sqrt{2}} \]
\[ = -8 \]

The values of \( L, k, \) and \( Latk \) for various members are given in Table 5.9.

**TABLE 5.9**

<table>
<thead>
<tr>
<th>Member</th>
<th>( L )</th>
<th>( A )</th>
<th>( k )</th>
<th>( Latk )</th>
<th>( \frac{k^2L}{A} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6\sqrt{2}</td>
<td>15</td>
<td>-4\sqrt{2}</td>
<td>-105.6 \times 10^{-4}</td>
<td>+12.8 \sqrt{2}</td>
</tr>
<tr>
<td>2</td>
<td>6\sqrt{2}</td>
<td>15</td>
<td>-4\sqrt{2}</td>
<td>-105.6 \times 10^{-4}</td>
<td>+12.8 \sqrt{2}</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>10</td>
<td>+5</td>
<td>+110 \times 10^{-4}</td>
<td>+25</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>10</td>
<td>+5</td>
<td>+110 \times 10^{-4}</td>
<td>+25</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>8</td>
<td>-8</td>
<td>-35.2 \times 10^{-4}</td>
<td>+16</td>
</tr>
<tr>
<td>( \Sigma )</td>
<td></td>
<td></td>
<td></td>
<td>-26.4 \times 10^{-4}</td>
<td>66 + 25.6 \sqrt{2} = 102.2</td>
</tr>
</tbody>
</table>

Movement of roller = \( \Sigma Latk \)
\[ = -26.4 \times 10^{-4} \times 100 \]
\[ = -0.264 \text{ cm.} \]

Movement of roller is 0.264 cm. inwards.

If the end B is hinged, horizontal thrust \( H \) will be exerted at the hinges A and B and the inward movement 0.264 cm. will be prevented.
Outward movement of roller due to unit load
\[ = \sum k \times \frac{kl}{AE} = \sum \frac{k^2 l}{AE} \]

Outward movement due to force \( H \)
\[ = H \sum \frac{k^2 l}{AE} \]
\[ H \sum \frac{k^2 l}{AE} = \Delta = 0.264 \]
\[ H \times \frac{102.2 \times 100}{2 \times 10^6} = 0.264 \]
\[ H = 51.66 \text{ kg.} \]

Horizontal thrust will be 51.66 kg. acting outwards.

**Ex. 5.14.** A vertical post \( ABD \) of height 12 m. built into ground at base \( A \) and a member \( BC \) is hinged to it at \( B \), and which is stayed by a steel wire \( CD \) as shown in Fig. 5.23 (a). The cross-sectional area of the wire is 0.5 cm\(^2\) and the second moment of area of the post is 1000 cm\(^4\). \( E \) for both post and wire is \( 2 \times 10^6 \text{ kg/cm}^2 \). Neglecting strains due to direct compressive stresses in the post and strut, determine the vertical displacement of \( C \) when a load of 100 kg. is suspended at the post.

**Solution.** Tension in wire \( CD = T = \frac{W}{\sin \theta} \)
\[ = \frac{W}{4/5} = \frac{5}{4} W. \]

Thrust in strut \( CB \)
\[ = T \cos \theta = \frac{5}{4} W \times \frac{3}{5} = \frac{3}{4} W \]

Partial derivative of strain energy of total system with respect to \( W \) gives deflection of point \( C \).

Strain energy of wire \( CD = \frac{T^2 l}{2AE} \)
\[ \frac{\partial U}{\partial W} \text{ for wire} \]
\[ = \frac{TL}{AE} \times \frac{\partial T}{\partial W} = \frac{5}{4} W \times \frac{5}{AE} \times \frac{1}{4} \]
\[ = \frac{125}{16} \frac{W}{AE}. \]

Free body diagram of portion \( ABD \) is shown in Fig. 5.23 (b).

**Portion BD**
\[ M = \frac{3}{4} W \times x \]
\[ \frac{\partial M}{\partial W} = \frac{3}{4} x. \]

**Portion AB**
\[ M = \frac{3}{4} W x - \frac{3}{4} W(x - 4) = \frac{3}{4} W \times 4 \]
\[ \frac{\partial M}{\partial W} = 3 \]
\[ \frac{\partial U}{\partial W} \text{ due to bending in post } ABD \]

\[ = \int \frac{M}{EI} \times \frac{\partial M}{\partial W} \, dx + \int_4^{12} \frac{M}{EI} \times \frac{\partial M}{\partial W} \, dx \]

\[ = \frac{1}{EI} \left[ \int_0^4 \frac{3}{4} Wx \times \frac{3}{4} x \, dx + \int_4^{12} 3W \times 3 \, dx \right] \]

\[ = \frac{9W}{16EI} \left[ \frac{x^4}{3} \right]_0^4 + \frac{9W}{EI} \left[ x \right]_4^{12} \]

\[ = \frac{9W}{16EI} \times 64 + \frac{9W}{EI} \times 8 \]

\[ = \frac{12W}{EI} + \frac{72W}{EI} = \frac{84W}{EI} \]

\[ \Delta c = \frac{\partial U}{\partial W} = \frac{125W}{16AE} + \frac{84W}{EI} \]

\[ = \frac{125 \times 100}{16 \times 0.5 \times 2 \times 10^8} + \frac{84 \times 100}{2 \times (10^8 \times 1000)} \]

\[ = \frac{42}{1280} \text{ m.} \]

\[ \Delta c = 100 \left( \frac{1}{1280} + \frac{42}{1000} \right) \text{ cm.} \]

\[ = 0.0781 + 4.2 \]

\[ = 4.2781 \text{ cm.} \]

5. Graphical method of deflection for Frames.

Consider a member \( AB \) of a frame. Let the member take position \( A'B' \) after deformation as shown in Fig 5.24. The total deformation of the bar will consist of displacement of the bar \( AB \), rotation of bar \( AB \) and finally stretching or shortening of the bar \( AB \). The deformation of the bar can be analysed in three steps, firstly movement of bar parallel to itself so that \( A \) takes position \( A' \) and \( B \) moves to \( B' \), secondly bar \( A'B' \) rotates about \( A' \) and takes position \( A'B' \), and finally the stretching of the bar due to axial force in it. In this case the member is elongated and hence it is in tension. In case of member subjected to compression there will be shortening of the member. In frames the rotation is very small and hence displacement \( B_1B_2 \), at right angles to \( A'B_1 \) is taken as rotation. These displacements can be taken in any order. After displacement of the bar parallel to itself, the deformation may be taken as shown by \( B_1B_2 \) and then rotation \( B_2B' \).
Next consider a part of a frame consisting of members $AB$ and $CB$, joined at $B$. Let $A$ get displaced to $A'$ and $C$ to $C'$ and let $AB$ be shortened by $\Delta_{AB}$ and $BC$ be elongated by $\Delta_{BC}$. $B'$ the final position of $B$ can be determined as follows:

The members $AB$ and $CB$ are separated out at $B$. $AB$ moves parallel to itself to position $A'B_1$ and $BC$ moves parallel to itself to position $C'B_2$. $B_1'B_2$ is made equal to the shortening of member $AB$ by $\Delta_{AB}$. $B_3B_4$ is made equal to the elongation of bar $CB$ by $\Delta_{BC}$. At $B_3$ perpendicular is drawn to the axis of member $AB$ i.e. at right angles to $A'B_1$ and at $B_4$ perpendicular is drawn to the axis of the member $BC$ i.e. at right angles to $C'B_2$. The intersection of these perpendiculars $B'$ will be the position of $B$. The movement of joint $B$ is $BB'$.

As the displacements of the members are very small compared to the length of the members, the displacement diagram is drawn showing only actual displacement to a bigger scale without drawing the lengths of the members. In Fig. 5.25, $BB_3B_4B_5B_6$ is the displacement diagram. Such diagram is called Williot diagram of displacement. This diagram is shown separately in Fig. 5.25 (b). It can be drawn as follows:

Take a pole $o$, draw $oa$ displacement of $A$ and $oc$ displacement of $C$. Draw $ab_1$ deformation of $AB$ equal to $\Delta_{AB}$ and parallel to $AB$, $\Delta_{AB}$ being drawn towards $A$ if there is shortening of the member as shown in Fig. 5.25 (b). Draw $cb_2$ deformation of $CB$ equal to $\Delta_{CB}$ and parallel to $CB$. $\Delta_{CB}$ is drawn away from $C$ as there is elongation of the member. Draw perpendiculars at $b_1$ and $c_2$. The intersection of these perpendiculars i.e. $b$ gives the position of the point. The displacement of $B$ will be given by $ob$.

The diagram drawn for finding deflection is known as Williot diagram.

Next consider truss consisting of three members $AB$, $BC$ and $CA$ as shown in Fig. 5.26. As the point $B$ is hinged, there:
will be no displacement of point B. Members AB and AC will be in compression and force in each of these members will be
\[ \frac{P}{2} \csc 60^\circ = \frac{P}{\sqrt{3}} \] and the force in BC will be tensile equal to
\[ \frac{P_{AB}}{\cos 60^\circ} = \frac{2}{\sqrt{3}}. \]

As the point C is supported on rollers at C, the joint can move only along BC by amount equal to the extension of the member BC.

Take any point o. As position of B is fixed, the point b will coincide with pole o. Draw bc equal to \( \triangle_{OB} \) in direction of BC. To get position of a on Williot diagram, \( \triangle_{AB} \) the shortening in member \( AB \) is drawn towards B in direction of \( AB \) as shown in Fig. 5.26 (b) by \( ba_2 \). Similarly \( \triangle_{AC} \) the shortening of member AC is drawn towards C in direction of AC as shown by \( ca_1 \). Perpendiculars are drawn at ends \( a_1 \) and \( a_2 \) of the deformations drawn. The intersection of these perpendiculars gives position of a. The displacement of joint A will be equal to \( oa \) in direction and magnitude.

![Diagram](image)

In some cases the displacement of two consecutive joints is at known and direction of displacement of one of the joints is
assumed and finally correction is applied. In the truss of Fig. 5-26 it will be assumed that $A$ moves along $AB$ and deflected form is drawn. Take any pole $o$. The point $B$ remains at the same position hence $b$ will be at $o$. The joint $A$ will move towards $B$ by $\Delta_{AB}$ as $AB$ is in compression. $oa$ is drawn in Williot diagram in direction of $AB$. At $'b'$, $\Delta_{BC}$ is drawn in the direction of $BC$ as $BC$ is in tension as shown by $bc$, $\Delta_{CA}$ is drawn at $a$ in direction of $CA$ as it is in compression as shown by $ac_a$. The position of $c$ will be given by intersection of perpendicularrs drawn at extremities $c_1$ and $c_2$ as shown in Fig. 5-27. The displacement of $C$ will be $oc$ and the deflected truss will be $A'BC'$. In actual case $C$ can have only horizontal movement and hence correction is to be applied for the rotation of the truss. The deflected truss is rotated by $\theta$ so that $C'$ will come to position $C''$ in line with $BC$ and $A'$ will come to $A''$ so that deflected truss will be $A''BC''$. The displacements of $C'$ and $A'$ due to rotation $\theta$ will be proportional to the lengths $CB$ and $AB$ respectively. This correction is applied to Williot diagram from consideration of the fact that movement of $C$ should be only horizontal and that vertical component of movement should be zero. The displacement $oc$ is equal to $oc'$ plus $c'c'$. By applying $c'o$ correction, the remaining displacement will be the required displacement $c'c'$. The correction for other joints is obtained by drawing the shape of the truss on $oc'$ as the base. Thus $a'b'c'$ is the correction applied, $a'c'$ is the rotation correction for $AB$ and it will be at right angles to $AB$. Similarly $c'o$ is the rotation correction for $BC$ and it will be at right angles to $BC$. The deflection of $A$ will be $a'a$ and deflection of $C$ will be $c'c'$. The correction applied is known as Mohr’s correction and final deflection diagram is known as Williot-Mohr diagram.
Next consider the truss shown in Fig. 5-28, where movement of C is to be parallel to the support. The polar diagram is drawn assuming A to move along AB. \( oc \) will be the displacement of joint C. As C can move parallel to the support due to rollers, C can move only along \( CC' \). \( CC' \) is equal to displacement \( CC'' \) plus \( CC'' \). By imposing displacement \( CC'' \), the correction can be applied. \( C'C'' \) is obtained by rotating deflected truss \( A'BC' \) about \( B \) by \( \theta \) as shown in Fig. 5-31. In the polar diagram \( oc \) is equal to vertical, and \( c'c'' \) parallel to \( CC'' \). By imposing correction \( c'o' \), the net displacement will be \( c'o' \). The correction for other joints obtained by drawing truss \( a'b'c' \) on \( oc' \) as base as shown in the figure. The displacement of A will be \( a'a'' \) and of C will be \( c'o' \).

Next consider the truss shown in Fig. 5-29. The area of cross-section of all members is same and all members have same lengths. It is assumed that D does not get displaced and \( h \) is displaced along BD. Taking any pole \( o, d \) will coincide with \( o \). Point 'b' will be downwards as BD is in compression. From B, \( ba_1 \) is drawn equal to \( \Delta AB \) in direction \( AB \) and from \( d, da_2 \) is drawn equal to \( \Delta AD \) in direction \( DA \). The intersection of perpendiculars drawn \( a_1 \) and \( a_2 \) gives position of a. From \( d, ac_1 \) is drawn equal to \( \Delta AD \) in direction \( DC \) and from b, \( bc_2 \) is drawn in direction \( OB \). The intersection of perpendiculars drawn at \( c_1 \) and \( c_2 \) gives position of c. The position of A cannot change but because of initia
assumptions that $D$ does not displace and $B$ displaces along $BD$, the displacement of $A$ comes to $oa$. The displacement diagram is shown as $A'B'C'D$. By moving the displacement diagram parallel to itself by $A'A$ the truss will take position $AB'C'D'$. This in the polar diagram is obtained by shifting the pole from $o$ to $a$. The final displacements will be for $B$, $ab$, for $C$, it will be $ac$ and for $D$, $ad$.

![Diagram of truss with displacements](image)

Fig. 5.29

When the frame is symmetrical and symmetrically loaded the above mentioned method of starting with central member is more convenient as the diagram will be symmetrical about centre line.

**Ex 5.15.** Find graphically the horizontal and vertical deflection of joint $A$ of the truss shown in Fig. 5.30 (a).

**Solution.** Forces in various members and values of $A$, $L$ and $\frac{1}{AL}$ are tabulated in Table 5.10.

Any pole $o$ is taken. Points $D$ and $G$ will coincide with $o$ as there is no change in position of $D$ and $G$ after loading.
Williot diagram is completed as shown in Fig. 5.30.
### Table 5.10

<table>
<thead>
<tr>
<th>Member</th>
<th>Length</th>
<th>Area</th>
<th>$P$</th>
<th>$PL/AE$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>2</td>
<td>8</td>
<td>-4000</td>
<td>-1000/E</td>
</tr>
<tr>
<td>$BC$</td>
<td>2</td>
<td>8</td>
<td>-4000</td>
<td>-1000/E</td>
</tr>
<tr>
<td>$CD$</td>
<td>2</td>
<td>8</td>
<td>-4000</td>
<td>-1000/E</td>
</tr>
<tr>
<td>$AE$</td>
<td>$\sqrt{5}$</td>
<td>12</td>
<td>$2000\sqrt{5}$</td>
<td>+833.3/E</td>
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<tr>
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<td>$\sqrt{5}$</td>
<td>12</td>
<td>$2000\sqrt{5}$</td>
<td>+833.3/E</td>
</tr>
<tr>
<td>$FO$</td>
<td>$\sqrt{5}$</td>
<td>12</td>
<td>$2000\sqrt{5}$</td>
<td>+833.3/E</td>
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<td>1</td>
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<td>0</td>
<td>0</td>
</tr>
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<td>$CE$</td>
<td>$\sqrt{5}$</td>
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<td>0</td>
<td>0</td>
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<td>$CF$</td>
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<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$FD$</td>
<td>$2\sqrt{2}$</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Horizontal deflection of joint $A = 0.15$ cm.
Vertical deflection = 0.58 cm.

**Ex. 5.16.** Find graphically the horizontal deflection of joint $E$ of the truss shown in Fig 5.31 (a). Figures in parenthesis show the areas of cross-section of various members. $E = 2 \times 10^6$ kg/cm².

**Solution.** Forces in various members and values of $PL/AE$ are tabulated in Table 5.11.

![Diagram](image)

*Fig. 5.31(a)*
It is seen that $D$ has rotated anti-clockwise with respect to $A$. Mohr correction is superposed on the Williot diagram. The horizontal movement of joint $E$ will be horizontal component of displacement $e'^{e}$. 

Fig. 5.31 (b), (c)
The horizontal deflection of joint $E$ is 0.03 cm.

**Ex. 5.17.** Find the vertical deflection of point $B$ of the truss shown in Fig. 5.32 and also find the horizontal deflection of the support $D$. $A$ is hinged and $D$ rests on rollers. Area of cross-section of each member is 48.5 cm$^2$ and $E = 2020$ t/cm$^2$.

*(A.M.I.E. May, 1967)*

**Solution.**

Deflection of $B$ \( \frac{487 \times 300}{9 \times 48.5 \times 2020} \) cm.

0.1657 cm.

Displacement of $D$ \( \frac{42 \times 300}{\sqrt{3} \times 48.5 \times 2020} \)

\[ = 0.7423 \text{ cm.} \]
Fig. 5.32
### Table 5.12

<table>
<thead>
<tr>
<th>Member</th>
<th>Length $L$</th>
<th>Area $A = 48.5 \text{ cm}^2$</th>
<th>$k_1$</th>
<th>$P_{k_1}$</th>
<th>$P_{k_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>$L = 3 \text{ m}$</td>
<td>- $\frac{11}{\sqrt{3}}$</td>
<td>- $\frac{2}{3\sqrt{3}}$</td>
<td>$-1$</td>
<td>$\frac{22}{9}$</td>
</tr>
<tr>
<td>$BC$</td>
<td>$\frac{21}{\sqrt{3}}$</td>
<td>- $\frac{1}{\sqrt{3}}$</td>
<td>$-1$</td>
<td>$63$</td>
<td>$\frac{21}{\sqrt{3}}$</td>
</tr>
<tr>
<td>$CD$</td>
<td>$\frac{10}{\sqrt{3}}$</td>
<td>- $\frac{1}{3\sqrt{3}}$</td>
<td>$-1$</td>
<td>$10$</td>
<td>$\frac{10}{\sqrt{3}}$</td>
</tr>
<tr>
<td>$DE$</td>
<td>$\frac{20}{\sqrt{3}}$</td>
<td>+ $\frac{2}{3\sqrt{3}}$</td>
<td>$0$</td>
<td>$40$</td>
<td>$\frac{40}{9}$</td>
</tr>
<tr>
<td>$EF$</td>
<td>$\frac{20}{\sqrt{3}}$</td>
<td>+ $\frac{2}{3\sqrt{3}}$</td>
<td>$0$</td>
<td>$40$</td>
<td>$\frac{40}{9}$</td>
</tr>
<tr>
<td>$FG$</td>
<td>$\frac{22}{\sqrt{3}}$</td>
<td>+ $\frac{4}{3\sqrt{3}}$</td>
<td>$0$</td>
<td>$88$</td>
<td>$\frac{88}{9}$</td>
</tr>
<tr>
<td>$OA$</td>
<td>$\frac{22}{\sqrt{3}}$</td>
<td>+ $\frac{4}{3\sqrt{3}}$</td>
<td>$0$</td>
<td>$88$</td>
<td>$\frac{88}{9}$</td>
</tr>
<tr>
<td>$BG$</td>
<td>$\frac{22}{\sqrt{3}}$</td>
<td>- $\frac{4}{3\sqrt{3}}$</td>
<td>$0$</td>
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<td>$BF$</td>
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<td>- $\frac{2}{3\sqrt{3}}$</td>
<td>$0$</td>
<td>$4$</td>
<td>$\frac{4}{9}$</td>
</tr>
<tr>
<td>$CF$</td>
<td>$\frac{2}{\sqrt{3}}$</td>
<td>+ $\frac{2}{3\sqrt{3}}$</td>
<td>$0$</td>
<td>$4$</td>
<td>$\frac{4}{9}$</td>
</tr>
<tr>
<td>$CE$</td>
<td>$\frac{20}{\sqrt{3}}$</td>
<td>- $\frac{2}{3\sqrt{3}}$</td>
<td>$0$</td>
<td>$40$</td>
<td>$\frac{40}{9}$</td>
</tr>
</tbody>
</table>

Ex 5.18. The Warren girder as shown in Fig. 5.33 carries a load of 10t at the panel point B. The end A is hinged and D is on rollers. The members are so proportioned that all members have the same stress of 0.945 t/cm².

Find (a) the vertical deflection of the panel point C and (b) the horizontal deflection of the support D. $E = 2000$ t/cm².

(A.M.I.E. May 1963)
\text{Solution.}

\[ \Delta = \sum \frac{PLk}{AE}. \]

Deflection of \( U \), \( \Delta_U = \frac{94.5 \times 20}{\sqrt{3} \times 2000} = 0.5455 \text{ cm.} \]

Deflection of \( D \), \( \Delta_D = \frac{-850.5}{2000} = -0.42525 \text{ cm. (towards left)} \)
<table>
<thead>
<tr>
<th>Member</th>
<th>( \frac{PL}{A} )</th>
<th>( k_1 )</th>
<th>( k_2 )</th>
<th>( \frac{PLk_1}{A} )</th>
<th>( \frac{PLk_2}{A} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( AB )</td>
<td>-0.945 x 300</td>
<td>- ( \frac{1}{3\sqrt{3}} )</td>
<td>1</td>
<td>+ ( \frac{94.5}{\sqrt{3}} )</td>
<td>-94.5 x 3</td>
</tr>
<tr>
<td>( BC )</td>
<td>-0.945 x 300</td>
<td>- ( \frac{1}{\sqrt{3}} )</td>
<td>1</td>
<td>+ ( \frac{94.5 x 3}{\sqrt{3}} )</td>
<td>-94.5 x 3</td>
</tr>
<tr>
<td>( CD )</td>
<td>-0.945 x 300</td>
<td>- ( \frac{2}{3\sqrt{3}} )</td>
<td>1</td>
<td>+ ( \frac{94.5 x 2}{\sqrt{3}} )</td>
<td>-94.5 x 3</td>
</tr>
<tr>
<td>( EF )</td>
<td>+0.945 x 300</td>
<td>+ ( \frac{2}{3\sqrt{3}} )</td>
<td>0</td>
<td>+ ( \frac{94.5 x 2}{\sqrt{3}} )</td>
<td>0</td>
</tr>
<tr>
<td>( FG )</td>
<td>+0.945 x 300</td>
<td>+ ( \frac{4}{3\sqrt{3}} )</td>
<td>0</td>
<td>+ ( \frac{94.5 x 4}{\sqrt{3}} )</td>
<td>0</td>
</tr>
<tr>
<td>( AE )</td>
<td>+0.945 x 300</td>
<td>+ ( \frac{2}{3\sqrt{3}} )</td>
<td>0</td>
<td>+ ( \frac{94.5 x 2}{\sqrt{3}} )</td>
<td>0</td>
</tr>
<tr>
<td>( BE )</td>
<td>-0.945 x 300</td>
<td>- ( \frac{2}{3\sqrt{3}} )</td>
<td>0</td>
<td>+ ( \frac{94.5 x 2}{\sqrt{3}} )</td>
<td>0</td>
</tr>
<tr>
<td>( BF )</td>
<td>-0.945 x 300</td>
<td>+ ( \frac{2}{3\sqrt{3}} )</td>
<td>0</td>
<td>- ( \frac{94.5 x 2}{\sqrt{3}} )</td>
<td>0</td>
</tr>
<tr>
<td>( CF )</td>
<td>+0.945 x 300</td>
<td>- ( \frac{2}{3\sqrt{3}} )</td>
<td>0</td>
<td>- ( \frac{94.5 x 2}{\sqrt{3}} )</td>
<td>0</td>
</tr>
<tr>
<td>( CG )</td>
<td>-0.945 x 300</td>
<td>- ( \frac{4}{3\sqrt{3}} )</td>
<td>0</td>
<td>+ ( \frac{94.5 x 4}{\sqrt{3}} )</td>
<td>0</td>
</tr>
<tr>
<td>( DG )</td>
<td>+0.945 x 300</td>
<td>+ ( \frac{4}{3\sqrt{3}} )</td>
<td>0</td>
<td>+ ( \frac{94.5 x 4}{\sqrt{3}} )</td>
<td>0</td>
</tr>
</tbody>
</table>

\( \Sigma \) | \( \frac{94.5 x 20}{\sqrt{3}} \) | -850.5 |
PROBLEMS

1. Find the horizontal movement of the roller and $B$ of the frame shown in Fig. 5·34. Area of cross-section of all members is 20 cm$^2$. $E = 2 \times 10^8$ kg/cm$^2$.

![Fig. 5·34](image)

Find the value of horizontal thrust at $B$ when it is hinged.  
[Ans. $\Delta_H = 0·541$ cm. outwards.  
$H = 1061·2$ kg.]

2. All the members shown in Fig. 5·35 are so proportioned that under the given loading all compression members are stressed to 500 kg/cm$^2$ and tension members to 1000 kg/cm$^2$. Determine the horizontal and vertical movement of $A$.

$$E = 2 \times 10^8 \text{ kg/cm}^2.$$  
[Ans. $\Delta_V = 1·1375$ cm.  
$\Delta_H = 1·3667$ cm.]

3. A beam $ABC$ is supported by pin jointed members $AE$, $BE$ and $BD$ each having area of 40 cm$^2$. $I$ for the beam is 10,000 cm$^4$. The beam carries load of 5000 kg, inclined at 45° as shown in Fig. 5·36. Determine the vertical deflection of $C$ considering bending moment and thrust. $E = 2 \times 10^8$ kg/cm$^2$.

![Fig. 5·36](image)  
[Ans. $\Delta = 0·243$ cm.]
4. When equal loads act at the lower panel points of the steel truss shown in Fig. 5.37, the stress in all tension members is \( f \) kg/cm\(^2\) and in compression members is 0.8\( f \) kg/cm\(^2\). Find \( f \) if the ratio of maximum deflection to span is 1/900.

![Fig. 5.37]

[Ans. \( f = 1293 \) kg/cm\(^2\)].

5. All members of steel truss shown in Fig. 5.38 have cross-sectional area of 6 cm\(^2\). A vertical load at \( F \) causes the joint \( G \) to deflect 5 mm. downwards. Find the load at \( F \).

\[ E = 2.1 \times 10^6 \text{ kg/cm}^2. \]

[Ans. 3782.4 kg.]

![Fig. 5.38]

![Fig. 5.39]

6. Find the vertical deflection of joint \( X \) in the truss shown in Fig. 5.39 due to load of \( 2P \) there. All the members have cross-sectional area \( A \).

[Ans. \( \Delta x = 100.67 \frac{Pl}{AE} \)].

7. In the truss shown in Fig. 5.40 the cross-sectional area of the chord members is 2\( A \) and of the web members is \( A \). Find the vertical deflection of \( F \) caused by the vertical load of \( P \) at \( E \) and rise of \( t^\circ C \) in the temperature of lower chord members. Co-efficient of linear expansion is \( \alpha \).

![Fig. 5.40]

[Ans. \( \Delta F = \frac{2400P}{AE} - 720 \text{ cm.} \)]
8. Find the vertical deflection of joint $L_1$ in the truss shown in Fig. 5.41. The different numbers along members indicate their areas in $\text{cm}^2$. $E = 2.1 \times 10^6 \text{ kg/cm}^2$. [Ans. $\Delta = 0.038 \text{ cm}$.]

9. A Warren truss as shown in Fig. 5.42 is made up of 3 equilateral triangles, each side 4.69 m. Its left support is hinged, while the right support is on rollers. It is subjected to a horizontal force of 10 tonnes at $C$. The cross-sectional area of each of the tension members is 6.45 cm$^2$ while that of each of compression members is 12.9 cm$^2$. Taking $E$ for the material as 2000 tonnes/cm$^2$ calculate the horizontal deflection of $C$ and the vertical deflection of $E$. 
6.1. In a member if the slopes at the ends and the relative displacement of the ends are known, the moment at the ends can be found in terms of slopes, deflections and the stiffness of the member k which is equal to ratio of moment of inertia of the member to its length for a prismatic member i.e. a member having constant moment of inertia throughout its length.

In this method, clockwise moments at the ends are taken as positive and anti-clockwise moments as negative. The slope is +ve if the tangent rotates in a clockwise direction. If the tangent at the end rotates in anti-clockwise direction the slope is -ve. The displacement of the right end with respect to left end, if downwards will be +ve and if upwards will be negative.

Consider a prismatic member AB with certain load on it (Fig. 6.2). Considering the ends as fixed, the fixed end moments
can be calculated for the member. Let these moments be \( M_{AB} \) and \( M_{BA} \) at ends \( A \) and \( B \) respectively.

Let \( +\theta_A \) be slope at \( A \) and end \( B \) be considered fixed as shown in Fig. 6.2 (b). As already proved in Chapter 2, the moment caused by rotation \( \theta_A \) at \( A \) will be \( +4Ek\theta_A \) at \( A \) and \( +2Ek\theta_A \) at \( B \).

Let \( +\theta_B \) be slope at \( B \) and end \( A \) be considered fixed as shown in Fig. 6.2 (c). The moment caused by rotation \( +\theta_B \) at \( B \) will be \( +4Ek\theta_B \) at \( B \) and \( +2Ek\theta_B \) at \( A \).

Fig. 6.2

Let \( +\delta \) be the displacement of \( B \) with respect to \( A \) with both ends fixed as shown in Fig. 6.2 (d). The moments at ends \( A \) and \( B \) will be \( -\frac{6EI\delta}{l^2} = -\frac{6Ek\delta}{l} \).

Superposing all the four conditions, the total moment \( M_{AB} \) and \( M_{BA} \) will be,

\[
M_{AB} = 4Ek\theta_A + 2Ek\theta_B - 6EI \frac{\delta}{l} + M_{AB}
\]

\[
= 2Ek\left[ 2\theta_A + \theta_B - \frac{3\delta}{l} \right] + M_{AB} \quad \ldots (6.1)
\]

\[
M_{BA} = 4Ek\theta_B + 2Ek\theta_A - 6Ek \frac{\delta}{l} + M_{BA}
\]

\[
= 2Ek\left[ 2\theta_B + \theta_A - \frac{3\delta}{l} \right] + M_{BA} \quad \ldots (6.2)
\]

The final moments at ends \( A \) and \( B \) in terms of rotations at ends, relative displacement and fixed end moments are given by equations (6.1) and (6.2).
6.2. Analysis of Indeterminate Beams.

1. Propped cantilevers.

The propped cantilever $AB$ will have some slope $\theta_B$ at the end $B$. The slope at fixed end $A$ will be zero.

\[
M_{BA} = 2Ek \left[ 2\theta_B + \theta_A - \frac{3\delta}{l} \right] + \overline{M}_{BA}
\]

\[
= 2Ek \left[ 2\theta_B - \frac{3\delta}{l} \right] + \overline{M}_{BA}
\]

Fig. 6.3

As $\theta_A = 0$

In case there is no yielding of supports $\delta = 0$

\[
M_{BA} = 2Ek[2\theta_B] + \overline{M}_{BA}
\]

\[
= 4Ek\theta_B + \overline{M}_{BA}
\]

If the end $B$ is simply supported $M_{BA} = 0$

\[
\therefore 4Ek\theta_B + \overline{M}_{BA} = 0
\]

\[
\therefore \theta_B = - \frac{\overline{M}_{BA}}{4Ek}
\]

\[
M_{AB} = 2Ek[\theta_B] + \overline{M}_{AB}
\]

\[
= 2Ek[- \frac{\overline{M}_{BA}}{4Ek}] + \overline{M}_{AB}
\]

\[
= \frac{\overline{M}_{BA}}{2} + \overline{M}_{AB}
\]

If the end $B$ has an overhang on right side, the moment $M_{BA}$ will be equal to cantilever moment. Equating equation (1) equal to $M_{BA}$, the value of $\theta_B$ can be found. Substituting this value in equation (2), the value of $M_{AB}$ can be calculated.

6.3. Continuous beams.

Consider the continuous beam shown in Fig. 6.4. The beam will have slopes $\theta_A$, $\theta_B$, $\theta_C$, and $\theta_D$ at $A$, $B$, $C$ and $D$ respectively.

Let $+\delta_1$, $+\delta_2$ and $+\delta_3$ be relative displacements of end $B$ with respect to $A$, $C$ with respect to $B$ and $D$ with respect to $C$ respectively. The equations for moments $M_{AB}$, $M_{BA}$, $M_{BC}$, $M_{CB}$, $M_{CD}$ and $M_{DC}$ can be written in terms of slopes and displacements.

If the ends $A$ and $D$ are simply supported the moments at ends $A$ and $D$ will be zero. The sum of the moments at $B$ i.e., $M_{BA} + M_{BC}$ and the sum of the moments at $C$ i.e., $M_{CB} + M_{CD}$ will be zero.

\[
\therefore M_{AB} = 0
\]

\[
M_{DC} = 0
\]

\[
M_{BA} + M_{BC} = 0
\]

\[
M_{CB} + M_{CD} = 0
\]

\[
\cdots (1)
\]

\[
\cdots (2)
\]

\[
\cdots (3)
\]

\[
\cdots (4)
\]
With the help of these four equations, four unknowns $\theta_A$, $\theta_B$, $\theta_C$ and $\theta_D$ can be solved and moments determined.

In the above continuous beam if the end is fixed slope $\theta_A = 0$ and there will be only three unknowns $\theta_B$, $\theta_C$ and $\theta_D$ which can be evaluated from the equations—

\[
\begin{align*}
M_{DC} &= 0 \quad \ldots(1) \\
M_{BA} + M_{BC} &= 0 \quad \ldots(2) \\
M_{CB} + M_{CD} &= 0 \quad \ldots(3)
\end{align*}
\]

In case ends $A$ and $D$ are fixed $\theta_A = 0$ and $\theta_D = 0$ and there are only two unknowns $\theta_B$ and $\theta_C$. These can be evaluated from the equations

\[
\begin{align*}
M_{BA} + M_{BC} &= 0 \quad \ldots(1) \\
M_{CB} + M_{CD} &= 0 \quad \ldots(2)
\end{align*}
\]

**Ex. 6.1.** Find the moment at support $A$ of the propped cantilever shown in Fig. 6.5 (a), when

(a) the supports are at the same level.

(b) the support $B$ sinks by 1 cm. $EI = 2 \times 10^6$ kg. cm$^3$.

**Solution.** (a) When the supports are at the same level.

\[
\begin{align*}
\dd{M}_{AR} &= -2 \times 4 \\
&= -8 \\
\dd{M}_{RA} &= 2 \times 4 \\
&= +8 \\
\theta_A &= 0.
\end{align*}
\]

$M_{BA}$ will be zero as support $B$ is simply supported.

\[
\begin{align*}
M_{BA} &= 2Ek[2\theta_B + \theta_A] + 1 \\
0 &= 2Ek[2\theta_B] + 1
\end{align*}
\]

\[\therefore \theta_B = -\frac{1}{4Ek}\]

\[
\begin{align*}
M_{AB} &= 2Ek[2\theta_A + \theta_B] + M_{AB} \\
&= 2Ek\left(-\frac{1}{4Ek}\right) - 1 \\
&= -\frac{1}{2} - 1 \\
&= -\frac{3}{2} \text{ t.m.}
\end{align*}
\]

Taking moments about $A$,

\[
\begin{align*}
\dd{M}_{AB} &= 2 \times 13 \times 4 \times 10^{-3} \times 5/81 \times 2 \\
&= 1.875 \text{ t.m.}
\end{align*}
\]

Fig. 6.5
\[ M_{AB} + R_B \times 4 = 2 \times 2 \]

\[ R_B = 1 - \frac{M_{AB}}{4} = 1 - \frac{8}{8} \]

\[ = 5/8 \text{ t.} \]

S.F. diagram is shown in Fig. 6·5 (b).

(b) When support B sinks by 1 cm.

\[ \frac{l}{l} \cdot \frac{1}{400} \]

\[ E_k = \frac{EI}{l} \]

\[ - \frac{2 \times 10^6}{400} \text{ kg. cm.} \]

\[ = 0.5 \times 10^7 \text{ kg. cm.} \]

\[ = 50 \text{ t-m.} \]

\[ M_{BA} = 0 \]

\[ M_{BA} = 2E_k \left[ 2\theta_B + \theta_A - \frac{38}{l} \right] + 1 \]

\[ 0 = 2 \times 50 \left[ 2\theta_B - \frac{3}{400} \right] + 1 \]

\[ 2\theta_B = -\frac{1}{100} + \frac{3}{400} \]

\[ = \frac{1}{800} \]

\[ M_{AB} = 2E_k \left[ 2\theta_A + \theta_B - \frac{38}{l} \right] - 1 \]

\[ = 2 \times 50 \left[ -\frac{1}{800} - \frac{3}{400} \right] - 1 \]

\[ = 100 \left[ -\frac{7}{800} \right] - 1 \]

\[ = -0.875 - 1 = -1.875 \text{ t-m.} \]

Taking moments about A,

\[ 1.875 \times R_B \times 4 = 2 \times 2 \]

\[ R_B = 1 - \frac{1.875}{4} \]

\[ = 1 - 0.4687 \]

\[ = 0.5313 \text{ t.} \]

S.F. and B.M. diagrams and deflected form are shown in Fig. 6·5 (d, e, f).
Ex. 6.2. Analyse the continuous beam shown in Fig. 6.6 (a) by slope deflection method. The supports are at same level and the beam is of constant stiffness throughout.

Solution. The slope at end A will be zero. Let \( \theta_B \) and \( \theta_C \) be the slopes at B and C respectively. Let \( k \) be the stiffness of beam

\[
\overline{M_{AB}} = -\frac{wl^2}{12}
\]

\[
= -\frac{1 \times 6 \times 6}{12}
\]

\[
= -3 \text{ t.m.}
\]

\[
\overline{M_{BA}} = +3 \text{ t.m.}
\]

\[
\overline{M_{BC}} = -\frac{3 \times 4 \times 4}{12}
\]

\[
= -4 \text{ t.m.}
\]

\[
\overline{M_{CB}} = +\frac{3 \times 4 \times 4}{12}
\]

\[
= +4 \text{ t.m.}
\]

\( \theta_A = 0, \quad \delta = 0 \)

\[
M_{AB} = 2Ek(\theta_B) - 3
\]

\[
= 2Ek\theta_B - 3
\]

\[
M_{BA} = 2Ek(2\theta_B) + 3
\]

\[
= 4Ek\theta_B + 3
\]

\[
M_{BC} = 2Ek[2\theta_B + \theta_C] - 4
\]

\[
M_{CB} = 2Ek [\theta_B + 2\theta_C] + 4
\]

As the end C is simply supported \( M_{CB} = 0 \).

\[
\therefore \quad 2Ek[\theta_B + 2\theta_C] + 4 = 0
\]

\[
\theta_B + 2\theta_C = -\frac{2}{Ek}
\]  \( \ldots (1) \)

Sum of the moments at joint B is zero.

\[
\therefore \quad M_{BA} + M_{BC} = 0
\]

\[
4Ek\theta_B + 3 + 2Ek[2\theta_B + \theta_C] - 4 = 0
\]

\[
4Ek\theta_B + 4Ek\theta_B + 2Ek\theta_C = 1
\]

\[
\therefore \quad 8\theta_B + 2\theta_C = \frac{1}{Ek}
\]  \( \ldots (2) \)
(2 - 1) gives
\[ \theta_B = \frac{\omega}{E k} \]
\[ \theta_B = \frac{\omega}{7E k} \]
\[ 2\theta_C = \frac{1}{E k} \theta_B - 3 \theta_B \]
\[ = \frac{1}{E k} \times \frac{\omega}{7E k} - 3 \]
\[ = \frac{6}{7} - 3 = -\frac{15}{7} \]
\[ = -2.143 \text{ t.m.} \]
\[ M_{AB} = 4E k \theta_B + 3 \]
\[ = 4E k \left[ -\frac{3}{7E k} \right] + 3 \]
\[ = \frac{12}{7} + 3 = 3.33 \]
\[ = +4.714 \text{ t.m.} \]

B.M. diagram and deflected form are shown in Fig. 6.6.

**Ex. 6.3.** Analyse the continuous beam shown in Fig. 6.7 by method of slope deflection. The beam is of constant section through out its length and supports remain at same level after loading.

**Solution.** Let \( \theta_A, \theta_B, \theta_C \) and \( \theta_D \) be the slopes at ends \( A, B, C \) and \( D \) respectively. As the beam is symmetrical and is symmetrically loaded about centre line,

\[ \theta_A = -\theta_D, \]

and, \( \theta_B = -\theta_C. \)

\[ M_{AB} = -\frac{wl^2}{12} \]

\[ M_{BA} = +\frac{wl^2}{12} \]

\[ M_{AB} = 2Ek\left(2\theta_A + \theta_B\right) \]

\[ \frac{wl^2}{12} - \frac{wl^2}{12} \]

Fig. 6.7
\[ M_{BA} = 2Ek \left( 2\theta_B + \theta_A \right) + \frac{wl^2}{12} \]
\[ M_{BC} = 2Ek \left( 2\theta_B + \theta_C \right) - \frac{wl^2}{12} \]

But

\[ \theta_C = -\theta_B \]

\[ \therefore \quad M_{BC} = 2Ek\theta_B - \frac{wl^2}{12} \]

As the end \( A \) is simply supported, \( M_{AB} = 0 \).

\[ \therefore \quad 2Ek \left( 2\theta_A + \theta_B \right) - \frac{wl^2}{12} = 0 \]

\[ 2\theta_A + \theta_B = +\frac{wl^2}{24Ek} \quad \text{(1)} \]

Sum of the moments at the joint \( B \) must be zero.

\[ \therefore \quad M_{BA} + M_{BC} = 0 \]

\[ 2Ek \left( 2\theta_B + \theta_A \right) + \frac{wl^2}{12} + 2Ek\theta_B - \frac{wl^2}{12} = 0 \]

\[ \therefore \quad 3\theta_B + \theta_A = 0 \]

\[ \theta_A = -3\theta_B \]

Substituting in (1)

\[ -6\theta_B + \theta_B = +\frac{wl^2}{24Ek} \]

\[ \theta_B = -\frac{wl^2}{120Ek} \]

\[ \theta_A = -3\theta_B = +\frac{wl^2}{40Ek} \]

\[ M_{BA} = 2Ek \left( 2\theta_B + \theta_A \right) + \frac{wl^2}{12} \]

\[ = 2Ek \left[ -2 \times \frac{wl^2}{120Ek} + \frac{wl^2}{40Ek} \right] + \frac{wl^2}{12} \]

\[ = \frac{wl^2}{60} + \frac{wl^2}{12} \]

\[ = \frac{wl^2}{10} \]

From symmetry \( M_{CB} \) will also be \( \frac{wl^2}{10} \).

Taking moments about \( B \),

\[ R_A \times l + \frac{wl^2}{10} - \frac{wl^2}{2} = 0 \]

\[ \therefore \quad R_A = \frac{wl}{2} - \frac{wl}{10} = \frac{2}{5} \cdot \frac{wl}{5} \]
\[ R_D = \frac{2}{5}wl \]
\[ R_B = R_C = \frac{1}{2} \left( 3wl - 2 \times \frac{2}{5}wl \right) \]
\[ = \frac{1}{2} \times \frac{11}{5}wl \]
\[ = \frac{11}{10}wl \]

S.F. and B.M. diagrams are shown in Fig. 6·7 (b) and (c).

**Ex. 6·4.** A continuous beam of constant moment of inertia is loaded as shown in Fig. 6·8 (a). Find the support moments and draw the B.M. diagram.

**Solution.** Slope at fixed end \(A\), \(\theta_A\) will be zero. Let \(\theta_B\) and \(\theta_C\) be the slopes at \(B\) and \(C\) respectively.

The loading in the span \(BC\) will produce clockwise moment of \(4000 \times 1\) kg. m. at the centre of span. The fixed end moments due to this will be

\[ 4000\text{kg}. \]
\[ 3m. \]
\[ 3m. \]

\[ \begin{align*}
M_{BC} &= M_{CB} = + \frac{M}{4} - + \frac{4000}{4} + 1000 \text{ kg. m.} \\
M_{CD} &= -2000 \times 3 = -6000 \text{ kg. m.} \\
M_{AB} &= \frac{2EI}{8} \left( 0 + \theta_B \right) + 0 = \frac{E}{4} \theta_B \\
M_{BA} &= \frac{2EI}{c} \left( 2\theta_B \right) + 0 = \frac{E}{2} \theta_B
\end{align*} \]
\[ M_{BC} = \frac{2EI}{6} (2\theta_B + \theta_C) + 1000 = \frac{EI}{3} (2\theta_B + \theta_C) + 1000 \]
\[ M_{CB} = \frac{2EI}{6} (\theta_B + 2\theta_C) + 1000 = \frac{EI}{3} (\theta_B + 2\theta_C) + 1000 \]

Sum of the moments at joint B is zero.
\[ \therefore M_{BA} + M_{BC} = 0 \]
\[ \frac{EI\theta_B}{2} + \frac{EI}{3} (2\theta_B + \theta_C) + 1000 = 0 \]
\[ 3\theta_B + 4\theta_B + 2\theta_C + \frac{6000}{EI} = 0 \]
\[ 7\theta_B + 2\theta_C = -\frac{6000}{EI} \quad \text{(1)} \]

Sum of the moments at joint C is zero.
\[ \therefore M_{CB} + M_{CD} = 0 \]
\[ EI (\theta_B + 2\theta_C) + 1000 - 6000 = 0 \]
\[ \therefore \theta_B + 2\theta_C = \frac{15,000}{EI} \quad \text{(2)} \]

Eqn. (1) – Eqn. (2) gives
\[ 6\theta_B = -\frac{21,000}{EI} \]
\[ \therefore \theta_B = -\frac{3500}{EI} \]

Substituting in (2),
\[ 2\theta_C = \frac{150,000}{EI} + \frac{3500}{EI} \]
\[ \therefore \theta_C = \frac{9,250}{EI} \]

\[ M_{AB} = \frac{EI\theta_B}{4} = \frac{EI}{4} \left( -\frac{3500}{EI} \right) = -875 \text{ kg} \cdot \text{m}. \]
\[ M_{BA} = \frac{EI\theta_B}{2} = \frac{EI}{2} \times \left( -\frac{3500}{EI} \right) = -1750 \text{ kg} \cdot \text{m}. \]

B.M. diagram is shown in Fig. 6.8 (b).

Ex. 6.5. A continuous beam ABCD 18 m. long is loaded as shown in Fig. 6.9. During loading support B sinks by 1 cm. Find support moments. \( I = 8,000 \text{ cm}^4, \ E = 2 \times 10^6 \text{ kg/cm}^2. \)

Solution. Let \( \theta_A, \theta_B, \theta_C \) and \( \theta_D \) be slopes at A, B, C and D respectively.

\[ EI = 8000 \times 2 \times 10^4 \text{ kg} \cdot \text{cm}^3 = 16 \times 20^5 \text{ kg} \cdot \text{m}^3. \]

\[ M_{AB} = -\frac{Wab^3}{4} = -\frac{4,000 \times 3 \times 1 \times 1}{4 \times 4} \]
\[ = -750 \text{ kg} \cdot \text{m}. \]
\[ M_{BA} = + \frac{Wa^2b}{l^2} = + \frac{4,000 \times 1 \times 3^2}{4 \times 4} = +2250 \text{ kg} \cdot \text{m.} \]

![Diagram](image)

\[ M_{BC} = -\frac{wL^2}{12} = -\frac{1000 \times 8 \times 8}{12} = -5333 \text{ kg} \cdot \text{m.} \]

\[ M_{CB} = +\frac{wL^2}{12} = +5333 \text{ kg} \cdot \text{m.} \]

\[ M_{CD} = -\frac{Wl}{8} = -\frac{4000 \times 6}{8} = -3000 \text{ kg} \cdot \text{m.} \]

\[ M_{DC} = +\frac{Wl}{8} = +\frac{4000 \times 6}{8} = +3000 \text{ kg} \cdot \text{m.} \]

Support B sinks by 1 cm. \[ \frac{1}{100} \]

\[ M_{AB} = \frac{2E \times I}{4} \left( 2\theta_A + \theta_B - \frac{3\delta}{4} \right) - 750. \]

\[ = \frac{EI}{2} \left( 2\theta_A + \theta_B - \frac{3}{400} \right) - 750 \]

\[ M_{BA} = \frac{2E \times I}{4} \left( 2\theta_B + \theta_A - \frac{3\delta}{4} \right) + 2250 \]

\[ - \frac{EI}{2} \left( \theta_A + 2\theta_B - \frac{3}{400} \right) + 2250 \]

\[ M_{BC} = \frac{2E \times 2I}{8} \left( 2\theta_B + \theta_C + \frac{3\delta}{8} \right) - 5333 \]

\[ = \frac{EI}{2} \left( 2\theta_B + \theta_C + \frac{3}{800} \right) - 5333 \]

\[ M_{CB} = \frac{2E \times 2I}{8} \left( \theta_B + 2\theta_C + \frac{3\delta}{8} \right) + 5333 \]

\[ = \frac{EI}{2} \left( \theta_B + 2\theta_C + \frac{3}{800} \right) + 5333 \]

\[ M_{CD} = \frac{2E \times 1.5I}{6} \left( 2\theta_C + \theta_D - 0 \right) - 3000 \]

\[ = \frac{EI}{2} \left( 2\theta_C + \theta_D \right) - 3000 \]

\[ M_{DC} = \frac{2E \times 1.5I}{6} \left( \theta_C + 2\theta_D - 0 \right) + 3000 \]

\[ - \frac{EI}{2} \left( \theta_C + 2\theta_D \right) + 3000 \]
The end $A$ is simply supported

\[
M_{AB} = 0 \quad \therefore \quad \frac{EI}{2} \left( 2\theta_A + \theta_B - \frac{3}{400} \right) - 750 = 0
\]

\[
2\theta_A + \theta_B - \frac{3}{400} = \frac{1500}{EI}
\]

\[
2\theta_A = \frac{1500}{16 \times 10^5} + \frac{3}{400} - \theta_B
\]

\[
= \frac{27}{3200} - \theta_B
\]

\[
\therefore \quad \theta_A = \frac{27}{6400} - \theta_B/2 \quad \ldots(1)
\]

The end $D$ is simply supported

\[
M_{DC} = 0 \quad \therefore \quad \frac{EI}{2} \left( \theta_C + 2\theta_D \right) + 3000 = 0
\]

\[
\theta_C + 2\theta_D = - \frac{6000}{EI}
\]

\[
2\theta_D = - \frac{6000}{16 \times 10^5} - \theta_C
\]

\[
\therefore \quad \theta_D = - \frac{3}{1600} - \frac{\theta_C}{2} \quad \ldots(2)
\]

Sum of the moments at support $B$ is zero.

\[
M_{BA} + M_{BC} = 0
\]

\[
\therefore \quad \frac{EI}{2} \left( \theta_A + 2\theta_B - \frac{3}{400} \right) + 2250 + \frac{EI}{2} \left( 2\theta_B + \theta_C + \frac{3}{800} \right) - 5333 = 0
\]

\[
\theta_A + 4\theta_B + \theta_C - \frac{3}{800} = \frac{3083 \times 2}{-EI}
\]

\[
\theta_A + 4\theta_B + \theta_C = - \frac{3}{800} + \frac{3083 \times 2}{16 \times 10^5}
\]

\[
\theta_A + 4\theta_B + \theta_C = \frac{6.083}{800}
\]

Substituting $\theta_A$ from (1) in above equation

\[
\frac{27}{6400} - \frac{\theta_B}{2} + 4\theta_B + \theta_C = \frac{6.083}{800}
\]

\[
\frac{7}{2} \theta_B + \theta_C = \frac{6.083}{800} - \frac{27}{6400}
\]

\[
\frac{7}{2} \theta_B + \theta_C = \frac{2.708}{800}
\]

\[
7\theta_B + 2\theta_C = \frac{2.708}{400} \quad \ldots(3)
\]
Sum of the moments at joint $C$ is zero

\[
\frac{EI}{2} \left( \theta_B + 2\theta_C + \frac{3}{800} \right) + 5333 + \frac{EJ}{2} \left( 2\theta_C + \theta_D \right) - 3000 = 0
\]

\[
\theta_B + 4\theta_C + \theta_D = -\frac{3}{800} - \frac{2333 \times 2}{EJ}
\]

\[
\theta_B + 4\theta_C + \theta_D = \frac{3}{800} \times \frac{2333 \times 2}{16 \times 10^5}
\]

\[
\theta_B + 4\theta_C + \theta_D = \frac{5333}{800}
\]

Substituting $\theta_D$ from (2)

\[
\theta_B + 4\theta_C = \frac{3}{1600} \theta_C - \frac{5333}{800}
\]

\[
\theta_B + \frac{7}{2} \theta_C = -\frac{3833}{800}
\]

\[
2\theta_B + 7\theta_C = -\frac{3833}{400} \quad \ldots(4)
\]

Multiplying by 7,

\[
14\theta_B + 49\theta_C = -\frac{26831}{400} \quad \ldots(5)
\]

Multiplying (3) by (2),

\[
14\theta_B + 4\theta_C = \frac{5416}{800} \quad \ldots(6)
\]

Eqn. (5) - Eqn. (6) gives

\[
45\theta_C = -\frac{32247}{400}
\]

\[
\therefore \quad \theta_C = -\frac{07166}{400}
\]

\[
2\theta_B + 7\theta_C = -\frac{3833}{400}
\]

\[
\therefore \quad 2\theta_B = -\frac{3833}{400} - 7\theta_C = -\frac{3833}{400} + 7 \times \frac{07166}{400}
\]

\[
= \frac{11832}{400}
\]

\[
\therefore \quad \theta_B = \frac{05916}{400}
\]

\[
\theta_D = -\frac{3}{1600} \theta_C = -\frac{3}{1600} \theta_C + \frac{07166}{2 \times 400}
\]

\[
\therefore \quad \theta_D = -\frac{3917}{400}
\]

\[
\theta_A = \frac{27}{6400} \theta_B - \frac{16875}{400} - \frac{02958}{400}
\]
\[ \theta_A = \frac{1.3917}{400} \]
\[ M_{EA} = \frac{EI}{4} \left[ \theta_A + 2\theta_B - \frac{3}{400} \right] + 2250 \]
\[ = \frac{16 \times 10^5}{2} \left[ 1.3917 + 0.5916 \times 2 - \frac{3}{400} \right] + 2250 \]
\[ = -850.2 + 2250 \]
\[ = 1399.8 \text{ kg.m.} \]
\[ M_{CD} = \frac{EI}{2} (2\theta_C + \theta_D) = 3000 \]
\[ = -6649.8 \text{ kg.m.} \]

**Ex. 6-6.** A continuous beam ABCD, 20 m long is continuous over 3 spans. AB = 8 m, BC = 4 m, and CD = 8 m. Moment of inertia of AB is 21, that of BC is 1 and that of CD is 21. There is a uniformly distributed load of 1500 kg/m over spans AB and BC. On CD there is a central load of 4000 kg. The ends are fixed and during load B support B sinks by 1 cm. Find the fixed end moments. 

\[ = 16,000 \text{ cm}^4, E = 2 \times 10^5 \text{ kg/cm}^2. \]

**Solution.** Slope at ends A and D will be zero as the ends are fixed. Let \( \theta_B \) and \( \theta_C \) be the slopes at B and C respectively.

\[ \theta = \frac{1}{100} \text{ m.} \]

\[ EI = \frac{16,000 \times 2 \times 10^6}{100 \times 100} \text{ kg. m}^2 \]
\[ = 32 \times 10^6 \text{ kg. m}^2 \]
\[ \bar{M}_{AB} = -\frac{1500 \times 8 \times 8}{12} \]
\[ = -8000 \text{ kg.m.} \]
\[ \bar{M}_{BA} = + \frac{1500 \times 8 \times 8}{12} \]
\[ = +8000 \text{ kg.m.} \]
\[ \bar{M}_{BC} = -\frac{1500 \times 4 \times 4}{12} \]
\[ = -2000 \text{ kg.m.} \]
\[ \bar{M}_{CB} = + \frac{1500 \times 4 \times 4}{12} \]
\[ = +2000 \text{ kg.m.} \]
\[ \bar{M}_{CD} = \frac{8W}{8} \]
\[ = \frac{4000 \times 8}{8} = 4000 \text{ kg.m.} \]
\[ M_{BA} = \frac{EI}{3} \left( \theta_B - \frac{3}{800} \right) - 8000 \]

\[ M_{BC} = \frac{2EI \times I}{8} \left[ 2\theta_B + \theta_C - \frac{3}{4} \left( -\frac{1}{100} \right) \right] - 2000 \]

\[ M_{CB} = \frac{EI}{2} \left( \theta_B + 2\theta_C + \frac{3}{400} \right) + 2000 \]

\[ M_{CD} = \frac{2EI \times I}{8} \left( 2\theta_C - 0 - 0 \right) - 4000 \]

\[ = \frac{EI\theta_C}{2} - 4000 \]

\[ M_{DC} = \frac{2EI \times I}{8} \left( 0 + \theta_C - 0 \right) + 4000 \]

\[ = \frac{EI\theta_C}{2} + 4000 \]

**Sum of the moments at B is zero.**

\[ \therefore M_{BA} + M_{BC} = 0 \]

\[ \frac{EI}{2} \left( 2\theta_B - \frac{3}{800} \right) + 8000 + \frac{EI}{2} \left( 2\theta_B + \theta_C + \frac{3}{400} \right) - 2000 = 0 \]

\[ 2\theta_B - \frac{3}{800} + 2\theta_B + \theta_C + \frac{3}{400} = - \frac{12000}{EI} \]

\[ 4\theta_B + \theta_C = - \frac{12000}{32 \times 10^6} - \frac{3}{800} \]

\[ \therefore 4\theta_B + \theta_C = - \frac{3}{400} \]

**Sum of the moments at joint C is zero**

\[ \therefore M_{CB} + M_{CD} = 0. \]

\[ \frac{EI}{2} \left( \theta_B + 2\theta_C + \frac{3}{400} \right) + 2000 + EI\theta_C - 4000 = 0 \]
\[ \theta_B + 4\theta_C = -\frac{4000}{52 \times 10^5} \cdot \frac{3}{400} \]

\[ \theta_B + 4\theta_C = -\frac{5}{800} \]

Multiplying by 4,
\[ 4\theta_B + 16\theta_C = -\frac{20}{800} \]

Eq. (2) - Eq. (1) gives
\[ 15\theta_C = -\frac{7}{400} \]
\[ \theta_C = -\frac{6000}{5} \]
\[ \theta_B = \frac{19}{12,000} \]

\[ M_{AB} = \frac{EI}{2} \left[ \theta_B - \frac{3}{800} \right] - 8000 \]
\[ = \frac{32 \times 10^5}{2} \left[ \frac{-19}{12,000} - \frac{3}{800} \right] - 8000 \]
\[ = -8533 - 8000 \]
\[ = -16,533 \text{ kg. m.} \]

\[ M_{BA} = \frac{32 \times 10^5}{2} \left[ \frac{-19}{12,000} \times 2 - \frac{3}{800} \right] + 8000 \]
\[ = -11066 + 8000 \]
\[ = -3066 \text{ kg. m.} \]

\[ M_{DC} = \frac{EI\theta_C}{2} + 4000 \]
\[ = \frac{32 \times 10^5}{2} \times \left( -\frac{7}{6000} \right) + 4000 \]
\[ = -1868 + 4000 \]
\[ = 2132 \text{ kg. m.} \]

\[ M_{CD} = EI\theta_C - 4000 \]
\[ = 32 \times 10^5 \left( -\frac{7}{6000} \right) - 4000 \]
\[ = -3732 - 4000 \]
\[ = -7732 \text{ kg. m.} \]

6.3. Portal frames
Portal frames are structures consisting of a number of members having rigid joints at their junctions. Such frames when
loaded will deform in such a way that members meeting at a

junction will rotate by same magnitude and in the same direction, the angle between members remaining the same.

In the frame shown in Fig. 6.11 (a), the joint B is rigid. There will be slope \( \theta_B \) when the frame is loaded. The slopes at \( A \) and \( C \) will be zero as the ends are fixed. There will be no displacement of members \( AB \) and \( BC \) the ends \( A \) and \( C \) are fixed to foundation. The value of \( \theta_B \) can be obtained from the equation \( M_{BA} + M_{BC} = 0 \). Knowing \( \theta_B \), the moments at all joints can be worked out.

When the portal frames are unsymmetrical or when they are loaded unsymmetrically as shown in Fig. 6.11 (b) and (c) the frame will sway. When the frame is loaded, there will be slopes at \( B \) and \( C \), and also some displacement \( \delta \) of joints \( B \) and \( C \). The two equations for the two joints are \( M_{BA} + M_{BC} = 0 \) and \( M_{CB} + M_{CD} = 0 \). The third equation is obtained from the fact that horizontal shear in the vertical members...
will be equal and opposite to the external horizontal force. In Fig. 6.12 (a)

\[
\frac{M_{BA} + M_{BC}}{h_1} + \frac{M_{DC} + M_{CD}}{h_2} = 0 \\
\frac{M_{AB} + M_{BA}}{h_1} + \frac{M_{DC} + M_{CD}}{h_2} + P = 0
\]

In Fig. 6.12 (b)

\[
\frac{M_{BA} + M_{BC}}{h_1} + \frac{M_{DC} + M_{CD}}{h_2} = 0 \\
\frac{M_{AB} + M_{BA}}{h_1} + \frac{M_{DC} + M_{CD}}{h_2} = 0
\]

In Fig. 6.12 (c)

\[
\frac{M_{BA} + M_{BC}}{h_1} + \frac{M_{DC} - M_{CD}}{h_2} + wh_1 = 0
\]

The values of \( \theta_B \), \( \theta_C \) and \( \delta \) can be calculated from the three equations in each case.

**Ex. 6.7.** Analyse the frame shown in Fig. 6.13 (a).

**Solution.** Let \( \theta_B \) be the slope at B. Slopes at A and C will be zero.

\[
\bar{t}_{BD} = + 3000 \times 2 = 6000 \text{ kg. m.}
\]

\[
\bar{t}_{BC} = - \frac{wt^2}{12} = - \frac{3000 \times 4 \times 4}{12} = -4000 \text{ kg. m.}
\]

\[
\bar{t}_{CB} = + \frac{3000 \times 4 \times 4}{12} = +4000 \text{ kg. m.}
\]

\[
I_{AB} = \frac{2EI}{4} (0 + \theta_B) = \frac{EI\theta_B}{2}
\]

\[
I_{BA} = \frac{2EI}{4} (0 + 2\theta_B) = \frac{E I \theta_B}{2}
\]

\[
\bar{t}_{BC} = 2E \times 3I \quad (2\theta_B + 0) - 4000
\]

\[
3EI\theta_B - 4000
\]

\[
\bar{t}_{CB} = 2E \times 3I \quad (\theta_B) + 4000
\]

\[
\frac{3EI\theta_B}{2} = 4000
\]

Sum of the moments at joint B is zero.

\[
\therefore \quad M_{BD} + M_{BC} + M_{BA} = 0.
\]
\[ 6000 + 3EI\theta_B - 4000 + EI\theta_B = 0 \]
\[ 4EI\theta_B = -2000 \]
\[ \theta_B = -\frac{500}{EI} \]
\[ M_{AB} = \frac{EI\theta_B}{2} \]
\[ = \frac{EI}{2} \left( -\frac{500}{EI} \right) = -250 \text{ kg. m.} \]
\[ M_{BA} = EI\theta_B = EI \left( -\frac{500}{EI} \right) \]
\[ = -500 \text{ kg. m.} \]
\[ M_{BC} = 3EI\theta_B - 4000 = -1500 - 4000 \]
\[ = -5500 \text{ kg. m.} \]
\[ M_{CB} = \frac{3}{2} EI\theta_B + 4000 = -750 + 4000 \]
\[ = +3250 \text{ kg. m.} \]

The B.M. diagram is shown in Fig. 6.13 (b), B.M. is drawn on tension side.

**Ex. 6.8.** Analyse the frame shown in Fig. 6.14 (a) by slope deflection method.

**Solution.** The ends \( A \) and \( D \) are fixed and therefore

\[ \theta_A = \theta_D = 0. \]

As the portal frame is symmetrical and symmetrically loaded slope \( \theta_B = -\theta_C \) and there will be no sway, i.e. \( \delta = 0 \)

\[ \overline{M_{BC}} = -\frac{2000 \times 6 \times 6}{12} \]
\[ = -6000 \text{ kg. m.} \]
\[ \overline{M_{CB}} = +6000 \text{ kg. m.} \]

\[ M_{AB} = \frac{2EI}{4} \left( 0 + \theta_B - 0 \right) \]
\[ = \frac{EI\theta_B}{2} \]

\[ M_{AB} = \frac{2EI}{4} \left( 0 + 2\theta_B \right) \]
\[ = E\theta_B \]

\[ M_{BC} = \frac{2E \times 3I}{6} \left( 2\theta_B - \theta_B \right) \]
\[ = E\theta_B - 6000 \]

\[ M_{CB} = \frac{2E \times 3I}{6} \left( -2\theta_B + \theta_B \right) + 6000 \]
\[ = -E\theta_B + 6000 \]
Sum of the moments at joint B is zero.

\[ M_{BA} + M_{BC} = 0 \]
\[ EI\theta_B + EIB - 6000 = 0 \]

\[ \theta_B = \frac{3000}{EI} \]

\[ M_{AB} = \frac{EI}{2} \left( 3000 \times \frac{EI}{EI} \right) + 1500 \text{ kg. m.} \]

\[ M_{BA} = EI\theta_B = +3000 \text{ kg.m.} \]

\[ M_{BC} = EI\theta_B - 6000 = 3000 - 6000 \]
\[ = -3000 \text{ kg. m.} \]

\[ M_{CB} = -EI\theta_B + 6000 = -3000 + 6000 \]
\[ = +3000 \text{ kg. m.} \]

The B.M. diagram is shown in Fig. 6.14 (b). The B.M. diagram is shown on tension side.

**Ex. 6.9.** Analyse the frame shown in Fig. 6.15 (a) by slope deflection method.

**Solution.** As the ends A and D are fixed slopes \( \theta_A \) and \( \theta_D \) will be zero.

Let \( \theta_B \) and \( \theta_C \) be the slopes at B and C respectively. The frame is symmetrical but is unsymmetrically loaded hence the frame will sway. Let \( \delta \) be the sway of the frame, to the right.

\[ M_{BB} = +4000 \times 1 = +4000 \text{ kg. m.} \]

\[ M_{CB} = -2000 \times 1 = -2000 \text{ kg. m.} \]

\[ M_{AB} = \frac{2EI}{4} \left( 0 + \theta_B - \frac{3\delta}{4} \right) + 0 \]

\[ = \frac{EI}{2} \left( \theta_B - \frac{3\delta}{4} \right) \]

\[ M_{BA} = \frac{EI}{2} \left( \theta_B - \frac{3\delta}{4} \right) \]

\[ M_{DC} = \frac{EI}{2} \left( \theta_C - \frac{3\delta}{4} \right) \]

\[ M_{CD} = \frac{EI}{2} \left( 2\theta_C - \frac{3\delta}{4} \right) \]

\[ M_{BC} = \frac{2EI}{4} \left( 2\theta_B + \theta_C \right) = \frac{EI}{2} \left( 2\theta_B + \theta_C \right) \]

\[ M_{CB} = \frac{2EI}{4} \left( 2\theta_C + \theta_B \right) = \frac{EI}{2} \left( \theta_B + 2\theta_C \right) \]

Sum of the moments at joint B is zero.

\[ \therefore \ M_{BA} + M_{BC} + M_{BB} = 0 \]
\[
\frac{EI}{2} \left( 2\theta_B - \frac{3\delta}{4} \right) + \frac{EI}{2} \left( 2\theta_B + \theta_C \right) + 4000 = 0
\]
\[
4\theta_B + \theta_C - \frac{3\delta}{4} = \frac{8000}{EI}
\]

(a)

4,000 kg.

2000 kg.

\(\text{4,000 kg.m.}\)

\(\text{18571 kg.m.}\)

2142-86 kg.m.

\(\text{18571 kg.m.}\)

857 kg.m.

1142-86 kg.m.

(b)

Fig. 615. B.M Diagram

Sum of the moments at joint C is zero.

\[
\frac{EI}{2} \left( 2\theta_C + \theta_B \right) + \frac{EI}{2} \left( 2\theta_C - \frac{3\delta}{4} \right) - 2000 = 0
\]

\[
\theta_B + 4\theta_C - \frac{3\delta}{4} = \frac{4000}{EI}
\]

Total shear in vertical members is zero.

\[
\frac{M_{AB} + M_{BA}}{4} + \frac{M_{CD} + M_{DC}}{4} = 0
\]
\[
\frac{EI}{2} \left( \theta_B - \frac{33}{4} \right) + \frac{EI}{2} \left( 2\theta_B - \frac{33}{4} \right) + \frac{EI}{2} \left( \theta_C - \frac{33}{4} \right) + \frac{EI}{2} \left( 2\theta_C - \frac{33}{4} \right) = 0
\]
\[
\frac{EI}{2} \left( 3\theta_B - \frac{3}{2} \delta \right) + \frac{EI}{2} \left( 3\theta_C - \frac{3}{2} \delta \right) = 0
\]
\[
3\theta_B + 3\theta_C - 3\delta = 0
\]
\[
\text{Eqn. (1) - Eqn. (2) gives } 3\theta_B - 3\theta_C = -\frac{12,000}{EI} \quad \ldots(4)
\]
\[
4 \times \text{Eqn. (2) - Eqn. (3) gives } \theta_B + 13\theta_C = +\frac{16,000}{EI} \quad \ldots(5)
\]

Multiplying Eqn. (5) by 3, \[
3\theta_B + 39\theta_C = \frac{48,000}{EI} \quad \ldots(6)
\]
\[
3\theta_B - 3\theta_C = -\frac{12,000}{EI} \quad \ldots(7)
\]

Eqn. (6) - Eqn. (4) gives \[
42\theta_C = \frac{60,000}{EI}
\]
\[
\therefore \quad \theta_C = \frac{10,000}{7EI}
\]

From Eqn. (5), \[
\theta_B = \frac{16,000}{EI} - 13\theta_C = \frac{16,000}{EI} - \frac{130,000}{7EI}
\]
\[
\theta_B = -\frac{18,000}{7EI}
\]

From Eqn. (3) \[
\delta = \theta_B + \theta_C = -\frac{18,000}{7EI} + \frac{10,000}{7EI}
\]
\[
= \frac{8,000}{7EI} \quad \text{(The sway is to the left)}
\]
\[
\lambda_{AB} = \frac{EI}{2} \left( \theta_B - \frac{3}{4} \delta \right) = \frac{EI}{2} \left( -\frac{18,000}{7EI} + \frac{3}{4} \times \frac{8,000}{7EI} \right)
\]
\[
= \frac{6000}{7} = -857.1 \text{ kg. m.}
\]

\[
M_{BA} = \frac{EI}{2} \left( 2\theta_B - \frac{3}{4} \delta \right) = \frac{EI}{2} \left( -\frac{36,000}{7EI} + \frac{3}{4} \times \frac{8,000}{7EI} \right)
\]
\[
= -15,000 \quad -2142.86 \text{ kg. m.}
\]

\[
M_{BC} = \frac{EI}{2} \left( 2\theta_B + \theta_C \right) = \frac{EI}{2} \left( -\frac{36,000}{7EI} + \frac{10,000}{7EI} \right)
\]
\[
= -13,000 \quad -1857.14 \text{ kg. m.}
\]

\[
M_{CB} = \frac{EI}{2} \left( 2\theta_C + \theta_B \right) = \frac{EI}{2} \left( \frac{20,000}{7EI} - \frac{18,000}{7EI} \right)
\]
\[
= \frac{1000}{7} = 142.86 \text{ kg. m.}
\]
\[ M_{DC} = \frac{EI}{2} \left( \theta_C - \frac{3\delta}{4} \right) = \frac{EI}{4} \left( \frac{10,000}{7EI} + \frac{3}{4} \times \frac{8,000}{7EI} \right) \]
\[ = \frac{8,000}{7} = 1,142.86 \text{ kg.m.} \]

\[ M_{CD} = \frac{EI}{2} \left( 2\theta_C - \frac{3}{4} \delta \right) = \frac{EI}{2} \left( \frac{2000}{7EI} + \frac{3}{4} \times \frac{8,000}{7EI} \right) \]
\[ = \frac{13000}{1857.14} = 1857.14 \text{ kg.m.} \]

The B.M. diagram is shown in Fig. 6.15 (b). The B.M. diagram is drawn on tension side.

**Ex. 6.10.** Analyse the frame shown in Fig. 6.16 by slope deflection method.

**Solution.** Let \( \theta_A, \theta_B \) and \( \theta_C \) be slopes at \( A, B \) and \( C \) respectively. Slope at \( D \) will be zero. Let \( \delta \) be the sway of joints \( B \) and \( C \) to the right.

\[
\bar{M}_{CB} = -\frac{4000 \times 4}{8} = -2000 \text{ kg.m.}
\]

\[
\bar{M}_{BC} = \frac{4000 \times 4}{8} = 2000 \text{ kg.m.}
\]

\[
\bar{M}_{DC} = -\frac{1000 \times 6 \times 6}{12} = -3000 \text{ kg.m.}
\]

\[
\bar{M}_{CD} = +\frac{1000 \times 6 \times 6}{12} = +3000 \text{ kg.m.}
\]

\[
\bar{M}_{AB} = 2EI \left( \frac{2\theta_A + \theta_B - \frac{3\delta}{l}}{3} \right)
\]
\[
\bar{M}_{AB} = \frac{2EI}{3} \left( \frac{2\theta_A + \theta_B - \frac{3\delta}{3}}{3} \right)
\]
\[
2EI \left( 2\theta_A + \theta_B - \delta \right)
\]

\[ Fig. 6.16 \]
\[ M_{BA} = \frac{2EI}{3} \left( \theta_A + 2\theta_B - \delta \right) \]
\[ M_{CB} = 2E \times 2I \left( 2\theta_C + \theta_B \right) - 2000 \]
\[ M_{BC} = EI \left( 2\theta_C + \theta_B \right) - 2000 \]
\[ M_{DC} = \frac{3EI}{6} \left( 0 + \theta_C - \frac{3\delta}{6} \right) - 3000 \]
\[ = \frac{EI}{3} \left( \theta_C - \frac{\delta}{2} \right) - 3000 \]
\[ M_{CD} = \frac{EI}{3} \left( 2\theta_C - \frac{\delta}{2} \right) + 3000 \]

The unknowns \( \theta_A, \theta_B, \theta_C \) and \( \delta \) can be obtained from the following four equations:

\[ M_{AB} = 0 \quad \cdots (A) \]
\[ M_{BC} + M_{BA} = 0 \quad \cdots (B) \]
\[ M_{CB} + M_{CD} = 0 \quad \cdots (C) \]

\[ H_A + H_D + 6 \times 1000 = 0, \text{ where } H_A \text{ and } H_D \text{ are horizontal reactions at } A \text{ and } D \text{ respectively.} \]

\[ M_{CD} + M_{DC} = 6 \times 1000 \times 3 \frac{M_{BA}}{6} + 6000 \quad 0 \]

or

\[ M_{CD} + M_{DC} + 2M_{BA} = -18,000 \quad \cdots (D) \]

Eqn. (A) gives,

\[ \frac{2EI}{3} (2\theta_A + \theta_B - \delta) = 0 \]

or

\[ \theta_A = \frac{\delta - \theta_B}{2} \quad \cdots (1) \]

Equation (B) gives

\[ EI(2\theta_C + \theta_B) + 2000 + \frac{2EI}{3} (2\theta_B + \theta_A - \delta) = 0 \]

\[ 3\theta_C + 10\theta_B + 2\theta_A - 2\delta = \frac{6000}{EI} \quad \cdots (2) \]

Equation (C) gives,

\[ EI(2\theta_C + \theta_B) - 2000 + \frac{EI}{3} \left( 2\theta_C - \frac{\delta}{2} \right) + 3000 = 0 \]

\[ 6\theta_C + 3\theta_B + 2\theta_C - \frac{\delta}{2} = -\frac{3000}{EI} \]

\[ 6\theta_B + 16\theta_C - \delta = -\frac{6000}{EI} \quad \cdots (3) \]

Equation (D) gives,

\[ \frac{EI}{3} \left( \theta_C - \frac{\delta}{2} \right) - 3000 + \frac{EI}{3} \left( 2\theta_C - \frac{\delta}{2} \right) + 3000 \]

\[ + 2 \times \frac{2EI}{3} \left( 2\theta_B + \theta_A - \delta \right) = -18,000 \]

\[ 4\theta_A + 8\theta_B + 3\theta_C - 5\delta = -54,000 \quad \cdots (4) \]
Substituting value of $\theta_A$ from (1) in (2)

$$3\theta_C + 10\theta_B + \delta - \theta_B - 2\delta = -\frac{6000}{EI}$$

$$\therefore 3\theta_C + 9\theta_B - \delta = -\frac{6000}{EI} \quad \ldots (5)$$

Substituting values of $\theta_A$ from (1) in (4)

$$2\delta - 2\theta_B + 8\theta_B + 3\theta_C - 5\delta = -\frac{54,000}{EI}$$

$$\therefore 6\theta_B + 3\theta_C - 3\delta = -\frac{54,000}{EI}$$

$$\therefore 2\theta_B + \theta_C - \delta = \frac{18,000}{EI} \quad \ldots (6)$$

Eq. (5) — Eq. (6) gives

$$2\theta_C + 7\theta_B = +\frac{12,000}{EI} \quad \ldots (7)$$

Eq. (3) — Eq. (6) gives

$$15\theta_C + 4\theta_B = +\frac{12,000}{EI} \quad \ldots (8)$$

Multiplying Eq. (7) by 4,

$$8\theta_C + 28\theta_B = +\frac{48,000}{EI} \quad \ldots (9)$$

Multiplying Eq. (8) by 7,

$$105\theta_C + 28\theta_B = +\frac{84,000}{EI} \quad \ldots (10)$$

Eq. (9) — Eq. (10) gives

$$-97\theta_C = -\frac{36,000}{EI}$$

$$\therefore \theta_C = \frac{371.13}{EI}$$

From Eq. (8), $4\theta_B = \frac{12,000}{EI} - 15\theta_C$

$$12,000 = \frac{5566.95}{EI}$$

$$\therefore \theta_B = \frac{12,000}{EI} - \frac{5566.95}{EI}$$

$$\therefore \theta_B = \frac{6433.05}{EI}$$

From Eq. (6),

$$\delta = 2\theta_B + \theta_C + \frac{18,000}{EI}$$

$$\delta = \frac{3216.5}{EI} + \frac{371.13}{EI} + \frac{18,000}{EI}$$

$$\delta = \frac{21,587.63}{EI}$$

$$\theta_A = -\frac{\delta - \theta_B}{2} = \frac{21,587.63 - 1608.25}{2}$$

$$\therefore \theta_A = \frac{9,989.69}{EI}$$
\[ M_{DC} = \frac{EI}{3} \left( \theta_C - \frac{\delta}{2} \right) - 3000 \]
\[ = \frac{EI}{3} \left( \frac{371 \cdot 13}{EI} - \frac{10,793 \cdot 81}{EI} \right) - 3000 \]
\[ = -6474 \cdot 2 \text{ kg. m.} \]

\[ M_{CD} = \frac{EI}{3} \left( 2\theta_C - \frac{\delta}{2} \right) + 5000 \]
\[ = \frac{EI}{3} \left( \frac{742 \cdot 26}{EI} - \frac{10,793 \cdot 81}{EI} \right) + 5000 \]
\[ = -350 \cdot 5 \text{ kg. m.} \]

\[ M_{BC} = \frac{EI}{9} (\theta_C + 2\theta_B) + 2000 \]
\[ = 371 \cdot 13 + 3216 \cdot 5 + 2000 \]
\[ = 5587 \cdot 63 \text{ kg. m.} \]

\[ M_{CB} = EI (2\theta_C + r_B) - 2000 \]
\[ = 742 \cdot 26 + 1608 \cdot 25 - 2000 \]
\[ = 350 \cdot 5 \text{ kg. m.} \]

B.M. diagram is shown in Fig. 6.15.

**Ex. 6.11.** Analyse the frame shown in Fig. 6.17 by slope deflection method.

**Solution.** Let \( \theta_A, \theta_B, \theta_C \) and \( \theta_D \) be slopes at \( A, B, C \) and \( D \) respectively. Let \( \delta \) be sway of joints \( B \) and \( C \) to the right.

Applied loading is equivalent, to moment of \( 4000 \times 0 \cdot 5 = 2000 \) kg. m. clockwise.

Fixed end moments due to this loading will be

\[ \bar{M}_{AB} = \bar{M}_{BA} = +\frac{M}{4} \]
\[ = 500 \text{ kg. m.} \]

\[ M_{AB} = \frac{2EI}{6} \left( 2\theta_A + \theta_B - \frac{3\delta}{6} \right) + 500 \]
\[ = \frac{EI}{3} \left( 2\theta_A + \theta_B - \frac{\delta}{2} \right) + 500 \]

\[ M_{BA} = \frac{EI}{3} \left( \theta_A + 2\theta_B - \frac{\delta}{2} \right) + 500 \]

\[ M_{BC} = \frac{2EI \times 2I}{8} (2\theta_B + \theta_C) \]
\[ = \frac{EI}{2} (2\theta_B + \theta_C) \]

\[ M_{CB} = \frac{EI}{2} (\theta_B + 2\theta_C) \]

\[ M_{DC} = \frac{2EI}{3} \left( 2\theta_D + \theta_C - \frac{3\delta}{3} \right) \]
\[ = \frac{2EI}{3} (2\theta_D + \theta_C - \delta) \]
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\[ M_{CD} = \frac{2EI}{3} (\theta_D + 2\theta_C - \delta) \]

\[ M_{AB} = 0 \quad \ldots (A) \]
\[ M_{DC} = 0 \quad \ldots (B) \]
\[ M_{CB} + M_{CD} = 0 \quad \ldots (C) \]
\[ H_A + H_D = 0 \quad \text{where } H_A \text{ and } H_D \text{ are horizontal reactions at } A \text{ and } D \text{ respectively.} \]

\[ \frac{M_{BA} + 2000}{6} + \frac{M_{CD}}{3} = 0 \]
\[ M_{BA} + 2M_{CD} + 2000 = 0 \quad \ldots (E) \]

Equation (A) gives,
\[ \frac{2EI}{3} (2\theta_D + \theta_C - \delta) = 0 \]
\[ 2\theta_D + \theta_C - \delta = 0 \quad \ldots (1) \]