\[ \frac{\partial U}{\partial H} = \frac{1}{EI} \int_{0}^{3} (Hy - \frac{4000}{3} y) y \, dy + \frac{1}{EI} \int_{0}^{3/2} (Hy - 4000y + 4000(\frac{4}{3}y - 2)) \, dy = 0 \]

\[ \int_{0}^{3} \left( Hy^3 - \frac{4000}{3} y^3 \right) \, dy + \int_{0}^{3/2} (Hy^3 - 4000y^2) \, dy \]

\[ + \int_{3/2}^{3} 4000 \left( \frac{4}{3}y^2 - 2y \right) \, dy = 0 \]

\[ H \left[ \frac{y^3}{3} \right]_{0}^{3} - \frac{4000}{3} \left[ \frac{y^3}{3} \right]_{0}^{3/2} + H \left[ \frac{y^3}{3} \right]_{0}^{3/2} - 4000 \left[ \frac{y^3}{3} \right]_{3/2}^{3} + 4000 \left[ \frac{4y^2}{9} - y \right]_{3/2}^{3} = 0 \]

\[ 9H - 12000 + 9H - 36,000 + 4000\left[ 3^3 - 1 - \frac{4}{3} \right] - 9 \left[ 11 - \frac{4}{3} \right] = 0 \]

\[ 18H - 48,000 + 15,000 = 0 \]

\[ 18H = 33,000 \]

\[ H = 1833 \frac{3}{3} \text{ kg.} \]

Taking moments about \( B \),

\[ M_{BC} + H \times 3 - 1000 \times 4 = 0 \]

\[ M_{BC} = 4000 - 1833 \frac{3}{3} \times 3 \]

\[ = 4000 - 5500 \]

\[ = -1500 \text{ kg. m.} \]

B.M. diagram is shown in Fig. 7.12 (b).

**Ex. 7.12.** Analyse the frame shown in Fig. 7.13 for rotational yield of 0.002 radians anti-clockwise and vertical yield of 0.5 cm. at \( A \). \( EI = 3 \times 10^{10} \) kg. cm².

**Solution.** Let \( H, R \) and \( M_A \) be horizontal thrust, vertical reaction and moment respectively at \( A \). Vertical reaction is assumed downwards.

Taking moments about \( D \),

\[ M_D - M_A - R \times 6 = 0 \]

\[ \therefore M_D = M_A + 6R \]

Let \( U \) be the total strain energy of the frame.

As the supports at \( A \) and \( D \) do not move horizontally.

\[ \frac{\partial U}{\partial H} = 0. \]

Support \( A \) yields vertically by ½ cm.

\[ \therefore \quad \frac{\partial U}{\partial R} = \delta = \frac{1}{2} \text{ cm.} \]
Support $A$ rotates by $0.002$ radian anti-clockwise.

\[
\frac{\partial U}{\partial M_A} = 0.002.
\]

Equations for B.M. values of \(\frac{\partial M}{\partial H} \), \(\frac{\partial M}{\partial R} \), \(\frac{\partial M}{\partial M_A} \) and limits of integration are shown in Table 7.11.

### Table 7.11

<table>
<thead>
<tr>
<th>Origin at</th>
<th>Portion</th>
<th>Bending moment</th>
<th>(\frac{\partial M}{\partial H} )</th>
<th>(\frac{\partial M}{\partial R} )</th>
<th>(\frac{\partial M}{\partial M_A} )</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$AB$</td>
<td>$M_A + Hx'y'$</td>
<td>$y$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$0-4$</td>
</tr>
<tr>
<td>$D$</td>
<td>$DC$</td>
<td>$M_D + Hx'y'$</td>
<td>$y$</td>
<td>$6$</td>
<td>$1$</td>
<td>$0-4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= M_A + 6R + Hx'y'$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B$</td>
<td>$BC$</td>
<td>$M_A + 4H + Rx'x'$</td>
<td>$4$</td>
<td>$x$</td>
<td>$1$</td>
<td>$0-6$</td>
</tr>
</tbody>
</table>

(I) \[
\frac{\partial U}{\partial H} = \frac{1}{EI} \int_0^4 (M_A + Hy)dy + \frac{1}{EI} \int_0^4 (M_A + 6R + Hy)dy = 0
\]
\[
\Rightarrow M_A \left[ \frac{y^2}{2} \right]_0^4 + H \left[ \frac{y^2}{3} \right]_0^4 + M_A \left[ \frac{y^2}{2} \right]_0^4 + 6R \left[ \frac{y^2}{2} \right]_0^4
\]
\[
+ H \left[ \frac{y^3}{3} \right]_0^4 + \frac{4}{1.5} \left[ M_A x + 4Hx + R^2 \right]_0^6 = 0
\]
\[
\Rightarrow 8M_A + \frac{64}{3} H + 8M_A + 48R + \frac{64}{3} H + \frac{24}{1.5} M_A + \frac{96}{1.5} H + \frac{72}{1.5} R = 0
\]
\[
\Rightarrow 32M_A + 96R + \frac{320}{3} H = 0
\]
\[
\Rightarrow 3M_A + 9R + 10H = 0
\]
\[
\Rightarrow 12M_A + 36R + 40H = 0
\]
\(\ldots(1)\)

(II) \[
\frac{\partial U}{\partial R} = \frac{1}{EI} \int_0^4 (M_A + 6R + Hy) 6dy + \frac{1}{1.5EI} \int_0^6 (M_A + 4H + Rx)dx = \delta
\]
\[
\Rightarrow 6 \left[ M_A y + 6Ry + \frac{Hy^2}{2} \right]_0^4 + \frac{1}{1.5} \left[ M_A x^2 + 2Hx^2 + R \frac{x^3}{3} \right]_0^6 = EI\delta
\]
\[
\Rightarrow 24M_A + 144R + 48H + 12M_A + 48H + 48R
\]
\[
= 3 \times 10^{10} \times \frac{1}{100 \times 100 \times 2 \times 100}
\]
\[
\Rightarrow \frac{36M_A + 96H + 192R = 15,000}{12M_A + 32H + 64R = 5000}
\]
\(\ldots(2)\)
(III) \[ \frac{\partial U}{\partial M_0} = \frac{1}{EI} \int_0^4 (M_A + Hy) dx + \frac{1}{EI} \int_0^4 (M_A + 6R + Hy) dy \]
\[ + \frac{1}{1.5EI} \int_0^6 (M_A + 4H + Rx) \times 1 dx = 0 \]

\[ M_A \left[ y \right]^4_0 + H \left[ \frac{y^2}{2} \right]^4_0 + M_A \left[ y \right]^4_0 + 6R \left[ y \right]^4_0 + H \left[ \frac{y^2}{2} \right]^4_0 \]
\[ + \frac{M_A}{1.5} \left[ x \right]^6_0 + 4H \frac{1}{1.5} \left[ x \right]^6_0 + R \frac{1}{1.5} \left[ x^2 \right]^6_0 \]
\[ = 3 \times 10^{10} \times 0.002 \]
\[ = 6000 \quad \text{(3)} \]

Equation (2)—Equation (3) gives,
\[ 28R = -1000 \]
\[ R = -35.71 \text{ kg.} \]

Equation (1)—Equation (2) gives,
\[ 8H - 28R = -5000 \]
\[ 8H = -5000 + 1000 = -6000 \]
\[ H = -750 \text{ kg.} \]

From Eqn. (1), \[ 3M_A + 9R + 10H = 0 \]
\[ 3M_A = + (9 \times 35.71 + 7500) = +7821.39 \]
\[ M_A = + 2607.13 \text{ kg. m.} \]

Ex. 7.13. Determine the maximum B.M. and increase in vertical diameter of the ring under the action of forces \( P \) applied as shown in Fig. 7.14 (a).

Solution. The ring is cut along horizontal diameter at \( CD \). The free body diagram of half the ring is shown in Fig. 7.14 (b). At the cut portions moments \( M_0 \) and vertical loads \( P/2 \) are applied as shown in the figure. There will be no horizontal force as the ring is symmetrically loaded. As the ring is symmetrically loaded, cross-section \( CD \) will not rotate and hence \( \frac{\partial U}{\partial M_0} = 0 \).

B.M. at a cross-section at angle \( \phi \) is given by
\[ M = M_0 - \frac{P}{2}(R - R \cos \phi) \]

Strain energy of half ring
\[ U = 2 \int_0^\pi/2 \frac{M^2 ds}{2EI} \]
\[ \frac{\partial U}{\partial M_0} = 2 \int_0^\pi/2 \frac{2M}{2EI} \times \frac{3M}{\partial M_0} ds \]
$$\frac{\partial M}{\partial M_0} = 1, \quad ds = Rd\phi$$

$$\therefore \frac{\partial U}{\partial M_0} = \frac{2}{EI} \int_0^{\pi/2} [M_0 - \frac{PR}{2}(1 - \cos \phi)] \times 1 Rd\phi = 0$$

$$\therefore \left[ M_0\phi - \frac{PR}{2}(\phi - \sin \phi) \right]_0^{\pi/2} = 0$$

$$\therefore M_0 = \frac{PR}{2} \left( \frac{\pi}{2} - 1 \right) = 0.182PR$$

B.M. at any cross-section is given by

$$M = M_0 - \frac{PR}{2} (1 - \cos \phi)$$

$$\therefore \frac{PR}{2} \left( 1 - \frac{2}{\pi} \right) - \frac{PR}{2} (1 - \cos \phi)$$

$$\frac{PR}{2} \left[ \cos \phi - \frac{2}{\pi} \right]$$

Thus maximum +ve B.M. occurs at $\phi = 0$ and is $0.182\ PR$.

Maximum -ve B.M. will occur at $\phi = \pi/2$.

$$M = \frac{PR}{2} \left( \frac{2}{\pi} \right) = \frac{PR}{\pi} = -0.318PR.$$}

$$\frac{\partial U}{\partial P}$$ will give change in vertical diameter of the ring.

B.M. at any section is given by

$$M = \frac{PR}{2} \left( \cos \phi - \frac{2}{\pi} \right)$$

$$\frac{\partial M}{\partial P} = \frac{R}{2} \left( \cos \phi - \frac{2}{\pi} \right)$$

Total strain energy of the ring

$$= 4 \int_0^{\pi/2} \frac{M^2 ds}{2EI}$$

$$\frac{\partial U}{\partial P} = 4 \int_0^{\pi/2} \frac{M \partial M}{EI} ds$$

$$= \frac{4PR}{EI} \int_0^{\pi/2} \left[ \frac{\cos \phi - \frac{2}{\pi}}{2} \right] \frac{R}{2} d\phi$$

$$= \frac{PR}{EI} \left[ \frac{\cos^2 \phi + \frac{4}{\pi^2} - \frac{4}{\pi} \cos \phi}{2} \right] d\phi$$

$$= \frac{PR}{EI} \left[ \frac{1 + \cos 2\phi + \frac{4}{\pi^2} - \frac{4}{\pi} \cos \phi}{2} \right] d\phi$$
\[
\frac{PR^3}{EI} \left[ \frac{\phi}{2} + \frac{\sin 2\phi}{2} + \frac{4}{\pi^2} \phi - \frac{4}{\pi} \sin \phi \right]_{0}^{\pi/2}
\]

\[
= \frac{PR^3}{EI} \left[ \frac{\pi}{4} + 0 + \frac{4}{\pi^2} \times \frac{\pi}{2} - \frac{4}{\pi} \times 1 \right]
\]

\[
= \frac{PR^3}{EI} \left[ \frac{\pi}{4} - \frac{2}{\pi} \right]
\]

\[
= 0.149PR^3
\]

\[
= \frac{0.149PR^3}{EI}
\]

**Ex. 7.14.** A beam AB shown in Fig. 7.15 is simply supported at B, hinged at A and supported by pin joints members PQ, PR and PC. Moment of inertia of the beam is 80,000 cm^4 and areas of members PQ, PR and PC are 15 cm^2, 25 cm^2, and 15 cm^2 respectively. If the beam carries a uniformly distributed load of 4000 kg/m, determine the force in PC. \( E = 2 \times 10^8 \) kg/cm^2 for all members.

**Solution.** Let \( P \) be the force in member PC.

Total strain energy of the system will consist of strain energy due to bending in AB and strain energy due to direct forces in members PC, PR and PQ. The strain energy will be the minimum.

The force \( P \) can be obtained by putting \( \frac{\partial U}{\partial P} = 0 \), where \( U \) is strain energy of the frame.

Free body of beam AB is shown in Fig. 7.15 (b).

Reactions \( R_A = R_B \)

\[
= \frac{4000 \times 8 - P}{2}
\]

\[
= 16000 - \frac{P}{2}
\]

In the beam AB, B.M. at distance \( x \) from A is given by

\[
M = (16,000 - P/2)x - \frac{4000 \times x^3}{2} \quad \text{for portion AC.}
\]

\[
\frac{\partial M}{\partial P} = -x/2
\]

For the beam AB,

\[
\frac{\partial U}{\partial P} = 2 \times \int_0^4 \frac{M \partial M}{EI} \times dx
\]
\[
\begin{align*}
2 & \int_0^4 \left[ \frac{(16,000 - \frac{P}{2})x - 2000x^2}{EI} \right] \times \left( \frac{-x}{2} \right) \, dx \\
& = \frac{2}{EI} \left[ -\frac{1}{2} \left( 16,000 - \frac{P}{2} \right) \left( \frac{x^2}{3} \right) + 1000 \left( \frac{x^4}{4} \right) \right]_0^4 \\
& = \frac{2}{EI} \left[ \left( \frac{P}{2} - 8000 \right) \frac{64}{3} + 64,000 \right] \\
& = \frac{2}{EI} \left[ \frac{16}{3} P - \frac{320,000}{3} \right].
\end{align*}
\]

Consider frame \(CPQR\). Let \(P_1\) be force in \(PQ\) and \(P_2\) be force in \(PR\).

At joint \(P\), \(\Sigma H = 0\)

\[
P_1 = -P_2
\]

\(\Sigma V = 0\) gives \(2P_1 \sin \phi = P\)

\[
P_1 = \frac{P}{2 \sin \theta}
\]

\[
= \frac{\sqrt{17}}{2} P
\]

\[
\frac{\partial P_1}{\partial P} = \frac{\sqrt{17}}{2}
\]

\[
P_2 = -\frac{\sqrt{17}}{2} P
\]

\[
\frac{\partial P_2}{\partial P} = -\frac{\sqrt{17}}{2}
\]

We use direct force \(F_{\parallel}\)

\[
\frac{\partial U}{\partial P} = \frac{F_{\parallel}}{AE} \times \frac{\partial F}{\partial P}
\]

\[
\frac{\partial U}{\partial P} \text{ for members } CP, PR \text{ and } PQ \text{ will be}
\]

\[
P \times 2 \quad -\frac{\sqrt{17}}{2} P \times \sqrt{17} \times \frac{-\sqrt{17}}{2}
\]

\[
\frac{15}{(100)^2 E} \quad \frac{25}{(100)^2 E}
\]

\[
\sqrt{17} P \times \sqrt{17} \left( \frac{\sqrt{17}}{2} \right)
\]

\[
+ - \quad \frac{15}{(100)^2}
\]
\[
\frac{\partial U}{\partial P} = \frac{2}{E} \left[ \frac{2P}{15E} + \frac{17\sqrt{17}P}{100E} + \frac{17\sqrt{17}P}{60E} \right] \\
\frac{2}{3} P - \frac{80,000}{3} + \frac{2P}{15} + \frac{17\sqrt{17}P}{100} + \frac{17\sqrt{17}P}{60} = 0 \\
3.3333 P = 80,000 \\
\therefore P = 8000 \text{ kg.}
\]

**Ex. 7.15.** Find the fixing moments at A and B and draw the B.M. diagram for the beam shown in Fig. 7.16. There is a moment of 6 t.m. applied at C.

(A.M.I.E., May, 1967)

Solution. Let \( R_B \) and \( M_B \) be the reactions at B

\[
\begin{align*}
&\frac{8I}{6t m} \\
&\begin{array}{c}
4.86 m \\
\downarrow 2.43 m \\
\downarrow \text{C} \\
\downarrow \text{A}
\end{array}
\end{align*}
\]

\[
\begin{array}{c}
M_B \\
\downarrow 6t m \\
\downarrow \text{C} \\
\downarrow \text{A}
\end{array}
\]

\[
\begin{array}{c}
R_B \\
\downarrow \text{x} \\
\downarrow \text{x} \\
\downarrow \text{B}
\end{array}
\]

Fig. 7.16

**TABLE 7.12**

<table>
<thead>
<tr>
<th>Portion</th>
<th>Bending Moment</th>
<th>( \frac{\partial M}{\partial R_B} )</th>
<th>( \frac{\partial M}{\partial M_B} )</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>BC</td>
<td>( R_B x - M_B )</td>
<td>( x )</td>
<td>( -1 )</td>
<td>( 0 ) to ( \frac{l}{3} )</td>
</tr>
<tr>
<td>CA</td>
<td>( R_B x - M_B - 6 )</td>
<td>( x )</td>
<td>( -1 )</td>
<td>( \frac{l}{3} ) to ( l )</td>
</tr>
</tbody>
</table>

\[
(1) \quad \frac{\partial U}{\partial M_B} = 0 = \int_0^{l/3} \frac{(R_B x - M_B) \times (-1)}{EI} \, dx \\
+ \int_{l/3}^l \frac{(R_B x - M_B - 6) \times (-1)}{8EI} \, dx
\]
\[= \frac{1}{EI} \left[ -\frac{R_B x^2}{2} + M_B x \right]_{0}^{l/3} + \frac{1}{8EI} \left[ -\frac{R_B x^2}{2} + M_B x + 6x \right]_{0}^{l/3} + M_B \left( l - \frac{l}{3} \right) + 6 \left( l - \frac{l}{3} \right) \]

\[= \frac{R_B l^3}{18EI} + \frac{M_B l}{3EI} - \frac{R_B l^3}{18EI} + \frac{M_B l}{12EI} + \frac{l}{2EI} \]

\[= \frac{R_B l^3}{9EI} + \frac{5M_B l}{12EI} + \frac{l}{2EI} \]

Multiplying each term by \(12EI\) and arranging

\[M_B = \frac{R_B l}{15} - \frac{6}{5} \quad \ldots \text{(i)} \]

(II) \[\frac{\partial U}{\partial R_B} = 0 = \int_{0}^{l/3} \left( \frac{R_B x - M_B - 6}{8EI} \right) x \, dx + \int_{l/3}^{l} \left( \frac{R_B x - M_B - 6}{8EI} \right) x \, dx \]

\[= \frac{1}{EI} \left[ \frac{R_B x^3}{3} - \frac{M_B x^2}{2} \right]_{0}^{l/3} + \frac{1}{8EI} \left[ \frac{R_B x^3}{3} - \frac{M_B x^2}{2} + \frac{6x^2}{2} \right]_{0}^{l/3} \]

\[= \frac{R_B l^3}{81EI} - \frac{M_B l^3}{18EI} + \frac{13 R_B l^3}{4 \times 81EI} \]

Multiplying by \(4 \times 81EI\) each term,

\[0 = 4R_B l - 18M_B + 13R_B l - 18M_B - 108 \]

\[\Rightarrow 0 = 17R_B l - 36M_B - 108 \]

\[\Rightarrow M_B = \frac{17R_B l}{36} - 3 \quad \ldots \text{(ii)} \]

Equating (i) and (ii)

\[\frac{4R_B l}{15} - \frac{6}{5} = \frac{17R_B l}{36} - 3 \]
\[ 3 - \frac{6}{5} = R_B l \left[ \frac{17}{16} - \frac{4}{15} \right] \]

\[ R_B = \frac{36 \times 9}{37l} \]

\[ M_B = \frac{4}{15l} \times \frac{36 \times 9}{37l} \times \frac{6}{5} \]
\[ = 2.335 - 1.2 \]
\[ = 1.135 \text{ t.m.} \]

Assuming \( M_A \) as positive,

\[ R_R l - M_R - 6 - M_A = 0 \]

\[ M_A = R_B l - M_B - 6 \]
\[ = \frac{36 \times 9}{37l} \times l - 1.135 - 6 \]
\[ = 8.756 - 7.135 \]
\[ = 1.621 \text{ t.m.} \]

The final B.M. diagram is shown in Fig. 7.17.

---

**Ex. 7.16.** Draw the B.M. diagram for the following portal frame given in the Fig. 7.18. End A is fixed and D is hinged. There is load of 3.28 t/m on BC. The values of relevant M.I. are given against the members. (A.M.I.E. May, 1967)

**Solution.** Let \( R_D \) and \( H_D \) be the reactions at D.

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---

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<table>
<thead>
<tr>
<th>Portion</th>
<th>Bending Moment</th>
<th>$\frac{\partial U}{\partial R_D}$</th>
<th>$\frac{\partial U}{\partial H_D}$</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>DC</td>
<td>$-H_D y$</td>
<td>$-y$</td>
<td></td>
<td>0 to $h$</td>
</tr>
<tr>
<td>CB</td>
<td>$R_D x - H_D \frac{wx^2}{2}$</td>
<td>$x$</td>
<td>$-h$</td>
<td>0 to $\frac{h}{2}$</td>
</tr>
<tr>
<td>BA</td>
<td>$R_D \frac{h}{2} - H_D (h-y) - \frac{wh^2}{8}$</td>
<td>$\frac{h}{2z}$</td>
<td>$-h+y$</td>
<td>0 to $\frac{h}{2}$</td>
</tr>
</tbody>
</table>

(1) $\frac{\partial U}{\partial R_D} = 0 = \int_0^{\frac{h}{2}} \left( \frac{R_D x - H_D h - \frac{wx^2}{2}}{EI} \right) dx$

$+ \int_0^{\frac{h^2}{2}} \left[ \frac{R_D \frac{h}{2} - H_D (h-y) - \frac{wh^2}{8}}{4EI} \right] \frac{h}{2} dy$

$= \frac{1}{EI} \left[ \frac{R_D x^3}{3} - H_D h x^2 - \frac{wx^4}{8} \right]_0^{\frac{h}{2}}$

$+ \frac{1}{4EI} \left[ \frac{R_D h^2 y}{4} - H_D \frac{h}{2} (hy - \frac{y^2}{2}) - \frac{wh^2 y}{10} \right]_0^{\frac{h}{2}}$

$\therefore \ 0 = \left[ \frac{R_D h^3}{24} - \frac{H_D h^3}{8} - \frac{wh^4}{128} \right]$

$+ \left[ \frac{R_D h^3}{32} - \frac{3H_D h^3}{64} - \frac{wh^4}{128} \right]$

$\frac{7R_D h^3}{96} \quad \frac{11H_D h^3}{64} \quad \frac{wh^4}{64}$

$A \quad H_D = \frac{14R_D h}{3} \quad wh \quad \ldots \text{(i)}$

(II) $\frac{\partial U}{\partial H_D} = 0 = \int_0^{h} \frac{-H_D y \times (-y)}{8EI} dy$

$+ \int_0^{\frac{h^2}{2}} \left( \frac{R_D x - H_D h - \frac{wx^2}{2}}{EI} \right) (-h) dx$

$+ \int_0^{\frac{h^2}{2}} \left[ \frac{R_D \frac{h}{2} - H_D (h-y) - \frac{wh^2}{8}}{4EI} \right] (-h+y) dy$

$\therefore \ 0 = \frac{1}{8} \left[ \frac{H_D y^2}{3} - \frac{h}{2} \right]_0^h + \left[ \frac{-R_D h^2 x^2}{2} + H_D h^2 x + \frac{wx^2}{6} \right]_0^{\frac{h}{2}}$

$+ \frac{1}{4} \left[ R_D h (-hy + \frac{y^2}{2}) \right]_0^c + H_D \left( h^2 y - hy^2 + \frac{y^3}{3} \right)$

$- \frac{wh^2}{8} (-hy + \frac{y^2}{2})$
\[ \begin{align*}
= & \frac{H_D}{24} h^3 - \frac{R_D h^3}{8} + \frac{H_D h^3}{2} + \frac{wh^4}{48} - \frac{3R_D h^3}{64} + \frac{7H_D h^3}{96} \\
= & \frac{59 H_D h^3}{90} - \frac{11R_D h^3}{64} + \frac{25 wh^4}{3 \times 256} \\
\therefore \quad H_D = & \frac{33R_D}{118} - \frac{25wh}{8 \times 59} \quad \text{...(ii)}
\end{align*} \]

Equating equations (i) and (ii),
\[ \frac{14R_D}{33} - \frac{wh}{11} = \frac{33R_D}{118} - \frac{25wh}{8 \times 59} \]
\[ \therefore \quad R_D \left[ \frac{14}{33} - \frac{33}{118} \right] = wh \left[ \frac{1}{11} - \frac{25}{8 \times 59} \right] \]
\[ \therefore \quad R_D \left[ \frac{1652 - 1089}{3 \times 118} \right] = wh \left[ \frac{472 - 275}{11 \times 8 \times 59} \right] \]
\[ \therefore \quad R_D = \frac{33 \times 118}{11 \times 8 \times 59} \times \frac{197}{563} wh \]
\[ = \frac{591}{2252} wh \]
\[ = 0.2624 \text{ wh.} \]
\[ \therefore \quad H_D = \frac{14}{33} \times 2624 \text{ wh} - \frac{wh}{11} = \frac{wh}{33} \left[ 3.6736 - 3 \right] \]
\[ = \frac{6736}{33} \text{ wh} = 0.2040 \text{ wh.} \]

\[ M_O = -H_D \times h = -0.2040 \text{ wh}^2 \]
\[ = -0.2040 \times 3.28 \times 4.86^2 \]
\[ = -1.580 \text{ t.m.} \]

\[ M_A = R_D \frac{h}{2} - H_D \frac{h}{2} - \frac{wh^2}{8} \]
\[ = 1.312 \text{ wh}^2 - 0.0102 \text{ wh}^2 - \frac{wh^2}{8} \]
\[ = -0.040 \text{ wh}^2 = 0.0040 \times 3.28 \times 4.86^2 = -31 \]

- Fig. 7.19
\[ M_B = R_D \frac{h}{2} - H_D h - \frac{wh^3}{8} \]
\[ = 1.312 \, wh^2 - 0.0204 \, wh^2 - \frac{wh^3}{8} \]
\[ = -0.0142 \times 3.28 \times 4.86^2 = -1.1 \text{ t.m.} \]

Final B.M. are shown in Fig. 7.19 on tension side.

**Ex. 7.17.** Analyse the frame shown in Fig. 7.20 and draw the bending moment diagram. The end A is fixed and C rests on rollers. The moment of inertia of BC is twice the moment of inertia of AB.

(A.M.I.E. Nov. 1967)

**Solution.** Let \( R_C \) be the reaction at C.

![Fig. 7.20](image)

**TABLE 7.14**

<table>
<thead>
<tr>
<th>Portion</th>
<th>Moment</th>
<th>( \frac{3M}{\partial R_C} )</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>BC</td>
<td>( R_C x )</td>
<td>( x )</td>
<td>0 to 2.43</td>
</tr>
<tr>
<td>BA</td>
<td>( R_C \frac{2.43 - 6.6y^2}{2} )</td>
<td>2.43</td>
<td>0 to 4.86 m</td>
</tr>
</tbody>
</table>

\[
\frac{\partial U}{\partial R_C} = 0 = \int_0^{4.86} \left( \frac{R_C \frac{2.43 - 6.6y^2}{2}}{EI} \right) \times 2.43 \, dy + \int_0^{2.43} \frac{R_C \times x \, dx}{2EI} \\
= \frac{2.43}{EI} \left[ 2.43 \, R_C \, y - 1.1 \, y^3 \right]_0^{4.86} + \frac{R_C}{6EI} \left[ x^3 \right]_0^{2.43} \]
\[
\begin{align*}
2.43 \times 4.86 \times & \frac{\left(2.43 R_C - 1.1 \times 4.86 \times 6^2\right)}{EI} + \frac{R_C \times 2.43^3}{6EI} \\
\therefore 2.43 \times 4.86 \times & \frac{2.43}{6} \left[R_C - 1.1 \times 2 \times 4.86\right] + \frac{R_C \times 2.43^3}{6} = 0 \\
\therefore 2[R_C - 2.2 \times 4.86] + & \frac{R_C}{6} = 0 \\
12R_C - 12 \times 2.2 \times 4.86 + & R_C = 0 \\
\therefore R_C = & \frac{12 \times 2.2 \times 4.86}{13} 9.87t \\
M_B = & 2.43 \times 9.87 = 23.98 \text{ t.m.} \\
M_A = & 2.43 \times 9.87 - \frac{0.6 \times 4.86^3}{100} \\
& = 23.98 - 77.93 = -53.95 \text{ t.m.} \\
The bending moments have been shown in Fig. 7.21 on the tension side.
\end{align*}
\]

**Ex. 7.18.** Analyse the continuous beam shown in Fig. 7.22. It carries a uniformly distributed load of 6.6 t/m on AB and BC. The support B sinks by 6.35 mm. below A and C. EI is constant. \(E = 2000 \text{ t/cm}^2, I = 33,200 \text{ cm}^4\). \(A.M.I.E. \text{ May 1969}\)

**Solution.** Let \(R_A\) and \(R_B\) be the reactions at A and B.

\[EI = \frac{2000 \times 33,200}{(100)^4} = 6,640 \text{ t.m.}^2\]

**Table 7.15**

<table>
<thead>
<tr>
<th>Portion</th>
<th>Bending Moment</th>
<th>(\frac{\partial U}{\partial R_A})</th>
<th>(\frac{\partial U}{\partial R_B})</th>
<th>Limit</th>
</tr>
</thead>
</table>

\[\begin{align*}
AB & \quad R_A x - \frac{6.6 x^2}{2} \\
& \quad x \\
& \quad 0 \text{ to } 3.6 \\
BC & \quad R_A x - \frac{6.6 x^2}{2} + R_B \left(x - 3.6\right) \\
& \quad \left(x - 3.6\right) \\
& \quad 3.6 \text{ to } 8.4 \\
(1) & \quad \frac{\partial U}{\partial R_A} = 0 \\
& \quad \int_0^{3.6} \frac{(R_A x - 3.3x^2) x \, dx}{EI} \\
& \quad + \int_{3.6}^{8.4} \frac{\left[R_A x - 3.3x^2 + R_B \left(x - 3.6\right)\right] x \, dx}{EI}
\end{align*}\]
\[
\begin{align*}
0 & = \left[ \frac{R_A}{3} \cdot \frac{x^3}{3} - \frac{\frac{3}{4} \cdot x^4}{4} \right]_0^8 \frac{8^4}{EI} + \frac{1}{EI} \left[ \frac{R_B}{3} \cdot x^3 - R_B \cdot 1.8x^3 \right]_3^6 \frac{6^4}{3^6} \\
\Rightarrow & \quad 0 = \left[ \frac{R_A}{3} \cdot \frac{8^4}{3} + \frac{3 \times 8^4}{4} \right] \\
& \quad + \left[ \frac{R_B}{3} \left( 8^4 - 3.6^4 \right) - 1.8 \cdot R_B \left( 8^4 - 3.6^4 \right) \right] \\
& \quad = 197.6 \cdot R_A - 4107 + 78.34 \cdot R_B \\
\Rightarrow & \quad R_A = -3966 \cdot R_B + 20.79 \quad \ldots (i)
\end{align*}
\]

\(\frac{\partial U}{\partial R_B} = -6.35 \times 10^{-3}\)

\[
\begin{align*}
-6.35 \times 10^{-3} \times 6640 & = \left[ \frac{R_A}{3} \cdot x^3 - \frac{R_A}{3} \cdot 1.8 \cdot x^3 - \frac{3 \cdot 3^4}{4} \right] \\
& \quad + \left[ \frac{R_B}{3} \cdot x^3 - 3.6x^3 \cdot R_B + 12.96x \cdot R_B \right]_3^6 \frac{6^4}{3^6} \\
\Rightarrow & \quad -42.17 = \left[ \frac{R_A}{3} \left( 8^4 - 3.6^4 \right) - R_A \cdot 1.8 \left( 8^4 - 3.6^4 \right) \right] \\
& \quad - \frac{3 \cdot 3^4}{4} \left( 8^4 - 3.6^4 \right) + 3.96 \left( 8^4 - 3.6^4 \right) \right] \\
& \quad + \frac{R_B}{3} \left( 8^4 - 3.6^4 \right) - 3.6 \cdot R_B \left( 8^4 - 3.6^4 \right) + 12.96 \cdot R_B \left( 8^4 - 3.6^4 \right) \\
\Rightarrow & \quad -42.17 = 182.02 \cdot R_A - 103.68 \cdot R_A - 3966 + 2162 + 182.02 \cdot R_B \\
& \quad - 207.38 \cdot R_B + 62.22 \cdot R_B \\
\Rightarrow & \quad -42.17 = 78.34 \cdot R_A + 36.86 \cdot R_B - 1806 \\
\Rightarrow & \quad 78.34 \cdot R_A = -36.86 \cdot R_B + 1763.83 \\
\Rightarrow & \quad R_A = -4704 \cdot R_B + 2252 \quad \ldots (ii)
\end{align*}
\]

Equating equations (i) and (ii)

\(-4704 \cdot R_B + 2252 = -3966 \cdot R_B + 20.79\)

\[
\begin{align*}
1.73 & = 0.0738 \cdot R_B \\
\Rightarrow & \quad R_B = \frac{1.73}{0.0738} = 23.44 \text{ t.} \\
R_A & = -3966 \times 23.44 + 20.79 \\
& = -9296 + 20.79 \\
& = +11.504 \text{ t.}
\end{align*}
\]
\[ M_B = 11.504 \times 3.6 - \frac{6.6 \times 3.6^2}{2} = 41.44 - 42.77 = -1.33 \text{ t.m.} \]

\[ M_C = 11.504 \times 8.4 + 23.44 \times 4.8 - \frac{6.6 \times 8.4^2}{2} = 96.65 + 112.5 - 232.9 = 209.15 - 232.90 = -23.75 \text{ t.m.} \]

Final moments are shown in Fig. 7.23.

![Diagram](image)

**Ex. 7.19.** Analyse the portal frame shown in Fig. 7.24. Ends A and D are hinged. EI is constant. Draw the bending moment diagram.

(A.M.I.E. May 1968)

**Solution.** Let \( H \) be the horizontal reaction at \( D \).

\[ R_D l - H \frac{l}{2} - 8 \times \frac{l}{2} = 0 \]

\[ R_F = 4 + \frac{H}{2} \]

\[ R_A = 4 - \frac{H}{2} \]

![Diagram](image)

**TABLE 7.16**

<table>
<thead>
<tr>
<th>Portion</th>
<th>origin at</th>
<th>Moment</th>
<th>Moment rearranged</th>
<th>( \frac{\partial U}{\partial H} )</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>DC</td>
<td>D</td>
<td>(-Hy)</td>
<td></td>
<td>(-y)</td>
<td>0 to ( \frac{3l}{2} )</td>
</tr>
<tr>
<td>CE</td>
<td>O</td>
<td>(-H \times \frac{3l}{2} + \left( 4 + \frac{H}{2} \right)x )</td>
<td>( \frac{H}{2} ) ( (x-3l) + 4x )</td>
<td>( \frac{(x-3l)}{2} )</td>
<td>0 to ( \frac{l}{2} )</td>
</tr>
<tr>
<td>EB</td>
<td>B</td>
<td>(-Hl + \left( 4 - \frac{H}{2} \right)x )</td>
<td>(-\frac{H}{2} ) ( (x-2l) + 4x )</td>
<td>( \frac{(x-2l)}{2} )</td>
<td>0 to ( \frac{l}{2} )</td>
</tr>
<tr>
<td>BA</td>
<td>A</td>
<td>(-Hy)</td>
<td></td>
<td>(-y)</td>
<td>0 to ( l ).</td>
</tr>
</tbody>
</table>
\[
\frac{\partial U}{\partial H} = 0 = \int_0^{3l/2} \frac{(-Hy)(-y)}{EI} dy \\
+ \int_0^{l/2} \left\{ \frac{H}{2} \left( \frac{x}{2} - 3l + 4x \right)^{3l/2} \right\} dx \\
+ \int_0^{l/2} \left\{ \frac{H}{2} \left( x + 2l + 4x \right)^{3l/2} \right\} dx \\
+ \int_0^{l} \frac{(-Hy)(-y)}{EI} dy
\]

\[
0 = \left[ \frac{Hy^3}{3} \right]_0^{3l/2} + \left[ \frac{H}{4} \left( \frac{x^3}{3} - 3lx^2 + 9l^2x \right) + \frac{2x^3}{3} - 3lx^2 \right]_0^{l/2} \\
+ \left\{ \frac{H}{4} \left( \frac{x^3}{3} + 2x^2l + 4l^2x \right) \right\} - \frac{2x^3}{3} - 2lx^2 \right\} \left[ \frac{Hy^3}{3} \right]_0^{l/2}
\]

\[
0 = \frac{9Hl^3}{8} + \left[ \frac{H}{4} \left( \frac{l^3}{24} - \frac{3l^3}{4} + \frac{9l^3}{2} \right) + \frac{l^3}{12} - \frac{3l^3}{4} \right] \\
+ \left\{ \frac{H}{4} \left( \frac{l^3}{24} + \frac{l^3}{2} + 2l^3 \right) - \frac{l^3}{12} - \frac{l^3}{2} \right\} + \frac{Hl^3}{3}
\]

\[
0 = \frac{35Hl^3}{24} + \frac{91Hl^3}{96} - \frac{2l^3}{3} + \frac{61l^3}{12} - \frac{7l^3}{12} \\
= \frac{292Hl^3}{96} - \frac{5l^3}{4}
\]

\[
H = \frac{5}{4} \times \frac{96}{292} = \frac{30}{73}
\]

\[
M_C = -H \times \frac{3}{2} l \\
= \frac{-30}{73} \times 2.7 \\
= \frac{15 \times 2.7}{29} = -1.11 \text{ t.m.}
\]

\[
M_B = -H \times l \\
= \frac{-30}{73} \times 1.8 = -74 \text{ t.m.}
\]

B.M. diagram is drawn on tension side in Fig. 7.25.
PROBLEMS

1. Two cantilevers, one vertically above the other, are connected at the ends as shown in Fig. 7·26. Find the forces in tie rod, \( I \) for each cantilever = 5000 cm\(^4\). Area of rod = 2 cm\(^2\), \( E \) is same for all members.

\[ \text{Ans. } R = 590 \text{ kg.} \]

![Fig. 7·26](image)

2. Find the horizontal thrust at hinges \( A \) and \( D \) of two hinged portal frame shown in Fig. 7·27.

\[ \begin{align*}
\text{Ans. } H_A &= \frac{8}{9} \\
H_D &= \frac{28}{9} \text{ t.}
\end{align*} \]

![Fig. 7·27](image)

3. A link consisting of two semi-circles and two straight portions is submitted to the action of two equal opposite forces acting along the vertical axis of symmetry as shown in Fig 7·28. Determine the maximum B.M. and separation of loaded points, assuming that the cross-sectional dimensions of the link are small in comparison with the radius \( R \). Flexural rigidity = \( EI \).

\[ \begin{align*}
M &= \frac{2PR}{\pi + 2} \\
\delta &= \frac{0.23PR^3}{EI}
\end{align*} \]

![Fig. 7·28](image)
4. A circular steel pipe radius $2R$ and supported on soil is loaded with a uniformly distributed load of $w$ kg/m. Assuming the soil pressure to be uniform, find the maximum bending moment induced in the pipe.

\[ \text{Ans. } M = \frac{wR^2}{4} \]

5. A circular ring of mean diameter $D$ has a horizontal diametrical tie which may be considered inextensible. The ring is subjected to equal compressive forces $P$ along the vertical diameter. Show that the tension in the tie is

\[ \frac{2(4-\pi)P}{\pi^2-8} \]

6. A ring of radius $R$ and uniform cross-section hangs from a single support. Find the position and magnitude of the maximum bending moment due to its own weight $W$.

\[ \text{Ans. } \frac{3WR}{4\pi} \text{ at the support.} \]

7. A steel ring of mean diameter 20 cm. is used as a load measuring device. The thickness of ring in its own plane is 2 cm. and the breadth 0.5 cm. Estimate the maximum bending stress when the ring carries load of 1000 kg. Also calculate separation of loaded points. $E = 2 \times 10^6$ kg/cm$^2$.

\[ \text{Ans. Bending stress } = 955 \text{ kg/cm}^2. \]

\[ \delta = 0.223 \text{ cm.} \]

8. A link having constant section throughout and consisting of two semi-circles and two straight portions is subjected to the action of two equal and opposite axial forces $P$ as shown in Fig. 7.29. Assuming that the cross-sectional dimensions of the link are small in comparison with the radius $R$, determine the maximum B.M. in the link.

\[ \text{Ans. } M = 0.362 \text{ PE} \]
9. What is the value of maximum B.M. in the ring of problem 8 if the load is applied as shown in Fig. 7.30.

![Diagram of a ring with a load applied](image)

Fig. 7.30

10. A rectangular closed frame is 9.15 m. wide and 5.49 m. high. It carries a point load of 27,200 kg. at mid-point of its top horizontal member and is acted upon by horizontal earth pressure on the two vertical members varying in intensity from zero at top to 894 kg/m. at bottom. The upward ground reaction acts on the bottom horizontal member. The moment of inertia of top and bottom members is $I$ and that of side members is $0.4I$. Calculate the B.M. at the corners of the frame. \(\text{(A.M.I.E. Nov. 1968)}\)

11. A beam $ABCD$ is 9 m. long and is simply supported at $A$, $B$ and $C$. Span $AB$ is 3 m., $BC$ is 4.5 m. and overhang $CD$ is 1.5 m. Moment of inertia of beam in span $AB$ is $I$ and that in span $BC$, $2I$. It is loaded with a uniformly distributed load of 2000 kg/m. in span $AB$ and with a point load of 1,000 kg. at the free end $D$. Draw the bending moment and shear force diagrams for the beam. \(\text{(A.M.I.E. May 1969)}\)

12. A rectangular portal frame has a span of 6 m. and a height of 4.5 m. Its two vertical members are fixed to the ground at their bottom ends. The horizontal member of the portal carries a load of 3 t/m. The moment of inertia of vertical members is $I$ and that of horizontal member is $2I$. Calculate the support reactions and draw the B.M. diagram for the portal frame. \(\text{(A.M.I.E. Nov. 1969)}\)
MOMENT DISTRIBUTION METHOD

8.1. Analysis of indeterminate beams and rigidly jointed frames by methods of slope-deflection and strain energy discussed already, involve solving a number of simultaneous equations which makes the working tedious. The method of moment distribution is simple involving process of relaxation. This method is due to Hardy-Cross.

In this method all the members of structures are initially assumed fixed at the ends, in position and direction, and fixed end moments due to external loads are worked out. The joints are assumed to be locked and external moments and forces are applied to achieve fixity of members at the joints. The external moment is called unbalanced moment and the external forces called the sway forces. The external moment is applied to prevent rotation of the joint and external forces are applied to prevent displacement of the joints. The restraints provided at a joint are released and their effects on the joint and other joints are evaluated. One by one all the joints are released and the effects are evaluated. This process is continued till the external moments or forces at the joints are zero or negligible.

8.2. Definition of Terms.

Stiffness factor. For prismatic members, stiffness $k$ is defined as ratio of moment of inertia to length of the member, i.e.

$$k = \frac{I}{l}.$$  

Carry over factor. Consider a member $AB$, fixed at end $B$ and supported at end $A$. A moment $M_A$ applied at $A$, without displacing $A$, will produce moment at $B$. Let it be $M_B$. The ratio of $M_B/M_A$ is known as carry-over factor. The slope at $B$ is zero, therefore, the tangent drawn at $B$ to deflected form will pass through $A$. Thus the moment of B.M. diagram between $A$ and $B$ about $A$ will be zero. If moment $M_A$ applied at $A$ is $+\text{ve}$ (s moment produced at be $-\text{ve}$ (hoggign).

$$M_A \times \frac{l}{2} \times \frac{l}{3} - M_B \times \frac{2l}{3}$$

$$\therefore \quad M_B = \frac{M_A}{2}$$

$$\frac{M_R}{M_A} = \frac{1}{2}$$
Therefore, carry-over factor is \( \frac{1}{2} \).

It is seen that if a moment is applied at one end, half of this is in the same direction (clockwise or anti-clockwise) is produced at the other end.

Let \( \theta_A \) be slope at \( A \). Intercept on vertical at \( B \), above tangent at \( A \) is \( l \theta_A \)

\[
\frac{1}{EI} \left[ \frac{M_A}{3} \times \frac{l}{2} \times \frac{2l}{3} - \frac{M_B}{3} \times \frac{l}{2} \times \frac{l}{3} \right] = l \theta_A
\]

\[
\frac{M_A l^2}{3} - \frac{M_B l^2}{6} = M_l = l \theta_A EI
\]

\[
\frac{M_A l^2}{4} = \frac{M_l}{l} = 4EI\theta_A
\]

\[
M_l = 4EIk\theta_A
\]

Beam hinged at one end and supported at other end.
Consider a prismatic beam, supported at \( A \) and hinged at \( B \). Moment \( M_A \) is applied at \( A \). B.M. diagram is drawn in Fig. 8.2 (b). Let \( \theta_A \) be slope at \( A \).

\[
\theta_A = \frac{Z}{l}, \text{ where } Z \text{ is intercept on vertical at } B.
\]

\[
M_A \times \frac{l}{2} \times \frac{2}{3} l = Z
\]

\[
\frac{EI}{l \theta_A} = \frac{M_A}{l} = 3EI\theta_A = 3EIk\theta_A.
\]

**Sign convention.** For this method, clockwise moments at ends are considered positive and anti-clockwise moments are considered as \(-\)ve.

**Distribution factor.** Consider a frame with members, \( OA, OB, OC \) and \( OD \), rigidly connected at joint \( O \) as shown in Fig. 8.3. Let a moment \( M \) be applied at joint \( O \) in the clockwise direction. The joint will rotate clockwise and all the members \( OA, OB, OC \) and \( OD \) will rotate by same amount, say \( \theta \).

Let \( k_{OA}, k_{OB}, k_{OC} \) and \( k_{OD} \) be stiffnesses of members \( QA, OB, OC \) and \( OD \), respectively.
Moment \( M_{OA} = 4Ek_O\theta \)

\[ M_{OB} = 4Ek_{OB}\theta \]

\[ M_{OC} = 4Ek_{OC}\theta \]

\[ M_{OD} = 4Ek_{OD}\theta \]

\[ M_{OA} + M_{OB} + M_{OC} + M_{OD} = 4E\theta[k_{OA} + k_{OB} + k_{OC} + k_{OD}] \]

Sum of the moments in the members will be equal to external moment

\[ M = 4E\theta\Sigma k \]

where \( \Sigma k = k_{OA} + k_{OB} + k_{OC} + k_{OD} \)

\[ \frac{M_{OA}}{M} = \frac{4Ek_{OA}\theta}{4E\theta\Sigma k} \]

\[ \frac{M_{OA}}{\Sigma k} = r_{OA} \]

is known as distribution factor for OA and is denoted by \( r_{OA} \).

\[ \frac{M_{OA}}{\Sigma k} \]

Frame with one end hinged. Consider frame similar to Fig. 8.3 but with end A as hinged.

\[ M_{OA} = 3Ek_{OA}\theta \]

\[ M_{OB} = 4Ek_{OB}\theta \]

\[ M_{OC} = 4Ek_{OC}\theta \]

\[ M_{OD} = 4Ek_{OD}\theta \]

Fig. 8.4

\[ M_{OA} + M_{OB} + M_{OC} + M_{OD} = 4E\theta[k_{OA} + k_{OB} + k_{OC} + k_{OD}] \]

\[ M_{(OA)} = \frac{3Ek_{OA}\theta}{4E\theta[k_{OA} + k_{OB} + k_{OC} + k_{OD}]} \]

\[ M = \frac{\frac{3}{2}k_{OA}}{[\frac{1}{2}k_{OA} + k_{OB} + k_{OC} + k_{OD}]} \]

\[ r_{OA} = \frac{\frac{3}{2}k_{OA}}{\frac{3}{2}k_{OA} + k_{OB} + k_{OC} + k_{OD}} \]

The distribution factors can be worked out by assuming stiffness of member hinged at one end equal to \( \frac{1}{2} \) of \( I/l \).
8.3. Analysis of propped cantilevers

Consider a propped cantilever $AB$ loaded with uniformly distributed load of intensity $w/\text{unit length}$ throughout the span. To analyse the beam following steps are carried out.

(i) Assume ends $A$ and $B$ fixed so that there is no rotation. The fixed end moments will be $+\frac{wL^2}{12}$ at $B$ and $-\frac{wL^2}{12}$ at $A$.

(ii) The end $B$ is simply supported and, therefore the moment at $B$ should be zero. External moment $+\frac{wL^2}{12}$ has been applied at $B$ to stop it from rotation. This moment is called unbalanced moment.

(iii) The end $B$ is released by applying moment $-\frac{wL^2}{12}$, opposite to unbalanced moment. This process is called balancing.

(iv) The effect of applying $-\frac{wL^2}{12}$ at $B$ will be to produce moment $-\frac{wL^2}{24}$ at $A$. This process is known as carry-over.

(v) The final moment at $A$ will be

$$\frac{wL^2}{12} - \frac{wL^2}{24} = -\frac{wL^2}{8}$$

The whole process is shown in Table 8.1

<table>
<thead>
<tr>
<th>Joint</th>
<th>$A$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>$AB$</td>
<td>$BA$</td>
</tr>
<tr>
<td>Distribution Factors</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>F.E.M.</td>
<td>$-\frac{wL^2}{12}$</td>
<td>$+\frac{wL^2}{12}$</td>
</tr>
<tr>
<td>Balancing</td>
<td>-</td>
<td>$-\frac{wL^2}{12}$</td>
</tr>
<tr>
<td>Carry over</td>
<td>$-\frac{wL^2}{24}$</td>
<td>-</td>
</tr>
<tr>
<td>Final Moments</td>
<td>$-\frac{wL^2}{8}$</td>
<td>0</td>
</tr>
</tbody>
</table>
8.4. Analysis of Continuous Beams.

(I) Consider continuous beam ABC, having two equal spans and loaded with uniformly distributed load. Following steps are

Fig. 8.6

-carried out to analyse the beam.

(i) Calculate distributed factors at B.

\[
r_{BA} = \frac{k_{BA}}{k_{BA} + \frac{3}{4} k_{BC}}
\]

\[
= \frac{\frac{I}{l}}{\frac{3}{4} \frac{I}{l}} + \frac{1}{l}
\]

\[
= \frac{4}{7}
\]
\[ r_{BC} = \frac{1}{7} \]

(ii) The spans \( AB \) and \( BC \) are taken as fixed. The fixed end moments will be \(-\frac{wl^2}{12}\) at \( A \) and \(+\frac{wl^2}{12}\) at \( B \) for span \( AB \), and \(-\frac{wl^2}{12}\) at \( B \) and \(+\frac{wl^2}{12}\) at \( C \) for \( BC \).

(iii) The end \( C \) is released by applying \(-\frac{wl^2}{12}\) moment there. The carry over at \( B \) will be \(-\frac{wl^2}{24}\).

(iv) At joint \( B \), moment \( M_{BC} \) \(-\frac{wl^2}{12} - \frac{wl^2}{24} = -\frac{wl^2}{8}\) and moment \( M_{BA} \) \(+\frac{wl^2}{12}\). Thus there is unbalanced moment of \(-\frac{wl^2}{8}\) \(+\frac{wl^2}{12} = -\frac{wl^2}{24}\). Balancing moment of \(+\frac{wl^2}{24}\) is applied at \( B \). The balancing moment shared by \( BA \) and \( BC \) will be in proportion to distribution factors. The moment taken by \( BA \) at \( B \) will be \(+\frac{wl^2}{24} \times \frac{4}{7} = +\frac{wl^2}{42}\) and by \( BC \) at \( B \) will be \(+\frac{wl^2}{24} \times \frac{3}{7} = +\frac{wl^2}{56}\). The moment carried over to \( A \) will be \(+\frac{wl^2}{84}\). No moment is carried over to simply supported end \( C \) as stiffness of the member \( BC \) is taken as \( \frac{3}{4} \frac{I}{l} \).

(v) The final moments will be

\[
\begin{align*}
M_{AB} &= -\frac{wl^2}{12} + \frac{wl^2}{84} = -\frac{wl^2}{14} \\
M_{BA} &= +\frac{wl^2}{12} + \frac{wl^2}{42} = +\frac{3wl^2}{23} \\
M_{BC} &= -\frac{wl^2}{8} + \frac{wl^2}{56} = -\frac{3wl^2}{2} 
\end{align*}
\]

The process of moment distribution is shown in Table 8.2.


### Table 8.2

<table>
<thead>
<tr>
<th>Joints</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>(AB)</td>
<td>(BA)</td>
<td>(BC)</td>
</tr>
<tr>
<td>Distribution factors</td>
<td>(-)</td>
<td>4/7</td>
<td>3/7</td>
</tr>
<tr>
<td>F.E.M.</td>
<td>(-\frac{wl^2}{12})</td>
<td>(\frac{wl^2}{12})</td>
<td>(-\frac{wl^2}{12})</td>
</tr>
<tr>
<td>Release C</td>
<td>(-\frac{wl^2}{12})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Carry over</td>
<td>(-\frac{wl^2}{24})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total Moments</td>
<td>(-\frac{wl^2}{12})</td>
<td>(\frac{wl^2}{12})</td>
<td>(-\frac{wl^2}{8})</td>
</tr>
<tr>
<td>Balance B</td>
<td>(\frac{wl^2}{42})</td>
<td>(\frac{wl^2}{56})</td>
<td></td>
</tr>
<tr>
<td>Carry over</td>
<td>(+\frac{wl^2}{84})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final moments</td>
<td>(-\frac{wl^2}{14})</td>
<td>(+\frac{3}{28}wl^2)</td>
<td>(-\frac{3}{28}wl^2)</td>
</tr>
</tbody>
</table>

#### (II) Continuous beam with overhang

Consider continuous beam \(ABCD\) with overhang \(CD\).

(i) The distribution factors at \(B\) will be same as in first case. The overhang \(CD\) will have zero stiffness as the moment at \(C\) for \(CD\) will be cantilever moment and is not dependent on loading on the remaining portion of the continuous beam. The moment at \(C\) in \(CB\) will be equal and opposite to moment at \(C\) for \(CD\).

(ii) The fixed end moments considering all ends as fixed are worked out and are shown in Fig. 8.7 (b).

(iii) The moment at \(C\) for \(CB\) will be \(-\frac{wa^2}{2}\). Therefore unbalancing moment at \(C\) is \(\frac{wl^2}{12} - \frac{wa^2}{2}\). Balancing moment of \(-\left[\frac{wl^2}{12} - \frac{wa^2}{2}\right]\) is applied at \(C\) for \(CB\). Half of this moment is carried over to \(B\). The total moments are worked out and are shown in Fig. 8.6 (b).

(iv) At joint \(B\) there is unbalanced moment of \(-\frac{1}{2}\left[\frac{wl^2}{12} - \frac{wa^2}{2}\right]\). Balancing moment of \(+\frac{1}{2}\left[\frac{wl^2}{12} - \frac{wa^2}{2}\right]\) is applied at \(B\) and this is
shared by $BC$ and $BA$ in ratio of the stiffnesses of respective members. Half of the moment for $BC$ is carried over to $A$. No moment is carried over to $C$. Final moments are shown in Fig. 8.7 (c)

**Fixed End Moments**

\[
\begin{align*}
\frac{3A}{4} & \quad \frac{B}{4} + \frac{\omega l^2}{12} \\
-\frac{\omega l^2}{12} & \quad -\frac{\omega l^2}{12} \\
B & \quad C + \frac{\omega l^2}{12} \\
-\frac{\omega l^2}{12} + \frac{1}{2} \left( \frac{\omega l^2}{12} \frac{\omega a^2}{2} \right) & \quad \frac{\omega l^2}{12} + \frac{1}{2} \left( \frac{\omega l^2}{12} \frac{\omega a^2}{2} \right) \\
\frac{\omega l^2}{12} + \frac{1}{2} \left( \frac{\omega l^2}{12} \frac{\omega a^2}{2} \right) & \quad \frac{\omega l^2}{12} + \frac{1}{2} \left( \frac{\omega l^2}{12} \frac{\omega a^2}{2} \right) \\
\frac{3A}{4} & \quad \frac{B}{4} \quad \frac{C}{4} + \frac{\omega l^2}{12} \\
-\frac{\omega l^2}{12} + \frac{1}{2} \left( \frac{\omega l^2}{12} \frac{\omega a^2}{2} \right) & \quad \frac{3}{14} \left( \frac{\omega l^2}{12} \frac{\omega a^2}{2} \right) \\
\frac{\omega l^2}{12} + \frac{1}{2} \left( \frac{\omega l^2}{12} \frac{\omega a^2}{2} \right) & \quad \frac{3}{14} \left( \frac{\omega l^2}{12} \frac{\omega a^2}{2} \right) \\
\frac{\omega l^2}{12} + \frac{1}{2} \left( \frac{\omega l^2}{12} \frac{\omega a^2}{2} \right) & \quad \frac{3}{14} \left( \frac{\omega l^2}{12} \frac{\omega a^2}{2} \right) \\
\frac{\omega l^2}{12} + \frac{1}{2} \left( \frac{\omega l^2}{12} \frac{\omega a^2}{2} \right) & \quad \frac{3}{14} \left( \frac{\omega l^2}{12} \frac{\omega a^2}{2} \right) \\
\end{align*}
\]

**Release B and Carry Over to A**

Fig. 8.7

The process of moment distribution is shown in Table 8.3.
### Table 3.3

<table>
<thead>
<tr>
<th>Joints</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>$AB$</td>
<td>$BA$</td>
<td>$BC$</td>
</tr>
<tr>
<td>D.F.</td>
<td>$-$</td>
<td>$4/7$</td>
<td>$3/7$</td>
</tr>
</tbody>
</table>

**F.E.M.**

- $\frac{wl^2}{12}$
- $\frac{wl^2}{12}$
- $\frac{-wl^2}{12}$
- $\frac{-wl^2}{12}$
- $\frac{-wa^2}{2}$
- $\frac{-la^2}{2}$

**Balance C**

**Carry over**

**Total Moment**

- $\frac{-wl^2}{12}$
- $\frac{wl^2}{12}$
- $\frac{-wl^2}{12}$
- $\frac{-wl^2}{12}$
- $\frac{-wa^2}{2}$
- $\frac{-wa^2}{2}$

**Balance B**

**Carry over**

**Final Moments**

- $\frac{-wl^2}{14}$
- $\frac{-wa^2}{14}$
- $\frac{3wl^2}{28}$
- $\frac{-wa^2}{7}$
- $\frac{3wl^2}{28}$
- $\frac{-wa^2}{7}$

**Ex. 8.1**. Analyse the continuous beam shown in Fig. 8.8 by method of moment distribution.

**Solution**. Fixed end moments will be

$$M'_{AB} = \frac{-wl^2}{12} = \frac{-2000 \times 6 \times 6}{12} = -6000 \text{ kg.m.}$$
\[
\begin{align*}
M_{BA} &= \frac{wl^3}{12} = +6000 \text{ kg. m.} \\
M_{BC} &= -\frac{wab^3}{l^2} \\
&= -\frac{5000 \times 2 \times 3 \times 3}{5 \times 5} \\
&= -3600 \text{ kg. m.} \\
M_{CB} &= +\frac{wa^3b}{l} \\
&= +\frac{5000 \times 2 \times 2 \times 3}{5 \times 5} \\
&= 2400 \text{ kg. m.}
\end{align*}
\]

Distribution factors at \( B \) are

\[
\tau_{BA} = \frac{3}{4} \times \frac{I}{6} \times \frac{1}{\frac{4}{6} + \frac{3}{4} \times \frac{30}{11} \times \frac{11}{11}} \\
= \frac{5}{6} \times \frac{6}{11} = \frac{6}{11}
\]

Process of moment distribution is shown in Table 8.4.

<table>
<thead>
<tr>
<th>Joints</th>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>( AB )</td>
<td>( BA )</td>
<td>( BC )</td>
</tr>
<tr>
<td>Distribution factors</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fixed end moments</td>
<td>(-6,000)</td>
<td>(+6,000)</td>
<td>(-3,600)</td>
</tr>
<tr>
<td>Release ( A ) and ( C )</td>
<td>(+6,000)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Carry over</td>
<td>0</td>
<td>(+3,000)</td>
<td>(-1,200)</td>
</tr>
<tr>
<td>Total Moments Balance ( B )</td>
<td>(+9,000)</td>
<td>(-1,909)</td>
<td></td>
</tr>
<tr>
<td>Final Moments</td>
<td>0</td>
<td>(+7,091)</td>
<td>(-7,091)</td>
</tr>
</tbody>
</table>
Ex. 8'2. Analyse the continuous beam shown in Fig. 8'9 by moment distribution method.

Solution. Due to applied moment of \( +4000 \times 1 = +4000 \text{ kg. m.} \) at centre of BC, fixed end moments

\[
\overline{M_{AC}} = \overline{M_{CB}} - \frac{M}{4} = +\frac{4000}{4} \text{ kg. m.}
\]

\( M_{CD} = -2000 \times 3 = -6000 \text{ kg. m.} \)

Distribution factors at B are

\[
\frac{I}{k_{AB} + \frac{3}{4} k_{BC}} = \frac{1}{8 + 3 \times \frac{1}{4} \times \frac{1}{6}}
\]

\( r_{BA} = \frac{1}{2} \)

\( r_{BC} = \frac{1}{2} \)

Process of moment distribution is shown in Table 8'5.

**TABLE 8'5**

<table>
<thead>
<tr>
<th>Joints</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>AB</td>
<td>BA</td>
<td>BC</td>
</tr>
<tr>
<td>D.F.</td>
<td>1/2</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>E.M. Balance C</td>
<td></td>
<td>+1000</td>
<td>+1000</td>
</tr>
<tr>
<td>Carry over</td>
<td></td>
<td>+2500</td>
<td></td>
</tr>
<tr>
<td>Total Moments Balance B</td>
<td></td>
<td>-1750</td>
<td>+3500</td>
</tr>
<tr>
<td>Carry over</td>
<td></td>
<td>-875</td>
<td></td>
</tr>
<tr>
<td>Final Moments</td>
<td></td>
<td>-875</td>
<td>-1750</td>
</tr>
</tbody>
</table>

Ex. 8'3. Analyse the continuous beam shown in Fig. 8'10 by moment distribution method.

Solution. Fixed end moments are

\[
\overline{M'_{BA}} = +2000 \times 1 = 2000 \text{ kg. m.}
\]

\[
\overline{M'_{DE}} = -1000 \times 2 = -2000 \text{ kg. m.}
\]
\[-\overline{M}_{\text{BC}} = +\overline{M}_{\text{CB}} = \frac{wL^2}{12} = \frac{2000 \times 4 \times 4}{12} = 2666.66 \text{ kg.m.}\]

\[-\overline{M}_{\text{CD}} = \frac{5WL}{48}\]

\[
\frac{5}{48} \times \frac{1}{2} \times 4000 \times 6 \times 6 = 7500 \text{ kg.m.}
\]

Fig. 8.10

Distribution factors at C are

\[
\begin{align*}
\overline{r}_{\text{CB}} &= \frac{3}{4} \times \frac{1}{4} \\
\overline{r}_{\text{CD}} &= \frac{1}{4} \\
\overline{r}_{\text{DE}} &= \frac{1}{4} \\
\frac{3}{4} \times \frac{1}{4} + \frac{3}{4} \times \frac{1}{6} - \frac{1}{4} + \frac{1}{6} \\
\frac{1}{4} \times 4 \times 6 &= 0.6
\end{align*}
\]

\[
\overline{r}_{\text{CD}} = 0.4.
\]

Process of moment distribution is shown in Table 8.6.

**TABLE 8.6**

<table>
<thead>
<tr>
<th>Joints</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>BA</td>
<td>BC</td>
<td>CB</td>
</tr>
<tr>
<td>D.F.</td>
<td></td>
<td>1</td>
<td>0.6</td>
</tr>
<tr>
<td>F.E.M.</td>
<td>+2000</td>
<td>-2666.66</td>
<td>+2666.66</td>
</tr>
<tr>
<td>Balance B and D</td>
<td>+666.66</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Carry over</td>
<td></td>
<td></td>
<td>+333.33</td>
</tr>
<tr>
<td>Balance C</td>
<td></td>
<td></td>
<td>+4350</td>
</tr>
</tbody>
</table>
Ex. 8-14. Continuous beam ABCD is loaded as shown in Fig. 8-11. During loading support B sinks by 1 cm. Determine the support moments. \( I = 16,000 \text{ cm}^4 \): \( E = 2 \times 10^8 \text{ kg/cm}^2 \).

Solution. Fixed moments are

**Span AB**

Due to uniformly distributed load

\[
\overline{M_{AB}} = -\frac{wl^2}{12} = -\frac{1500 \times 8 \times 8}{12} = -8,000 \text{ kg.m.}
\]

Due to sinking of support,

\[
\overline{M_{AB}} = \overline{M_{BA}} = -\frac{6EI\delta}{8} = -\frac{6 \times 2 \times 10^6 \times 16,000}{8 \times 8 \times (100)^3} \times 100 = -6,000 \text{ kg.m.}
\]

Total fixed end moments are

\[
\overline{M_{AB}} = -8,000 - 6,000 = -14,000 \text{ kg.m.}
\]

\[
\overline{M_{BA}} = +8,000 - 6,000 = +2,000 \text{ kg.m.}
\]

**Span BC**

Due to uniformly distributed load,

\[
\overline{M_{BC}} = \overline{M_{CB}} = \frac{1500 \times 4 \times 4}{12} = -2,000 \text{ kg.m.}
\]

Due to sinking of support,

\[
\overline{M_{BC}} = \overline{M_{CB}} = +\frac{6EI\delta}{12} = +\frac{6 \times 2 \times 10^6 \times 16,000}{4 \times 4 \times (100)^3} \times \frac{1}{100} = +12,000 \text{ kg.m.}
\]

Total fixed end moments are

\[
\overline{M_{BC}} = \overline{M_{BC}} + \overline{M_{BC}} = -2,000 + 12,000 = +10,000 \text{ kg.m.}
\]

\[
\overline{M_{CB}} = +2,000 + 12,000 = +14,000 \text{ kg.m.}
\]
Span CD

\[-M_{CD} = -\frac{WL}{8} - \frac{4.000 \times 8}{8} = -1000 \text{ kg. m.}\]

\[M_{DC} = +\frac{WL}{8} = +4000 \text{ kg. m.}\]

Distribution factors at B are

\[r_{BA} = \frac{2I}{8 + \frac{I}{4}} = \frac{1}{\frac{1}{4}} \cdot \frac{1}{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2}\]

\[r_{BC} = \frac{1}{4}\]

Similarly distribution factors at C are

\[r_{CB} = r_{CD} = \frac{1}{4}\]

Process of moment distribution is shown in Table 8.7.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>AB</td>
<td>BA</td>
<td>BC</td>
<td>CB</td>
</tr>
<tr>
<td>D.F.</td>
<td></td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>F.E.M. Balance</td>
<td>-14,000</td>
<td>+2,000</td>
<td>+10,000</td>
<td>+14,000</td>
</tr>
<tr>
<td>Carry over Balance</td>
<td>-3000</td>
<td></td>
<td>-2500</td>
<td>-3000</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+625</td>
<td>-375</td>
<td>+750</td>
<td>+625</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>-188</td>
<td>+78</td>
<td>-158</td>
<td>-188</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+39</td>
<td>-24</td>
<td>+47</td>
<td>+39</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>-12</td>
<td>+5</td>
<td>-10</td>
<td>-2</td>
</tr>
<tr>
<td>C.O.</td>
<td>+3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final moments</td>
<td>-16533</td>
<td>-3066</td>
<td>+3066</td>
<td>+772</td>
</tr>
</tbody>
</table>
Ex. 8.5. Continuous beam ABCD is loaded as shown in Fig. 8.12. During loading support B sinks by 1 cm. Determine support moments. \( I = 8,000 \text{ cm}^4 \). \( E = 2 \times 10^6 \text{ kg/cm}^2 \).

Solution. Fixed end moments are

**Span AB**

Due to loading

\[
\frac{M'_{AB}}{l^2} = -\frac{Wab^2}{4} = -\frac{4,000 \times 3 \times 1 \times 1}{4 \times 4}
\]

\[= -750 \text{ kg. m.}\]

\[
\frac{M'_{BA}}{l^2} = \frac{Wab^2}{4} = \frac{4000 \times 1 \times 3 \times 3}{4 \times 4}
\]

\[= +2250 \text{ kg. m.}\]

Due to sinking of support

\[
\frac{M''_{AB}}{l^2} = \frac{M'_{BA}}{l^2} = -\frac{6EI8}{4 \times 4 \times (100)^2} \times \frac{1}{100}
\]

\[= -6,000 \text{ kg. m.}\]

Total fixed end moments are

\[
\frac{M_{AB}}{l^2} = -6,000 - 750 = -6,750 \text{ kg. m.}
\]

\[
\frac{M_{BA}}{l^2} = -6,000 + 2250 = -3,750 \text{ kg. m.}
\]

**Span BC**

Due to loading

\[
\frac{M'_{BC}}{l^2} = -\frac{wl^2}{12}
\]

\[= -\frac{1000 \times 8 \times 8}{12} = -5333 \text{ kg. m.}\]

\[
\frac{M'_{CB}}{l^2} = +\frac{wl^2}{12}
\]

\[= +\frac{1000 \times 8 \times 8}{12} = +5333 \text{ kg. m.}\]

Due to sinking of support

\[
\frac{M''_{BC}}{l^2} = \frac{M''_{CB}}{l^2} = +\frac{6EI8}{l^2}
\]

\[= +\frac{6 \times 2 \times 10^6 \times 8000 \times \frac{1}{100} \times 2}{8 \times 8 \times (100)^2} = +3000 \text{ kg. m.}
\]

\[\therefore 2l\]
Total fixed end moments are
\[ M_{BO} = -5333 + 3000 = -2333 \text{ kg. m.} \]
\[ M_{CB} = +5333 + 3000 = +8333 \text{ kg. m.} \]

**Span CD**

\[ M_{CD} = -\frac{Wl}{8} \]
\[ = -\frac{4000 \times 6}{8} = -3000 \text{ kg. m.} \]
\[ M_{DC} = +\frac{Wl}{8} \]
\[ = +\frac{4000 \times 6}{8} = +3000 \text{ kg. m.} \]

Distribution factors at \( B \) are
\[ r_{BA} = \frac{3}{4} \frac{I_{BA}}{l_{AB}} - \frac{I_{BC}}{l_{CB}} \]
\[ = \frac{3}{4} \frac{l_{AB}}{l_{AB} + l_{CB}} \]
\[ = \frac{3}{4} \times \frac{1}{4} \]
\[ = \frac{16}{3} \]
\[ = \frac{3}{7} \]
\[ r_{BC} = 1 - \frac{3}{7} = \frac{4}{7} \]

Distribution factors at \( C \) are
\[ r_{OB} = \frac{I_{CB}}{l_{CB}} - \frac{3}{4} \frac{I_{CD}}{l_{CD}} \]
\[ = \frac{2l}{8} \]
\[ = \frac{3}{4} \times \frac{1.5}{6} \]
\[ = \frac{2}{8} \]
\[ = \frac{3}{16} \]
\[ r_{OD} = 1 - r_{OB} = 1 - \frac{4}{7} = \frac{3}{7} \]
Process of moment distribution is shown in Table 8-8.

### TABLE 8-8

<table>
<thead>
<tr>
<th>Joints</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>( AB )</td>
<td>( RA )</td>
<td>( BC )</td>
<td>( CB )</td>
</tr>
<tr>
<td>D.F.</td>
<td></td>
<td>3/7</td>
<td>4/7</td>
<td>4/7</td>
</tr>
<tr>
<td>F.E.M.</td>
<td>(-6750)</td>
<td>(-3750)</td>
<td>(-2333)</td>
<td>(+8333)</td>
</tr>
<tr>
<td>Release</td>
<td>(+6750)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A and D</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Carry over</td>
<td></td>
<td>(+3375)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total moments</td>
<td>0</td>
<td>(-375)</td>
<td>(-2333)</td>
<td>(+8333)</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td>(+1161)</td>
<td>(+1547)</td>
<td>(-2109)</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td></td>
<td>(+469)</td>
<td>(-1069)</td>
<td>(+774)</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td></td>
<td>(+95)</td>
<td>(-221)</td>
<td>(+313)</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td></td>
<td>(+39)</td>
<td>(-90)</td>
<td>(+63)</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td></td>
<td>(+8)</td>
<td>(-18)</td>
<td>(+26)</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td></td>
<td>(+3)</td>
<td>(-7)</td>
<td>(+5)</td>
</tr>
<tr>
<td>Final Moments</td>
<td>0</td>
<td>(+1400)</td>
<td>(-1400)</td>
<td>(+6649)</td>
</tr>
</tbody>
</table>

### 8-5. Analysis of Single Storey Portal Frames

A portal frame consists of number of members rigidly connected at joints so that angle between members remains the same after loading.

In case the frame is symmetrical and symmetrically loaded, the joints will rotate and there will be no displacement of the supports. The analysis by moment distribution for such frames is carried out in a manner as for continuous beams. The fixed end moments...
are calculated by assuming various members fixed at ends and unbalanced moments at joints are found out. The joints are then balanced and half the moments are carried over to the opposite ends of the members. The process of balancing the joints and carry over is continued till the carryover moments are negligible.

In case the frame is unsymmetrical or the loading is unsymmetrical, there will be rotation of joints as well as displacement of the joints. As the displacement is not known the fixed end moments due to displacement cannot be calculated. The moment distribution is carried out by assuming that the joints do not get displaced. The moments thus obtained are called non-sway moments. Next the correction is applied for the displacement, i.e. sway of the frame. Moments are caused due to the displacement of the frame. For certain assumed value of displacement \( \delta \), the fixed end moments for members undergoing displacement are calculated. (In case both ends of a member are fixed, the fixed end moments caused will be \( \frac{6EI\delta}{L} \)). By moment distribution, moments in different members are calculated. If the actual displacement \( \Delta \) of the frame is \( x \) times the assumed displacement \( \delta \), the sway moments will be \( x \) times the moment found with \( \delta \) displacement. The final moments for various members will be non-sway moments plus sway moments.

To determine the value of \( x \), the horizontal shear in the frame will be made equal and opposite to the total horizontal force on the frame above the section where shear is calculated.

![Fig. 8.13](image)

In the unsymmetrical portal frame \( ABCD \) shown in Fig. 8.13 under unsymmetrical load \( W \), first the non-sway moments \( m_{AB}, m_{BA}, \ldots \) and so on are worked out. Then for a particular value of sway \( \delta \), fixed end moments are applied at the ends.
Fig. 8.14

Fixed end moments are \( \bar{M}_{AB} = \bar{M}_{BA} \)

\[ = \frac{6EI_s}{H_1^2} \]

and \( \bar{M}_{DC} = \bar{M}_{CD} = \frac{6EI_s}{H_2^2} \)

\[ \therefore \frac{\bar{M}_{AB}}{\bar{M}_{DC}} = \frac{I_1 H_2}{I_2 H_1^2} \]

With these assumed sway moments, moment distribution is carried out. Let the moments in various members be \( m'_{AB}, m'_{BA}, m'_{BO} \) and so on.

The final moments will be

\[ M_B = m_{BA} + xm'_{AB} \]
\[ M_{BA} = m_{BA} + xm'_{BA} \]
\[ M_{BC} = m_{BC} + xm'_{BC} \]

The value of \( x \) is found from the fact that the total horizontal shear in the frame will be zero.

i.e. \( H_A + H_D = 0 \)

\[ \therefore \frac{M_{AB} + M_{BA}}{H_1} + \frac{M_{DC} + M_{CD}}{H_2} = 0. \]

If there is external force \( P \) acting on the frame as shown in Fig. 8.12.

\[ H_A + H_D + P = 0 \]

\[ \frac{M_{AB} + M_{BA}}{H_1} + \frac{M_{DC} + M_{CD}}{H_2} + P = 0. \]

Substituting value of \( M_{AB}, M_{BA} \) etc. in above equation, value of \( x \) can be evaluated. In case \( x \) comes \(-ve\), actual sway will be opposite in direction to the assumed one.

Knowing \( x \), total moments at the ends of various members can be calculated and B.M. diagram drawn.
Portal frame with one base fixed and other hinged
Let $A$ be the hinged base and $D$ be fixed base of the frame. Due to sway $\delta$, to the right.

![Diagram](image)

$$
\frac{M_{DC}}{M_{CD}} = -\frac{6EI_1\delta}{H_2^3},
$$
$$
M_{AB} = 0,
$$
$$
M_{BA} = -\frac{3EI_1\delta}{H_1^3},
$$
$$
\frac{M_{BA}}{M_D} = \frac{I_1H_3^3}{2I_2H_1^2}.
$$

Portal frame with both bases hinged
Due to sway $\delta$,
\( M_{AB} = 0, \quad M_{DC} = 0 \)

\[ M_{BA} = -\frac{3EI_1\delta}{H_1^2} \]

Similarly

\[ M_{CD} = -\frac{3EI_2\delta}{H_2^2} \]

\[ \therefore \quad M_{BA} = \frac{I_1H_2^2}{H_1^2}, \quad M_{CD} = \frac{I_1H_2^2}{I_1H_1^2}. \]

**Ex. 8.6.** Analyse the frame shown in Fig. 8.17 by moment distribution method.

![Frame diagram](image)

**Fig. 8.17**

**Solution.** Fixed end moments are

\[ M_{BD} = +3000 \times 2 = +6000 \text{ kg.m.} \]

\[ M_{BC} = \frac{wi^3}{12} = \frac{3000 \times 4 \times 4}{12} = -4000 \text{ kg.m.} \]

\[ M_{CB} = +\frac{wi^3}{12} = +\frac{3000 \times 4 \times 4}{12} = +4000 \text{ kg.m.} \]

Stiffness of portion BD is taken as zero as it is cantilever.

\[ r_{BC} = \frac{I_{BC}}{I_{BA} + I_{BC}} = \frac{3I}{4 + 3I} = \frac{3}{4} \]

\[ r_{BA} = 1 - r_{BC} = 1 - \frac{3}{4} = \frac{1}{4} \]
Process of moment distribution is shown in Table 8.9.

**TABLE 8.9**

<table>
<thead>
<tr>
<th>Joints</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>AB</td>
<td>BA</td>
<td>BD</td>
</tr>
<tr>
<td>DF</td>
<td>-</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>F E M</td>
<td>-</td>
<td>-</td>
<td>+6000</td>
</tr>
<tr>
<td>Balance B</td>
<td>-500</td>
<td>-</td>
<td>-1500</td>
</tr>
<tr>
<td>Carry over</td>
<td>-250</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Final moments</td>
<td>-250</td>
<td>-500</td>
<td>+6000</td>
</tr>
</tbody>
</table>

B.M. diagram is shown in Fig. 8.17 (b).

**Ex. 8.7.** Analyse the portal frame shown in Fig. 8.18 by moment distribution method.

![Portal Frame Diagram](image)

**Solution.** As the frame is symmetrical and symmetrically loaded, there will be no sway of the frame.
<table>
<thead>
<tr>
<th>Members</th>
<th>( AB )</th>
<th>( BA )</th>
<th>( BC )</th>
<th>( CB )</th>
<th>( CD )</th>
<th>( DC )</th>
</tr>
</thead>
<tbody>
<tr>
<td>D.F.</td>
<td></td>
<td>1/3</td>
<td>2/3</td>
<td>2/3</td>
<td>1/3</td>
<td></td>
</tr>
<tr>
<td>F.E.M.</td>
<td></td>
<td>+2000</td>
<td>-6000</td>
<td>+6000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td></td>
<td>+4000</td>
<td>-4000</td>
<td>-2000</td>
<td></td>
</tr>
<tr>
<td>C.O.</td>
<td>+1000</td>
<td></td>
<td>-606.67</td>
<td>+606.67</td>
<td>+1333.33</td>
<td>-1333.33</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td>+666.67</td>
<td>+1333.33</td>
<td>-1333.33</td>
<td>-666.67</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+333.33</td>
<td></td>
<td>-606.67</td>
<td>+666.67</td>
<td>+444.45</td>
<td>-444.45</td>
</tr>
<tr>
<td></td>
<td>-333.33</td>
<td>-222.22</td>
<td>+444.45</td>
<td>-222.22</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>+111.11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>74.08</td>
<td>-222.23</td>
<td>-222.23</td>
<td>-74.08</td>
<td>-148.15</td>
<td>-74.08</td>
</tr>
<tr>
<td></td>
<td>+148.15</td>
<td>+222.23</td>
<td>-148.15</td>
<td>+222.23</td>
<td>-74.08</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-37.04</td>
<td>+24.69</td>
<td>-49.39</td>
<td>-24.69</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>+49.39</td>
<td>+74.08</td>
<td>+74.08</td>
<td>+49.39</td>
<td>-24.69</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-12.35</td>
<td>+8.2</td>
<td>-24.7</td>
<td>+24.7</td>
<td>-16.5</td>
<td>-8.2</td>
</tr>
<tr>
<td></td>
<td>+16.5</td>
<td>+8.2</td>
<td>-24.7</td>
<td>+24.7</td>
<td>-16.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-4.1</td>
<td>+2.75</td>
<td>-5.25</td>
<td>+8.25</td>
<td>-5.50</td>
<td>-2.75</td>
</tr>
<tr>
<td></td>
<td>+5.50</td>
<td>+2.75</td>
<td>-5.25</td>
<td>+8.25</td>
<td>-5.50</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1.38</td>
<td>0.92</td>
<td>-2.75</td>
<td>+2.75</td>
<td>-1.83</td>
<td>-0.92</td>
</tr>
<tr>
<td></td>
<td>0.92</td>
<td>-2.75</td>
<td>+2.75</td>
<td>-1.83</td>
<td>-0.92</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+0.46</td>
<td>+0.3</td>
<td>-0.91</td>
<td>+0.91</td>
<td>+0.61</td>
<td>-0.3</td>
</tr>
<tr>
<td></td>
<td>+0.61</td>
<td>-0.91</td>
<td>+0.91</td>
<td>+0.61</td>
<td>-0.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+0.15</td>
<td>+0.1</td>
<td>-0.3</td>
<td>+0.3</td>
<td>+0.2</td>
<td>-0.1</td>
</tr>
<tr>
<td></td>
<td>+0.2</td>
<td>-0.3</td>
<td>+0.3</td>
<td>+0.2</td>
<td>-0.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.05</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-0.05</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final moments</td>
<td>+1499.97</td>
<td>+2999.93</td>
<td>-2999.93</td>
<td>+2999.93</td>
<td>-2999.93</td>
<td>-1499.97</td>
</tr>
</tbody>
</table>
Fixed end moments are

\[ \frac{2000 \times 6 \times 6}{12} - 6000 \text{ kg. m.} \]

\[ M_{CB} = \frac{w l^2}{12} + \frac{2000 \times 6 \times 6}{12} + 6000 \text{ kg. m.} \]

Distribution factors at \( B \) are

\[ \frac{I_{BC}}{l_{BC}^2 + I_{BA}^2} = \frac{3I}{6} \]

\[ r_{BC} = \frac{I_{BC}}{l_{BC}^2 + I_{BA}^2} = \frac{3I}{6} \]

Similarly

\[ r_{CB} = \frac{3}{8} \quad \text{and} \quad r_{CD} = \frac{1}{4} \]

Process of moment distribution is shown in Table 8.10.

B.M. diagram is shown in Fig. 8.18 (b)

Ex. 8.8. Analyse the frame shown in Fig. 8.19 by moment distribution method. The end \( A \) is hinged and end \( D \) is fixed.

Solution. Non-sway moments

\[ M_{CB} = - \frac{Wl}{8} = - \frac{4000 \times 4}{8} \]

\[ = -2000 \text{ kg. m.} \]

\[ M_{BC} = + \frac{Wl}{8} = + \frac{4000 \times 4}{8} \]

\[ = +2000 \text{ kg. m.} \]

\[ M_{DC} = - \frac{w l^2}{12} = - \frac{1000 \times 6 \times 6}{12} \]

\[ = -3000 \text{ kg. m.} \]

\[ M_{CD} = + \frac{w l^2}{12} = + \frac{1000 \times 6 \times 6}{12} \]

\[ = +3000 \text{ kg. m.} \]

Distribution factors at \( B \) are

\[ r_{BA} = \frac{3}{4} \frac{I_{BA}}{l_{BA}^2 + I_{BA}^2} \]

\[ = \frac{3}{4} \frac{I_{BA}}{l_{BA}^2 + I_{BA}^2} = \frac{3}{4} \]

\[ r_{BC} = 1 \]

\[ i = r_{BA} = 1 \rightarrow \frac{3}{4} = \frac{3}{4} \]
The distribution factors at C are

\[
\begin{align*}
 I_{CD} & \quad I_{BC} & \quad I \\
 l_{CD} & \quad l_{BC} & \quad 6 & \quad 6 + \frac{2I}{4} & \quad \frac{1}{4} \\
 r_{CD} = & \frac{I_{CD}}{l_{CD}} + \frac{I_{BC}}{l_{BC}} & 1 - \frac{1}{I} &= \frac{3}{4}
\end{align*}
\]

Moment distribution for fixed end moments due to external loading, i.e., non-sway moments, is shown in Table 8.11.

### Table 8.11

<table>
<thead>
<tr>
<th>Joints</th>
<th>D</th>
<th>C</th>
<th>B</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Members</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D.F.</td>
<td>--</td>
<td>1/4</td>
<td>3 4</td>
<td>2/3</td>
</tr>
<tr>
<td>F.E.M. Balancing</td>
<td>-3000</td>
<td>+3000</td>
<td>-2000</td>
<td>+2000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-250</td>
<td>-750</td>
<td>-1333</td>
</tr>
<tr>
<td>C.O. Balancing</td>
<td>-125</td>
<td>--</td>
<td>-667</td>
<td>-375</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+167</td>
<td>+500</td>
<td>+250</td>
</tr>
<tr>
<td>C.O. Balancing</td>
<td>+84</td>
<td>--</td>
<td>+125</td>
<td>+250</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-31</td>
<td>-94</td>
<td>-167</td>
</tr>
<tr>
<td>C.O. Balancing</td>
<td>-16</td>
<td>+21</td>
<td>-83</td>
<td>-47</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final moments</td>
<td>-0.047</td>
<td>+2906</td>
<td>-2906</td>
<td>+617</td>
</tr>
</tbody>
</table>

As the frame is unsymmetrical and unsymmetricaly loaded, there will be sway of the frame and hence sway correction is to be applied to the above moments.

Assume that the frame sways towards right. Let 8 be the

Assumed sway.
\[ M_{CD} = M_{DC} \]
\[
\frac{6E18}{6 \times 6} = \frac{\epsilon_1 \delta}{C}
\]
\[ M_{AB} = 0, \quad M_{BA} = -\frac{3E18}{3 \times 3} = -\frac{\epsilon_1 \delta}{3}
\]
\[ \frac{M_{CD}}{M_{BA}} = \frac{I}{2} = \frac{\frac{2E18}{E18}}{\frac{\epsilon_1 \delta}{E18}} \cdot \frac{1}{2}
\]

Assume \( M_{CD} = M_{DC} = -100 \text{ kg. m.} \)

\[ M_{BA} = -200 \text{ kg. m.} \]

Distribution of assumed sway fixed moments is shown in Table 8.12.

<table>
<thead>
<tr>
<th>Joints</th>
<th>D</th>
<th>C</th>
<th>B</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>CD</td>
<td>CB</td>
<td>BC</td>
<td>BA</td>
</tr>
<tr>
<td>DF</td>
<td>1/4</td>
<td>3/4</td>
<td>2/3</td>
<td>1/3</td>
</tr>
<tr>
<td>FLM Balance</td>
<td>-100</td>
<td>-100</td>
<td>+25</td>
<td>+75</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+12.5</td>
<td>-16.67</td>
<td>+66.67</td>
<td>+37.5</td>
</tr>
<tr>
<td>C.O Balance</td>
<td>-8.34</td>
<td>+3.12</td>
<td>-12.5</td>
<td>-25</td>
</tr>
<tr>
<td>C.O Balance</td>
<td>+1.56</td>
<td>-2.08</td>
<td>+8.34</td>
<td>+4.69</td>
</tr>
<tr>
<td>C.O Balance</td>
<td>-1.04</td>
<td>+0.39</td>
<td>-1.56</td>
<td>-3.12</td>
</tr>
<tr>
<td>C.O Balance</td>
<td>+0.2</td>
<td>-0.26</td>
<td>+1.05</td>
<td>+0.59</td>
</tr>
<tr>
<td>CO</td>
<td>-0.13</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final moments</td>
<td>-95.25</td>
<td>-90.51</td>
<td>+90.51</td>
<td>+138.22</td>
</tr>
</tbody>
</table>

Note: actual sway moments be \( x \) times the moments for assumed sway.

\[ M_{DC} = -5047 - 95.22x \]
\[ M_{CD} = 2005 - 90.51x \]
\[ M_{BA} = -617 - 138.22x \]
\[ H_D + H_A + 1000 \times 6 = 0 \]
\[ \frac{M_{DC} + M_{CD} - 1000 \times 6 \times 3}{6} + \frac{M_{BA}}{3} + 6000 = 0 \]
\[ \therefore \quad M_{DC} + M_{CD} - 18,000 + 2M_{BA} + 36,000 = 0 \]
\[ \therefore \quad M_{DC} + M_{CD} + 2M_{BA} + 18,000 = 0 \]
\[ \therefore \quad -3047 - 95.25x + 2906 - 90.51x + 2(-617 - 138.22x) + 18,000 = 0 \]
\[ \therefore \quad 462.20x = 16,625 \]
\[ \therefore \quad x = \frac{16,625}{462.20} = 35.96 \]

: Actual sway moments will be 35.96 times the assumed sway moments. The final moments are shown in Table 8.13.

**Table 8.13**

<table>
<thead>
<tr>
<th>Member</th>
<th>( DC )</th>
<th>( CD )</th>
<th>( CB )</th>
<th>( BC )</th>
<th>( BA )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-sway moments</td>
<td>-3047</td>
<td>+2900</td>
<td>-2900</td>
<td>+617</td>
<td>-617</td>
</tr>
<tr>
<td>Sway moments</td>
<td>-3425</td>
<td>-3255</td>
<td>+3255</td>
<td>+4971</td>
<td>-4971</td>
</tr>
<tr>
<td>Final moments</td>
<td>-6472</td>
<td>-349</td>
<td>+349</td>
<td>+5588</td>
<td>-5588</td>
</tr>
</tbody>
</table>

Ex. 8.9. Analyse the frame shown in Fig. 8.20 by moment distribution method.

Solution. Given loading is equivalent to moment of \( 4300 \times 0.5 = 2150 \) kg. m. at \( E \). Non-sway fixed end moments are

\[ \overline{M_{AB}} = \overline{M_{BA}} = \frac{M}{4} \]
\[ = 500 \text{ kg. m.} \]

Distribution factors at \( B \) are

Fig. 8.20

\[ r_{BA} = \frac{3}{4} \times \frac{1}{6} = \frac{1}{8} - \frac{2}{8} = \frac{1}{3} \]

\[ r_{BC} = 1 - r_{BA} = \frac{2}{3} \]

Distribution factors at \( C \) are

\[ r_{CA} = \frac{21}{8} + \frac{3}{4} \times \frac{1}{3} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \]

\[ r_{CD} = \frac{1}{2} \]
MOMENT DISTRIBUTION METHOD

Process of moment distribution for fixed end moments is shown in Table 8.14.

<table>
<thead>
<tr>
<th>Joints</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>AB</td>
<td>BA</td>
<td>BC</td>
<td>OB</td>
</tr>
<tr>
<td>1/3</td>
<td>2/3</td>
<td>1/2</td>
<td>1/2</td>
<td>-</td>
</tr>
<tr>
<td>1' M</td>
<td>+500</td>
<td>+500</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Release A</td>
<td>-500</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Carry over</td>
<td>-250</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Total moments</td>
<td>0</td>
<td>250</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Balance</td>
<td>-83.33</td>
<td>+41.67</td>
<td>+41.67</td>
<td>-</td>
</tr>
<tr>
<td>(O) Balance</td>
<td>-6.95</td>
<td>+20.64</td>
<td>+13.89</td>
<td>-</td>
</tr>
<tr>
<td>C (O) Balance</td>
<td>-6.04</td>
<td>+3.47</td>
<td>+3.47</td>
<td>-</td>
</tr>
<tr>
<td>(O) Balance</td>
<td>-0.53</td>
<td>+1.74</td>
<td>-1.16</td>
<td>-</td>
</tr>
<tr>
<td>C (O) Balance</td>
<td>-0.58</td>
<td>+0.29</td>
<td>+0.29</td>
<td>-</td>
</tr>
<tr>
<td>(O) Balance</td>
<td>-0.05</td>
<td>+0.15</td>
<td>-0.10</td>
<td>-</td>
</tr>
<tr>
<td>C (O) Balance</td>
<td>-0.05</td>
<td>+0.03</td>
<td>+0.02</td>
<td>-</td>
</tr>
<tr>
<td>Final moments</td>
<td>0</td>
<td>159.09</td>
<td>-159.09</td>
<td>-45.45</td>
</tr>
</tbody>
</table>

Assume that the frame sways towards right be δ.

Fixed end moments due to assumed sway are

\[
\begin{align*}
M_{AB} &= 0, \\
M_{BA} &= \frac{3EI \delta}{b}
\end{align*}
\]
\[ \frac{M_{DC}}{M_{CD}} = 0, \quad \frac{M_{CD}}{3EI} = 3^3 \]

\[ \frac{M_{BA}}{M_{CD}} = \frac{1}{4} \]

Assume \( M_{BA} = -100 \text{ kg m.} \)

\( M_{CD} = -400 \text{ kg m.} \)

Distribution of assumed sway moments is shown in Table 8.15.

**TABLE 8.15**

<table>
<thead>
<tr>
<th>Joints</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>AB</td>
<td>BA</td>
<td>BC</td>
<td>OB</td>
</tr>
<tr>
<td>D.F.</td>
<td>1/3</td>
<td>2/3</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>P.E.M. Balance</td>
<td>-100</td>
<td>+33.33</td>
<td>+66.67</td>
<td>+200</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>-33.33</td>
<td>+100</td>
<td>-66.67</td>
<td>-16.67</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+2.78</td>
<td>-8.34</td>
<td>-33.34</td>
<td>+16.67</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>-2.78</td>
<td>+8.34</td>
<td>+2.78</td>
<td>-1.39</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+0.23</td>
<td>-0.69</td>
<td>-2.78</td>
<td>+1.39</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>-0.23</td>
<td>+0.69</td>
<td>+0.23</td>
<td>-0.12</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+0.02</td>
<td>-0.06</td>
<td>-0.23</td>
<td>+0.12</td>
</tr>
<tr>
<td>Final moments</td>
<td>0</td>
<td>-100</td>
<td>+100</td>
<td>+200</td>
</tr>
</tbody>
</table>

Let actual sway moments be \( x \) times the moments for assumed sway.

\[ M_{BA} = +150.09 - 100x, \quad M_{CD} = +45.45 - 200x \]

\[ B_D + H_A = 0 \]

\[ \frac{M_{BA}}{5} + \frac{M_{CD}}{5} = 0 \]
MOMENT DISTRIBUTION METHOD

\[ M_{BA} + 2000 + 2M_{CD} = 0 \]
\[ 159.09 - 100x + 2000 + 90.90 - 400x = 0 \]
\[ 500x = 2249.99 \]
\[ x = 4.5 \]

Final moments are given in Table 8.16.

<table>
<thead>
<tr>
<th>Member</th>
<th>AB</th>
<th>BA</th>
<th>BC</th>
<th>CB</th>
<th>OD</th>
<th>DC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-sway moments</td>
<td>0</td>
<td>+159.09</td>
<td>-159.09</td>
<td>-45.45</td>
<td>+45.45</td>
<td>0</td>
</tr>
<tr>
<td>Sway moments</td>
<td>0</td>
<td>-450</td>
<td>+450</td>
<td>+900</td>
<td>-900</td>
<td>0</td>
</tr>
<tr>
<td>Final moments</td>
<td>0</td>
<td>-290.91</td>
<td>+290.91</td>
<td>+854.55</td>
<td>-854.55</td>
<td>0</td>
</tr>
</tbody>
</table>

Ex. 8.10. Analyse the frame shown in Fig. 8.21 by moment distribution method.

![Diagram of the frame](image)

Solution. Non-sway fixed end moments are

\[ M_{BE} = +4000 \times 1 = 4000 \text{ kg. m.} \]
\[ M_{CF} = -2000 \times 1 = -2000 \text{ kg. m.} \]
Distribution factors at $B$ are

$$r_{BC} = \frac{l_{BC}}{l_{BC}} + \frac{l_{RA}}{l_{BA}} = \frac{I}{4} + \frac{I}{4} = \frac{1}{2}$$

Similarly $r_{CA} = \frac{1}{3}$ and $r_{CA} = \frac{1}{2}$

Distribution of fixed end moments is shown in Table 8-17.

**Table 8-17**

<table>
<thead>
<tr>
<th>Joints</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>$AB$</td>
<td>$BA$</td>
<td>$BE$</td>
<td>$BJ$</td>
</tr>
<tr>
<td>D.F.</td>
<td>-</td>
<td>$1/2$</td>
<td>-</td>
<td>$1/2$</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>$-1000$</td>
<td>-250</td>
<td>$+500$</td>
<td>$-1000$</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>$-125$</td>
<td>-125</td>
<td>$+250$</td>
<td>$-125$</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>$-7.81$</td>
<td>-7.81</td>
<td>$+15.62$</td>
<td>$-7.82$</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>$-3.91$</td>
<td>-0.98</td>
<td>+1.98</td>
<td>$-3.91$</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>$-0.49$</td>
<td>$-0.49$</td>
<td>$+0.49$</td>
<td>$-0.49$</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>$-0.25$</td>
<td>$-0.06$</td>
<td>$+0.12$</td>
<td>$-0.24$</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>$-0.03$</td>
<td>$-0.03$</td>
<td>$+0.08$</td>
<td>$-0.03$</td>
</tr>
<tr>
<td>C.O.</td>
<td>$-0.01$</td>
<td>$+0.4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final moments</td>
<td>$-1200$</td>
<td>$-2400$</td>
<td>+4000</td>
<td>$-1600$</td>
</tr>
</tbody>
</table>
MOMENT DISTRIBUTION METHOD

Assume that the frame sways towards right.

\[
\begin{align*}
\bar{M}_{AB} &= \bar{M}_{BA} = -\frac{6EI\delta}{l^2} = -\frac{6EI\delta}{4^2} \\
\bar{M}_{DC} &= \bar{M}_{CD} = \frac{-6EI\delta}{l^2} = \frac{6EI\delta}{4^2} \\
\frac{\bar{M}_{AB}}{\bar{M}_{DC}} &= 1
\end{align*}
\]

Assume \( \bar{M}_{AB} = \bar{M}_{BA} = \bar{M}_{DC} = \bar{M}_{CD} = -100 \) kg. m.

Distribution of assumed fixed end moments due to sway is shown in Table 8.18.

<table>
<thead>
<tr>
<th>Members</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AB</td>
<td>BA</td>
<td>BE</td>
<td>BC</td>
</tr>
<tr>
<td>D.F.</td>
<td>-</td>
<td>1/2</td>
<td>-</td>
<td>1/2</td>
</tr>
<tr>
<td>F.E.M. Balance</td>
<td>-100</td>
<td>-100</td>
<td>+50</td>
<td>+50</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+25</td>
<td>-12.5</td>
<td>+25</td>
<td>+25</td>
</tr>
<tr>
<td>CO Balance</td>
<td>-0.25</td>
<td>+3.13</td>
<td>-6.25</td>
<td>+3.12</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+1.56</td>
<td>-0.78</td>
<td>+1.56</td>
<td>+1.56</td>
</tr>
<tr>
<td>CO Balance</td>
<td>-0.39</td>
<td>+0.20</td>
<td>-0.39</td>
<td>+0.19</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+0.10</td>
<td>-0.05</td>
<td>+0.10</td>
<td>+0.10</td>
</tr>
<tr>
<td>CO</td>
<td>-0.02</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final moments</td>
<td>-80</td>
<td>-60</td>
<td>+60</td>
<td>+60</td>
</tr>
</tbody>
</table>
Let actual sway moments be \( x \) times the moments for assumed sway

\[
M_{AB} = -1200 - 80 x, \quad M_{BA} = -2400 - 60 x \\
M_{DC} = +800 - 80 x, \quad M_{OD} = +1600 - 60 x
\]

\[
H_A + H_D = 0
\]

\[
\frac{M_{AB} + M_{BA}}{4} + \frac{M_{CD} + M_{DC}}{4} = 0
\]

\[
\therefore \quad M_{AB} + M_{BA} + M_{OD} + M_{DC} = 0
\]

\[
-1200 - 80 x - 2400 - 60 x + 800 - 80 x + 1600 - 60 x = 0
\]

\[
-1200 - 280 x = 0
\]

\[
x = -\frac{1200}{280} = -4.285
\]

Sway will take place to the left.

Actual moments are given in Table 8.19.

<table>
<thead>
<tr>
<th>Members</th>
<th>( AB )</th>
<th>( BA )</th>
<th>( BB )</th>
<th>( BC )</th>
<th>( CB )</th>
<th>( CF )</th>
<th>( CD )</th>
<th>( DC )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-sway</td>
<td>-1,200</td>
<td>-2400</td>
<td>+4000</td>
<td>-1600</td>
<td>+400</td>
<td>-2000</td>
<td>+1600</td>
<td>+800</td>
</tr>
<tr>
<td>Sway moments</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 342.8 )</td>
<td>+257.1</td>
<td></td>
<td>-257.1</td>
<td>-257.1</td>
<td></td>
<td>+257.1</td>
<td></td>
<td>+342.8</td>
</tr>
<tr>
<td>Total moments</td>
<td>-357.2</td>
<td>-2142.9</td>
<td>+400</td>
<td>-1857.1</td>
<td>+1422.6</td>
<td>-2000</td>
<td>+1857.1</td>
<td>+1142.8</td>
</tr>
</tbody>
</table>

B.M. diagram is shown in Fig. 8.21 (b).

Ex. 8.11. Analyse the frame shown in Fig. 8.22 for a rotational yield of 0.002 radian anticlockwise and vertical yield of 0.5 cm. at \( A \). \( EI = 3 \times 10^{10} \text{ kg. cm}^2 \).

Solution. Due to anticlockwise rotational yield at \( A \), fixed end moment at \( A \) will be \(-\frac{4EI\text{ }\theta}{l}\) and at \( B \) will be \(-\frac{2EI\text{ }\theta}{l}\)

\[
M_{AB} = \frac{4 \times 3 \times 10^{10}}{(100)^2} \times \frac{0.002}{4} = -6000 \text{ kg. m.}
\]

\[
M_{BA} = \frac{2EI\text{ }\theta}{l} = \frac{-2 \times 3 \times 10^{10}}{(100)^2} \times \frac{0.003}{4} = -3000 \text{ kg. m.}
\]
Due to vertical yield of \( \frac{1}{2} \) cm. at \( A \), \( B \) settles down by \( \frac{1}{2} \) cm. relative to \( C \). Fixed end moments will be

\[
\frac{M_{BC}}{M_{CB}} = \frac{6EI\delta}{l^4}
\]

\[
\frac{M_{BC}}{M_{CB}} = \frac{6 \times 1.5 \times 3 \times 10^{10}}{6 \times 6 \times (100)^3} \times \frac{0.5}{100} = 3750 \text{ kg. m.}
\]

![Diagram of the structure with labeled dimensions and an arrow indicating the yield at B.](image)

Fig. 8.22

Distribution factors at \( B \) are

\[
r_{BC} = \frac{I_{BC}}{l_{BC} + l_{BA}} = \frac{1.5I}{6} + \frac{I}{4} = \frac{1}{2}
\]

\[
r_{BA} = \frac{I}{1}. \text{ Also, } r_{OB} = \frac{1}{1} \text{ and } r_{OB} = \frac{1}{1}
\]

Distribution of non-awry moments is shown in Table 8.20.
### TABLE 8-20

<table>
<thead>
<tr>
<th>Joints</th>
<th>Members</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$AB$</td>
<td>$BA$</td>
<td>$BJ$</td>
<td>$CB$</td>
<td>$CD$</td>
</tr>
<tr>
<td>D.F.</td>
<td></td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>F.E.M.</td>
<td></td>
<td>$-6000$</td>
<td>$-3000$</td>
<td>$+3750$</td>
<td>$+3750$</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td>$-375$</td>
<td>$-375$</td>
<td>$-1875$</td>
<td>$-1875$</td>
</tr>
<tr>
<td>C.O.</td>
<td></td>
<td>$-187.5$</td>
<td>$+465.8$</td>
<td>$-237.5$</td>
<td>$+93.7$</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C.O.</td>
<td></td>
<td>$+234.4$</td>
<td>$-23.4$</td>
<td>$+46.9$</td>
<td>$+234.4$</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C.O.</td>
<td></td>
<td>$-11.7$</td>
<td>$+29.3$</td>
<td>$-58.6$</td>
<td>$-11.8$</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C.O.</td>
<td></td>
<td>$+14.7$</td>
<td>$-1.5$</td>
<td>$+3.0$</td>
<td>$+14.7$</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C.O.</td>
<td></td>
<td>$-0.8$</td>
<td>$+1.9$</td>
<td>$-3.7$</td>
<td>$-0.8$</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C.O.</td>
<td></td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>Final moments</td>
<td>$-5080$</td>
<td>$-2900$</td>
<td>$+2900$</td>
<td>$+1900$</td>
<td>$-1900$</td>
</tr>
</tbody>
</table>

As the rotation and displacement in the frame are unsymmetrical, there will be sway of the frame and hence sway correction is to be applied. Assumed that the frame sways towards right.

**Assumed sway moments are**

\[
\frac{M_{AB}}{M_{BA}} = \frac{M_{BA}}{M_{AB}} = \frac{6EI \delta}{l^4} = \frac{6EI \delta}{4^4} = -
\]

\[
\frac{M_{DC}}{M_{CD}} = \frac{\frac{-3^2 \cdot \gamma_{CD}}{l^4}}{6EI \delta} = \frac{-3 \gamma_{CD} \delta}{6EI 
\]

\[
\frac{M_{AB}}{M_{DC}} = 1
\]

**Assume**

\[
\frac{M_{AB}}{M_{BA}} = \frac{M_{CD}}{M_{CD}} = \frac{M_{CD}}{M_{CD}} = -1000 \text{ kg, m.}
\]
**Distribution of assumed sway moments is shown in Table 8.21.**

### TABLE 8.21

<table>
<thead>
<tr>
<th>Joints</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>AB</td>
<td>BA</td>
<td>BC</td>
<td>OB</td>
</tr>
<tr>
<td>DF</td>
<td>-</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>FEM Balance</td>
<td>-1000</td>
<td>-1000</td>
<td>+500</td>
<td>-500</td>
</tr>
<tr>
<td>CO Balance</td>
<td>+250</td>
<td>-125</td>
<td>-250</td>
<td>+250</td>
</tr>
<tr>
<td>CO Balance</td>
<td>-62.5</td>
<td>+31.3</td>
<td>-62.5</td>
<td>+31.2</td>
</tr>
<tr>
<td>CO Balance</td>
<td>+15.6</td>
<td>-7.8</td>
<td>+15.6</td>
<td>-7.8</td>
</tr>
<tr>
<td>CO Balance</td>
<td>-3.9</td>
<td>+2.0</td>
<td>-3.9</td>
<td>+1.9</td>
</tr>
<tr>
<td>CO Balance</td>
<td>+1.0</td>
<td>-0.5</td>
<td>+1.0</td>
<td>-0.5</td>
</tr>
<tr>
<td>CO</td>
<td>-0.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final moments</td>
<td>-800</td>
<td>-600</td>
<td>+600</td>
<td>+600</td>
</tr>
</tbody>
</table>

Let actual sway moments be \( x \) times the moments for assumed sway.

\[
M_{AB} = -5940 - 800 x, \quad M_{BA} = -2900 - 600 x
\]

\[
M_{DC} = -950 - 800 x, \quad M_{OD} = -1900 - 600 x
\]

\[
H_A + H_D = 0
\]

\[
\frac{M_{AB} + M_{BA}}{A} + \frac{M_{DC} + M_{OD}}{D} = 0
\]

\[
M_{AB} + M_{BA} + M_{DC} + M_{OD} = 0
\]

\[
-5940 - 800 x - 2900 - 600 x - 950 - 800 x - 1900 - 600 x = 0
\]

\[
2800 x = -11,690
\]

\[
x = \frac{-11,690}{2800} = -4.174
\]

(Sway will be to the left)
### Ex. 8.12

Analyse the frame shown in Fig. 8.23 (a) by moment distribution method.

**Solution.** There is no loading on spans and only loading at the joint B. Thus, the fixed end moments due to loading are zero. There will be sway of the frame and the moments will be only due to sway of the frame. The frame will sway towards right.

<table>
<thead>
<tr>
<th>Non-sway moments</th>
<th>AB</th>
<th>BA</th>
<th>DC</th>
<th>CD</th>
<th>DC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-2000</td>
<td>-2000</td>
<td>2000</td>
<td>1900</td>
<td>900</td>
</tr>
<tr>
<td>Sway moments</td>
<td>+339.2</td>
<td>+2504.4</td>
<td>-2504.4</td>
<td>-2504.4</td>
<td>+2504.4</td>
</tr>
<tr>
<td>Final moments</td>
<td>-2600.8</td>
<td>-885.6</td>
<td>+395.6</td>
<td>-604.4</td>
<td>+604.4</td>
</tr>
</tbody>
</table>

\[
M_{AB} = M_{BA} = -\frac{6EI\delta}{6^2} \\
M_{DC} = M_{CD} = -\frac{6EI\delta}{3^2} \\
\frac{M_{AB}}{M_{DC}} = \frac{1}{4}
\]

Assume \( M_{AB} = M_{BA} = -100 \text{ kg. m.} \)

\( M_{DC} = M_{CD} = -400 \text{ kg. m.} \)

Distribution factors at B are

\[
r_{BC} = \frac{I_{BC}}{I_{BC} + I_{BA}} = \frac{\frac{2l}{6}}{\frac{2l}{6} + \frac{l}{6}} = \frac{2}{3}
\]

\[
r_{BA} = 1 - r_{BC} = 1
\]

![Diagram of the frame](image_url)
The distribution of assumed sway moments is shown in Table 8.23.

<table>
<thead>
<tr>
<th>Points</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>AB</td>
<td>BA</td>
<td>BC</td>
<td>CB</td>
</tr>
<tr>
<td>Distribution factors</td>
<td>-</td>
<td>1/3</td>
<td>2/3</td>
<td>1/2</td>
</tr>
<tr>
<td>F . M. balance</td>
<td>-100</td>
<td>-100</td>
<td>+33.3</td>
<td>+66.7</td>
</tr>
<tr>
<td>O balance</td>
<td>+16.7</td>
<td>-33.3</td>
<td>+100</td>
<td>+33.4</td>
</tr>
<tr>
<td>O balance</td>
<td>-16.7</td>
<td>+2.8</td>
<td>-8.4</td>
<td>+16.7</td>
</tr>
<tr>
<td>O balance</td>
<td>+1.4</td>
<td>-2.8</td>
<td>+8.4</td>
<td>+2.8</td>
</tr>
<tr>
<td>O balance</td>
<td>-1.4</td>
<td>+0.2</td>
<td>-0.7</td>
<td>-2.8</td>
</tr>
<tr>
<td>C O balance</td>
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<td>-0.2</td>
<td>+0.7</td>
<td>+0.2</td>
</tr>
<tr>
<td>C O</td>
<td>-0.1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final moments</td>
<td>-100</td>
<td>-100</td>
<td>+100</td>
<td>+200</td>
</tr>
</tbody>
</table>

Let actual sway moments be \( z \) times the assumed moments:

\[ M_{AB} = -100z, \quad M_{BA} = -100z, \quad M_{CD} = -200z, \quad M_{DC} = -300z \]

\[ H_A^t + H_D + l' = 0 \]
\[
\frac{M_{AB} + M_{BA}}{6} + \frac{M_{DC} + M_{CD}}{3} + 1000 = 0
\]
\[-100x - 100x + -200x + 300x + 1000 = 0
\]
\[- \frac{100}{3} x - \frac{500}{3} x + 1000 = 0
\]
\[x = \frac{1000}{200} = 5\]

Final moments are given in Table 8.24.

<table>
<thead>
<tr>
<th>Members</th>
<th>AB</th>
<th>BA</th>
<th>BC</th>
<th>CB</th>
<th>OD</th>
<th>DC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Final moments</td>
<td>-500</td>
<td>-500</td>
<td>-500</td>
<td>+1000</td>
<td>-1000</td>
<td>-1500</td>
</tr>
<tr>
<td>= 5 x assumed sway moments</td>
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<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

B.M. diagram is shown Fig. 8.23 (b).

**Ex. 8.13.** Analyse the frame shown in Fig. 8.24 by moment distribution method.

**Solution.** Non-sway fixed end moments are

\[
\begin{align*}
\bar{M}'_{BC} &= - \frac{Wl}{8} = - \frac{5W}{8} \\
\bar{M}'_{CB} &= + \frac{Wl}{8} = + \frac{5W}{8}
\end{align*}
\]

**Distribution factors at B are**

\[
\begin{align*}
\tau_{BC} &= \frac{3}{4} \frac{I_{BO}}{l_{BO}} \\
\tau_{BO} &= \frac{3}{4} \frac{I_{BO}}{l_{BO}} + \frac{l_{E}}{l_{E}} \\
\tau_{DA} &= \frac{3}{4} \frac{2l}{5} + \frac{l}{5} \\
\tau_{DB} &= 1 - \tau_{BC} = 0.4
\end{align*}
\]

C is hinged hence moment at C will be zero.

**Distribution of fixed end moments is shown in Table 8.25.**
<table>
<thead>
<tr>
<th>Joints</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>AB</td>
<td>BA</td>
<td>BC</td>
<td>CB</td>
</tr>
<tr>
<td>D.F.</td>
<td>0.4</td>
<td>0.6</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Twin E.M.</td>
<td></td>
<td></td>
<td>$-\frac{5W}{8} + \frac{3W}{8}$</td>
<td></td>
</tr>
<tr>
<td>Release C</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CO</td>
<td></td>
<td></td>
<td>$\frac{5W}{16}$</td>
<td></td>
</tr>
<tr>
<td>Total moments</td>
<td>-</td>
<td>-</td>
<td>$-\frac{15W}{16}$</td>
<td>-</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td></td>
<td>$+ \frac{6W}{16}$</td>
<td>$+ \frac{9W}{16}$</td>
</tr>
<tr>
<td>CO.</td>
<td>$+ \frac{3W}{16}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final moments</td>
<td>$+ \frac{3W}{16}$</td>
<td>$+ \frac{6W}{16}$</td>
<td>$- \frac{6W}{16}$</td>
<td></td>
</tr>
</tbody>
</table>

As the frame is unsymmetrical, the frame will sway due to symmetrical loading. Assume that the frame sways towards right.

$$\bar{M}_{AB} = \bar{M}_{BA} = -\frac{3EI\delta}{5^2}$$

$$\bar{M}_{CD} = 0, \bar{M}_{DC} = -\frac{3EI\delta}{5^2}$$

$$\frac{\bar{M}_{BA}}{\bar{M}_{DC}} = \frac{2}{1}$$

Assume, $$\bar{M}_{AB} = \bar{M}_{BA} = -2W$$

$$\bar{M}_{DC} = -Wl = -5W.$$  

Distribution of sway moments is shown in Table 8.25.
## Table 8.25

<table>
<thead>
<tr>
<th>Joints</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>$AB$</td>
<td>$BA$</td>
<td>$BC$</td>
<td>$CB$</td>
</tr>
<tr>
<td>D.F.</td>
<td>$-$</td>
<td>$0.4$</td>
<td>$0.6$</td>
<td>$-$</td>
</tr>
<tr>
<td>F.E.M.</td>
<td>$-10W$</td>
<td>$-10W$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Balance</td>
<td>$+4W$</td>
<td>$+6W$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C.O.</td>
<td>$+2W$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final moments</td>
<td>$-8W$</td>
<td>$-6W$</td>
<td>$+6W$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

Let actual sway moments be $x$ times the moments for assumed sway.

\[
M_{AB} = \frac{3W}{16} - 8Wx, \quad M_{BA} = \frac{6W}{16} - 6Wx
\]

\[
M_{DC} = -5Wx
\]

\[
H_A + \Pi_D = 0
\]

\[
\frac{M_{AB} + M_{BA} + M_{DC}}{5} = 0
\]

\[
\frac{3W}{16} - 8Wx + \frac{6W}{16} - 6Wx - 5Wx = 0
\]

\[
\frac{9W}{16} = 19Wx
\]

\[
x = \frac{9}{16 \times 19} = \frac{9}{304}
\]

Final moments are tabulated in Table 8.26.

## Table 8.26

<table>
<thead>
<tr>
<th>Members</th>
<th>$AB$</th>
<th>$BA$</th>
<th>$BC$</th>
<th>$CB$</th>
<th>$CD$</th>
<th>$DC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-sway moments</td>
<td>$+\frac{3W}{16}$</td>
<td>$\frac{6W}{16}$</td>
<td>$-\frac{6W}{16}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>Sway moments</td>
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<td>$0$</td>
<td>$-\frac{45W}{304}$</td>
</tr>
<tr>
<td>Final moments</td>
<td>$-\frac{V}{14}$</td>
<td>$\frac{15W}{78}$</td>
<td>$-\frac{15W}{78}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{45W}{304}$</td>
</tr>
</tbody>
</table>
Ex. 8.14. Analyse the frame shown in Fig. 8.25 by moment distribution method.

Solution. Fixed end moments are

\[
\bar{M}_B = -1000 \times \frac{6 \times 6}{12} = -3600 \text{ kg} \cdot \text{m}.
\]

\[
\bar{M}_D = + \frac{1000 \times 6 \times 6}{12} = +3000 \text{ kg} \cdot \text{m}.
\]

Distribution factors at B are

\[
r_{BA} = \frac{I_{BA}}{I_{BA} + I_{BD}} = \frac{\frac{I}{6}}{\frac{I}{6} + \frac{2I}{6}} = \frac{1}{3}
\]

\[
r_{BD} = 1 - \frac{1}{3} = \frac{2}{3}.
\]

Distribution factors at D are

\[
r_{DB} = \frac{I_{DB}}{I_{DB} + I_{DF} + I_{DC}} = \frac{\frac{2I}{6}}{\frac{2I}{6} + \frac{I}{6} + \frac{I}{6}} = \frac{2}{5}.
\]

Similarly,

\[
r_{DF} = \frac{2}{5} \text{ and } r_{DC} = \frac{1}{5}
\]

Distribution factors at F are

\[
r_{FD} = \frac{2I}{6} \quad \frac{2}{3}
\]

\[
r_{FE} = \frac{1}{3}.
\]

Distribution of fixed end moments is shown in Table 8.27.
<table>
<thead>
<tr>
<th>Joints</th>
<th>A</th>
<th>B</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
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<tr>
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<td>DC</td>
<td>DF</td>
<td>FD</td>
<td>FE</td>
<td>EF</td>
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<td>Distribution factors</td>
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<td>2/5</td>
<td>1/5</td>
<td>2/5</td>
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<tr>
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<tr>
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<tr>
<td>C.O.</td>
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<td>+53.3</td>
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<tr>
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<td>-1923.2</td>
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<td>F</td>
<td>E</td>
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<td>+333.4</td>
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<td>+666.6</td>
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<td>-1000</td>
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<td>( BA )</td>
<td>( BD )</td>
<td>( DB )</td>
<td>( DC )</td>
<td>( DF )</td>
<td>( FD )</td>
<td>( FE )</td>
<td>( EF )</td>
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<tr>
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<td>----------</td>
<td>----------</td>
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<td>----------</td>
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<tr>
<td>Non-sway moments</td>
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<td>-1923.2</td>
<td>-616.1</td>
<td>+230.3</td>
<td>+384.8</td>
<td>+78.8</td>
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<td>Sway moments</td>
<td>-1447</td>
<td>-1158</td>
<td>+1158</td>
<td>+579</td>
<td>-1737</td>
<td>+1159</td>
<td>+2316</td>
<td>-2316</td>
<td>-2894</td>
</tr>
<tr>
<td>Final moments</td>
<td>-4985.5</td>
<td>+765.2</td>
<td>-765.2</td>
<td>-36.1</td>
<td>-1506.7</td>
<td>-1542.8</td>
<td>+2392.8</td>
<td>-2392.8</td>
<td>-2932.5</td>
</tr>
</tbody>
</table>
Assume that the frame sways towards right.

Sway moments will be

\[ M_{AB} = M_{BA} = \frac{6EI\delta}{6^2} \]
\[ M_{CD} = M_{DC} = \frac{6EI\delta}{6^2} \]
\[ M_{EF} = M_{FE} = \frac{6EI/2}{3^2} \]

\[ \therefore \frac{M_{AB}}{M_{CD}} : \frac{M_{EF}}{M_{CD}} = 1 : 1 : 2. \]

Assume \[ \frac{M_{AB}}{M_{BA}} = \frac{M_{CD}}{M_{DC}} = \frac{M_{EF}}{M_{FE}} = -1000 \text{ kg. m.} \]

\[ M_{EF} = M_{FE} = -2000 \text{ kg. m.} \]

Distribution of sway moments is shown in Table 8.28.

Let actual sway moments be \( x \) times the moments for assumed sway.

\[ M_{AB} = -3538.5 - 833.3x, \]
\[ M_{BA} = +1923.2 - 696.7x, \]
\[ M_{CD} = +115.2 - 1000x, \]
\[ M_{DC} = +230.3 - 1000x, \]
\[ M_{EF} = -38.5 - 1633.6x, \]
\[ M_{FE} = -76.8 - 1333.3x. \]

Sum of horizontal shears being zero gives,

\[ M_{AR} + M_{BA} - 1000 \times 6 \times 3 \]
\[ + \frac{M_{CD} + M_{DC}}{6} + \frac{M_{EF} + M_{FE}}{6} \]
\[ + 1000 \times 6 = 0 \]

\[ M_{AB} + M_{BA} - 18,000 + M_{CD} + M_{DC} + 2M_{EF} + 2M_{FE} + 36,000 = 0 \]

\[ \therefore -3538.5 - 833.3x + 1923.2 - 696.7x + 115.2 - 1000x + 230.3 \]
\[ - 1000x - 153.6 - 2666.6x - 77 - 3333.2x + 18,000 = 0 \]

\[ \therefore -9499.8x + 16499.6 = 0 \]

\[ \therefore x = 1.737. \]

Final moments are tabulated in Table 8.29.

8'6. Frames with sloping legs

In case of portal frames with sloping legs, both sloping members and horizontal members sway and hence sway correction is to be applied to all the members.

Consider frame \( ABCD \) having legs sloping at \( \theta_1 \) and \( \theta_2 \) degrees as shown in Fig. 8'26. The analysis of such a frame is carried out in two stages. Firstly the moment distribution is carried out without considering sway of the frame. Let the moments for members \( AB, BC, \ldots \) be \( m_{AB}, m_{BA}, m_{BC}, m_{CB} \) and so on. Next the frame is allowed to sway. Let \( B \) be displaced to \( B' \) by \( \Delta_1 \) and \( C \)
be displaced to \( C' \) by \( \Delta_1 \). Displacements \( BB' \) and \( CC' \) will be at right angles to members \( AB \) and \( CD \) respectively. The displacements \( \Delta_1 \) and \( \Delta_2 \) will be such that there is no change in the length of the members \( BC \). Therefore, projection of \( B'C' \) on \( BC \) will be equal to \( L \). The horizontal component of displacement \( BB' \) will be equal to horizontal component of displacement \( CC' \).

\[ BB'' = CC'' \]

\[ \Delta_1 \sin \theta_1 = \Delta_2 \sin \theta_2 \]

\[ \Delta_2 = \Delta_1 \frac{\sin \theta_1}{\sin \theta_2} \]

Displacement of \( C' \) with respect to \( B' \)

\[ = \Delta_1 \cos \theta_1 + \Delta_2 \cos \theta_2 \]

\[ = \Delta_1 \cos \theta_1 + \Delta_1 \frac{\sin \theta_1}{\sin \theta_2} \cos \theta_2 \]

\[ = \Delta_1 [\cos \theta_1 + \sin \theta_2 \cot \theta_2] \]
Some value of sway is assumed and fixed end moments due to this sway are calculated for various members. The moment distribution is carried out with these sway moments. Let the distributed moments be $m'_{AB}$, $m'_{BA}$ and so on. Let actual sway be $x$ times the assumed sway. The moments obtained above are to be multiplied by $x$ to get actual sway moments.

Final moments will be

$$M_{AB} = m_{AB} + x m'_{AB}$$
$$M_{BA} = m_{BA} + x m'_{BA}$$ and so on.

The values of horizontal shears $H_A$ and $H_D$ at $A$ and $D$ respectively are determined. The value of $x$ is found from the equation

$$H_A + H_D + P = 0,$$

where $P$ is the external horizontal force taken $+ve$ when acting towards right.

From the free body diagram of $BC$, vertical reactions at $B$ and $C$ are calculated. Next considering free body diagrams of $AB$ and $CD$, horizontal reactions at $A$ and $D$ are calculated.

**Ex. 8.15.** Analyse the frame shown in Fig. 8.27 (a) by moment distribution method.

**Solution.** Fixed end moments are

$$M_{BC} = -\frac{wl^3}{12} = -\frac{1200 \times 5 \times 5}{12} = -2500 \text{ kg m.}$$
$$M_{CB} = +\frac{1200 \times 5 \times 5}{12} = +2500 \text{ kg m.}$$

Distribution factors at $B$

$$r_{BA} = \frac{1}{3 \frac{25}{25} + \frac{21}{5}} = 0.435$$
$$r_{BC} = 1 - 0.435 = 0.565.$$
\[ r_{CB} = 1 - 0.5 = 0.5. \]
The distribution of fixed end moments is shown in Table 8.30.

### Table 8.30

<table>
<thead>
<tr>
<th>Joints</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>AB</td>
<td>BA</td>
<td>BU</td>
<td>CB</td>
</tr>
<tr>
<td>D.F.</td>
<td></td>
<td>0.435</td>
<td>0.565</td>
<td>0.5</td>
</tr>
<tr>
<td>F.E.M. Balance</td>
<td>+1087.5</td>
<td>-500</td>
<td>1412.5</td>
<td>+2600</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+543.8</td>
<td>+272</td>
<td>-353</td>
<td>+706.2</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+136</td>
<td>+76.8</td>
<td>-176.6</td>
<td>+176.5</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+38.4</td>
<td>+19.2</td>
<td>-44.2</td>
<td>+49.9</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+9.6</td>
<td>+5.4</td>
<td>-12.5</td>
<td>+12.5</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+2.7</td>
<td>+1.4</td>
<td>-3.2</td>
<td>+3.6</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+0.7</td>
<td>+0.4</td>
<td>-0.9</td>
<td>+0.9</td>
</tr>
<tr>
<td>C.O.</td>
<td>+0.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final moments</td>
<td>+731.4</td>
<td>+1462.7</td>
<td>-1462.7</td>
<td>+1724.7</td>
</tr>
</tbody>
</table>

Assume that the frame sways to wards right as shown in Fig. 8.27 (b.) Let \( \Delta_1 \) be the sway of member \( AB \) and \( \Delta_2 \) be the sway of member \( LC \).

\[
\Delta_1 \sin \theta_1 = \Delta_2 \sin \theta_2
\]

\[
\Delta_1 \times \frac{3}{25} = \Delta_2 \times \frac{4}{5}
\]

\[
\Delta_2 = \frac{3}{3.25} \times \frac{5}{4} \Delta_1
\]

\[
= \frac{1}{12} \Delta_1.
\]
Sway of member $BC = \Delta_1 \cos \theta_1 + \Delta_2 \cos \theta_2$

\[ = \Delta_1 \times \frac{125}{3.25} + \Delta_2 \times \frac{3}{5} \]

\[ = \frac{1}{13} \Delta_1 + \frac{15}{13} \Delta_1 \times \frac{3}{5} \]

\[ = \frac{14}{13} \Delta_1 \]

\[ M_{AB} : M_{BC} : M_{CD} = \frac{6EI}{6} \times \frac{14}{13} \Delta_1 : \frac{6EI}{5} \times \frac{15}{13} \Delta_1 \]

\[ = -4000 : 3640 : -3900. \]

Distribution of assumed sway moments is shown in Table 8.31.

Let actual sway moments be $x$ times the moments for assumed sway.

\[ M_{AB} = +731.4 - 3930.5x \]

\[ M_{RA} = +1412.7 - 3862x \]

\[ M_{BC} = -1462.7 + 3862x \]
### TABLE 6.31

<table>
<thead>
<tr>
<th>Joints</th>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>( AE )</td>
<td>( BA )</td>
<td>( BC )</td>
<td>( CB )</td>
</tr>
<tr>
<td>D.F.</td>
<td></td>
<td>0.435</td>
<td>0.565</td>
<td>0.5</td>
</tr>
<tr>
<td>F.E.M.</td>
<td>-4000</td>
<td>-4000</td>
<td>+3640</td>
<td>+3640</td>
</tr>
<tr>
<td>Balance</td>
<td>+156.6</td>
<td>+203.4</td>
<td>+130</td>
<td>+130</td>
</tr>
<tr>
<td>C.O.</td>
<td>+78.8</td>
<td>-28.3</td>
<td>+65</td>
<td>+101.7</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C.O.</td>
<td>-14.2</td>
<td>+11</td>
<td>-25.4</td>
<td>-18.4</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C.O.</td>
<td>+5.5</td>
<td>-2.1</td>
<td>+4.6</td>
<td>+7.2</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C.O.</td>
<td>-1.0</td>
<td>+0.9</td>
<td>-1.8</td>
<td>-1.8</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C.O.</td>
<td>+0.5</td>
<td>-0.1</td>
<td>+0.3</td>
<td>+0.4</td>
</tr>
<tr>
<td>Balance</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C.O.</td>
<td>-0.1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final moments</td>
<td>-3930.5</td>
<td>-3862</td>
<td>+3862</td>
<td>+3814.8</td>
</tr>
</tbody>
</table>

\[
M_{CB} = +1724.7 + 3814.8x
\]

\[
M_{CD} = -1724.7 - 3814.8x
\]

\[
M_{DC} = -862.3 - 3857.4x.
\]

Consider free body diagram of member \( BO \). Taking moment about \( C \),

\[
R_B \times 5 + M_{BC} + M_{CB} - 1200 \times 5 \times 5 = 0
\]

\[
R_B = 3000 \frac{M_{BC} + M_{CB}}{5}
\]

\[
R_C = 3000 + \frac{M_{BC} + M_{CB}}{5}
\]
Consider free body diagram of \( AB \). Taking moments about \( B \),

\[
M_{BA} + M_{AB} + R_B \times 1.25 - H_A \times 3 = 0
\]

\[
\therefore \quad H_A = \frac{M_{AB} + M_{BA}}{3} + \frac{1.25}{3} \left[ 3000 - \frac{M_{BC} + M_{CB}}{12} \right]
\]

\[
= \frac{M_{AB} + M_{BA}}{3} + 1250 \quad \frac{M_{BC} + M_{CB}}{12}
\]

Consider free body diagram of \( CD \). Taking moments about \( C \),

\[
M_{CD} + M_{DC} - R_D \times 3 - H_D \times 4 = 0
\]

\[
\therefore \quad H_D = \frac{M_{DC} + M_{CD}}{4} - \frac{3}{4} R_D
\]

\[
= \frac{M_{DC} + M_{CD}}{4} - \frac{3}{4} \left( 3000 + \frac{M_{BC} + M_{CB}}{5} \right)
\]

Sum of horizontal shears at base must be zero.

\[
\therefore \quad H_A + H_D = 0
\]

\[
\therefore \quad \frac{M_{AR} + M_{BA}}{3} + 1250 \quad \frac{M_{BC} + M_{CB}}{12} + \frac{M_{DC} + M_{CD}}{4} - \frac{2250}{3} \left( M_{BC} + M_{CB} \right) = 0.
\]

\[
\therefore \quad \frac{M_{AB} + M_{BA} - M_{CD} + M_{DC}}{3} - 1000 - \frac{7}{30} \left( M_{BC} + M_{CB} \right) = 0
\]

Substituting values of \( M_{AB}, M_{BA} \) etc.

\[
\begin{align*}
(731.4 - 3930.5x + 1462.7 + 3862x) + \frac{1}{4}(-1724.7 - 3814.8x - 862.3 - 3857.4x) + 1000 & \quad \frac{2}{3}(1462.7 - 3802x + 1724.7 + 3814.8x) = 0 \\
\end{align*}
\]

\[
\therefore \quad 731.37 - 2597.5x - 646.75 - 1918.65x - 1000 - 1791.24x - 61.11 = 0
\]

\[
x = \frac{976.49}{630.679} = 0.155. (Sway will be to the left)
\]

Final values of moments are tabulated in Table 8.32.

<table>
<thead>
<tr>
<th>Number</th>
<th>( AB )</th>
<th>( BA )</th>
<th>( BC )</th>
<th>( CB )</th>
<th>( CD )</th>
<th>( DC )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-sway moments</td>
<td>+731.4</td>
<td>+1462.7</td>
<td>-1462.7</td>
<td>+1724.7</td>
<td>-1724.7</td>
<td>-862.3</td>
</tr>
<tr>
<td>Sway moments</td>
<td>+609.6</td>
<td>+599</td>
<td>-599</td>
<td>-391.7</td>
<td>+591.7</td>
<td>+598.3</td>
</tr>
<tr>
<td>Total moments</td>
<td>+1841.0</td>
<td>+2661.7</td>
<td>-2061.7</td>
<td>+1183.0</td>
<td>-1183.0</td>
<td>-384.0</td>
</tr>
</tbody>
</table>
8.7. Multi-Storey Frames

The analysis of multi-storeyed frames by moment distribution is rather tedious. If the frame is unsymmetrical and unsymmetrically loaded, each storey will sway independently and hence sway correction is to be applied for each storey. Let the sway for first storey be $\Delta_1$, for second $\Delta_2$ and for third $\Delta_3$ as shown in Fig. 8·29.

First of all fixed end moments are calculated assuming that there is no sway in any storey and the process of moment distribution is carried out.

Then known displacement $\delta_1$ is given to first storey, no relative displacement being given to other two storeys. The process of moment distribution is carried out with assumed sway moments. If the actual displacement $\Delta_1 = x\delta_1$, the actual sway moments will be $x$ times the moments obtained by distributing assumed sway moments. Next a known displacement $\delta_2$ is given to second storey, no relative displacement being given to first and third storey. If the actual displacement or sway $\Delta_2 = y\delta_2$, the actual sway moments will be $y$ times the moments obtained by distributing assumed sway moments. Next displacement $\delta_3$ is given only to third storey and process of moment distribution carried out. Let actual sway moments be $'x'$ times the moments obtained by assumed sway $\delta_3$. 

---

The diagram in the text shows different storey configurations with sway corrections applied at each level, illustrating the sway calculation process in multi-storey frames.
Final moments for any member $AB$ will be

$$M_{AB} = m \cdot AB + x \cdot m'_{AB} + y \cdot m''_{AB}$$

where $m_{AB}$ is non-sway moment, $m'_{AB}$ is sway moment due to $\delta_1$, $m''_{AB}$ is the sway moment due to $\delta_2$ and $m''_{AB}$ is the sway moment due to $\delta_3$.

To get values of $x$, $y$ and $z$ total horizontal shear in each storey is made equal and opposite to the external horizontal force acting above the section and thus three simultaneous equations are obtained. These equations are solved for values of $x$, $y$ and $z$ and final moments calculated.

**Ex. 8-16.** Analyse the frame shown in Fig. 8-28 (a) by moment distribution method.

**Solution.** As the loading is only at the joints, there will be no fixed end moments in various members. Moments will be only due to sway.

Distribution factors at $C$ will be

$$\tau_{CD} = \frac{2I}{6}$$

$$\tau_{CB} = \frac{2I}{6} + \frac{1}{2}$$

Similarly at $D$

$$\tau_{DC} = \frac{1}{2}$$

$$\tau_{DE} = \frac{1}{2}$$

Similarly at $B$

$$\tau_{BE} = \frac{2I}{6} + \frac{1}{6} + \frac{1}{6}$$

$$\tau_{BA} = \frac{1}{2}$$

Similarly at $E$

$$\tau_{EB} = \frac{1}{2}$$

$$\tau_{ED} = \frac{1}{2}$$

and $\tau_{EF} = \frac{1}{2}$.

**Sway correction for top storey.** Let $\delta$ be assumed sway towards right.

$$M_{EC} = \frac{6EI\delta_1}{6^3}$$

$$M_{ED} = \frac{6EI\delta_1}{6^3}$$

$$M_{BC} = M_{DE} = 1:1$$

Assume $M_{BC} = M_{CD} = M_{DE} = M_{ED} = -1000 \text{ kg.m.}$

Distribution of assumed sway moments for top storey is shown in Table 8-33.
<table>
<thead>
<tr>
<th>Joints</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
<th>$E$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>$AB$</td>
<td>$BA$</td>
<td>$BE$</td>
<td>$BC$</td>
<td>$CB$</td>
<td>$CD$</td>
</tr>
<tr>
<td>D.F.</td>
<td>1/4</td>
<td>1/2</td>
<td>1/4</td>
<td>1/3</td>
<td>2/3</td>
<td>2/3</td>
</tr>
<tr>
<td>F.E.M. Balance</td>
<td>+250</td>
<td>+500</td>
<td>-1000</td>
<td>-1000</td>
<td>+125</td>
<td>+334</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>+104</td>
<td>-209</td>
<td>-104</td>
<td>-104</td>
<td>-104</td>
<td>-104</td>
</tr>
<tr>
<td>C.O.</td>
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<td>+90</td>
<td>+46</td>
<td>+46</td>
<td>+46</td>
<td>+46</td>
</tr>
<tr>
<td>Balance</td>
<td>+23</td>
<td>+45</td>
<td>+34</td>
<td>+34</td>
<td>+34</td>
<td>+34</td>
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<tr>
<td>C.O.</td>
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<td>+9</td>
<td>+9</td>
<td>+9</td>
</tr>
<tr>
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<td>+5</td>
<td>-8</td>
<td>+7</td>
<td>+7</td>
<td>+7</td>
<td>+7</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>-4</td>
<td>-4</td>
<td>-4</td>
<td>-4</td>
<td>-4</td>
<td>-4</td>
</tr>
<tr>
<td>C.O.</td>
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<td>-2</td>
<td>-3</td>
<td>-3</td>
<td>-3</td>
<td>-3</td>
</tr>
<tr>
<td>Balance</td>
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<td>+2</td>
<td>+2</td>
<td>+2</td>
<td>+2</td>
<td>+2</td>
</tr>
<tr>
<td>Final moments</td>
<td>+90</td>
<td>+178</td>
<td>+530</td>
<td>-708</td>
<td>-683</td>
<td>+683</td>
</tr>
<tr>
<td>Joints</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td>E</td>
<td>F</td>
</tr>
<tr>
<td>-------</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>Members A</td>
<td>AB</td>
<td>BA</td>
<td>BE</td>
<td>BC</td>
<td>CB</td>
<td>CD</td>
</tr>
<tr>
<td>D.F.</td>
<td>1/4</td>
<td>1/2</td>
<td>1/4</td>
<td>1/3</td>
<td>2/3</td>
<td>1/3</td>
</tr>
<tr>
<td>F.E.M. Balance</td>
<td>-1000</td>
<td>-1000</td>
<td>+100</td>
<td>+250</td>
<td>+125</td>
<td>+125</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>-63</td>
<td>-63</td>
<td>+125</td>
<td>-42</td>
<td>-83</td>
<td>-83</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>-63</td>
<td>-63</td>
<td>-42</td>
<td>-83</td>
<td>-42</td>
<td>-83</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>-63</td>
<td>-63</td>
<td>-42</td>
<td>-83</td>
<td>-42</td>
<td>-83</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>-63</td>
<td>-63</td>
<td>-42</td>
<td>-83</td>
<td>-42</td>
<td>-83</td>
</tr>
<tr>
<td>C.O. Balance</td>
<td>-63</td>
<td>-63</td>
<td>-42</td>
<td>-83</td>
<td>-42</td>
<td>-83</td>
</tr>
<tr>
<td>Final moments</td>
<td>-1000</td>
<td>125</td>
<td>+100</td>
<td>-250</td>
<td>+500</td>
<td>-1000</td>
</tr>
</tbody>
</table>

**MOMENT DISTRIBUTION METHOD**
Let the actual sway moments be \( x \) times the moments obtained for assumed sway in top storey.

**Sway correction for bottom storey.**

Let \( \delta \) be assumed sway towards right.

\[
M_{AB} - M_{BA} = -\frac{6EI\delta}{6z}
\]

\[
M_{EF} - M_{FE} = -\frac{6EI\delta}{6z}
\]

\[
\frac{M_{AB}}{M_{EF}} = \frac{M_{BA}}{M_{FE}} = 1 : 1.
\]

Assume \( M_{AB} = M_{BA} = M_{FE} = M_{EF} = -1000 \text{ kg m.} \)

Distribution of assumed sway moments for bottom storey is shown in Table 8.34.

Let actual sway moments be \( y \) times the moments obtained for assumed sway in lower storey.

Total moments in various members will be

\[
M_{AB} = 90x - 900y,
\]

\[
M_{BA} = 1^\circ 3x - 799y
\]

\[
M_{BO} = -708x + 190y,
\]

\[
M_{BE} = +530x + 60y
\]

\[
M_{CB} = -683x + 75y,
\]

\[
M_{CD} = +683x - 75y
\]

\[
M_{DC} = 683x - 75y,
\]

\[
M_{DE} = -683x + 75y
\]

\[
M_{ED} = -708x + 190y,
\]

\[
M_{EB} = 530x + 60y
\]

\[
M_{EF} = 178x - 799y,
\]

\[
M_{FE} = 90x - 900y
\]

Shear at the base of top storey must be zero

\[
H_B + H_R + P = 0
\]

\[
\frac{M_{BO} + M_{CB} + M_{DE} + M_{ED}}{6} + 1000 = 0
\]

\[
-708x + 190y - 630x + 75y - 683x + 75y - 708x + 190y = -600
\]

\[
-2752x + 530y = -600
\]

\[
-x + 0.1905y = -2.157
\]

Shear at the base of bottom storey must be zero

\[
H_A + H_F + P = 0
\]

\[
\frac{M_{AB} + M_{BA}}{6} + \frac{M_{FE} + M_{EF}}{6} + 1000 + 3000 = 0
\]

\[
M_{AB} + M_{BA} + M_{FE} + M_{EF} = -24,000
\]

\[
90x - 900y + 178x - 799y + 90x - 900y + 178x - 799y = -24,000
\]

\[
530x - 3398y = -24,000
\]

\[
x - 6339y = -4427
\]

...\(1\)

Eq. 1 + Eq. 2 gives, \( -61485y = -46927 \)

\[
y = +7.633
\]

\[
x = -44.77 + 6.339 \times 7.633
\]

\[
= -44.77 + 48.39 = +3.63
\]

Final moments are given in Table 8.35.
<table>
<thead>
<tr>
<th>Members</th>
<th>AB</th>
<th>BA</th>
<th>BE</th>
<th>BG</th>
<th>CB</th>
<th>CD</th>
<th>DC</th>
<th>DE</th>
<th>ED</th>
<th>EB</th>
<th>EF</th>
<th>FE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.63 x sway moments for top storey</td>
<td>+3 5.8</td>
<td>+645</td>
<td>+1918</td>
<td>-2563</td>
<td>-2473</td>
<td>+2473</td>
<td>+2473</td>
<td>-2473</td>
<td>-2563</td>
<td>+1918</td>
<td>+645</td>
<td>+3258</td>
</tr>
<tr>
<td>7.633 x sway moment for bottom storey</td>
<td>-6869.7</td>
<td>-6099</td>
<td>+4648</td>
<td>+1451</td>
<td>+572.5</td>
<td>-372.6</td>
<td>-372.5</td>
<td>+572.6</td>
<td>+1451</td>
<td>+4648</td>
<td>-6098</td>
<td>-6869.7</td>
</tr>
<tr>
<td>Final moments</td>
<td>-6543.9</td>
<td>-5454</td>
<td>+6566</td>
<td>-1112</td>
<td>-1900.5</td>
<td>+1900.5</td>
<td>+1900.5</td>
<td>-1900.5</td>
<td>-1112</td>
<td>+566</td>
<td>-5454</td>
<td>-6543.9</td>
</tr>
</tbody>
</table>
8.2. Box-type frames

These frames are used for box culverts and for tanks. These frames are continuous around. The top of the frame is subjected to load of earth and live load. Side walls are subjected to pressure of water and of earth pressure and base is subjected to pressure from soil. These frames are generally symmetrically loaded. These frames can be easily analysed by moment distribution method.

Ex. 8.17. Analyse the frame shown in Fig. 8.31 by moment distribution method. Take M.I. for BC and AD, 2I and AB and DC, I.

Solution. Fixed end moments are

\[ M_{BC} = - \frac{WL}{8} \times \frac{4000 \times 4}{8} = -2000 \text{ kg.m.} \]

\[ M_{CB} = \frac{WL}{8} \times \frac{4000 \times 4}{8} = +2000 \text{ kg.m.} \]

![Fig. 8.31](image)

\[ M_{AD} = + \frac{WL^2}{12} = + \frac{1000 \times 4 \times 4}{12} = +1333.3 \text{ kg.m.} \]

\[ M_{DA} = - \frac{1000 \times 4 \times 4}{12} = -1333.3 \text{ kg.m.} \]

\[ M_{AB} = - \frac{WL}{10} = - \frac{1}{2} \times 3600 \times \frac{2 \times 2}{10} = -720 \text{ kg.m.} \]

\[ M_{BA} = + \frac{WL}{15} = + \frac{1}{2} \times 3600 \times \frac{9 \times 2}{15} = +480 \text{ kg.m.} \]

Similarly,

\[ M_{DC} = +720 \text{ kg.m. and } M_{CD} = -480 \text{ kg.m.} \]

Distribution factors are

\[ r_{BC} = \frac{2I}{4} + \frac{L}{4} = \frac{1}{2} \]

\[ r_{BA} = \frac{1}{2} \]

As the frame is symmetrical, distribution factor for each member will be \( \frac{1}{2} \).

The process of moment distribution is shown in Table 8.36.
<table>
<thead>
<tr>
<th>Joints</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Members</td>
<td>AD</td>
<td>AB</td>
<td>BA</td>
<td>BC</td>
</tr>
<tr>
<td>D.F.</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>F.E.M.</td>
<td>+1333 33</td>
<td>-72 1</td>
<td>+48 0</td>
<td>-2000</td>
</tr>
<tr>
<td>Balance</td>
<td>-306 67</td>
<td>-306 66</td>
<td>+76 0</td>
<td>+76 0</td>
</tr>
<tr>
<td>C.O.</td>
<td>+153 33</td>
<td>+38 0</td>
<td>-153 33</td>
<td>-38 0</td>
</tr>
<tr>
<td>Balance</td>
<td>-266 66</td>
<td>-266 67</td>
<td>+266 67</td>
<td>+266 66</td>
</tr>
<tr>
<td>C.O.</td>
<td>+133 33</td>
<td>+133 33</td>
<td>+133 33</td>
<td>-133 33</td>
</tr>
<tr>
<td>Balance</td>
<td>-133 33</td>
<td>-133 33</td>
<td>+133 33</td>
<td>+133 33</td>
</tr>
<tr>
<td>Final moments</td>
<td>+913 33</td>
<td>-913 33</td>
<td>+1353 34</td>
<td>-1353 34</td>
</tr>
</tbody>
</table>
No further distribution is done after step 3 as it is seen that carry over and balancing moment at a joint is same.

8.9. Short-cut method of moment distribution

When frames and continuous beams are symmetrical about centre line short-cut method of moment distribution can be easily used and thus avoiding considerable labour.

Continuous Beams. When the continuous beam is symmetrical about centre line and symmetrically loaded, the beam can be analysed quickly as given below:

\[ \begin{align*}
\Phi \\
b & \quad c & \quad c & \quad b & \quad a \\
\end{align*} \]

\[ \begin{align*}
a & \quad b & \quad c & \quad D & \quad a \\
\end{align*} \]

Fig. 8.32 (a), (b).

\(a\) If the centre line of the beam is a support, the beam can be analysed by considering central support as fixed support and carrying out moment distribution for half of the beam. The equivalent beam is shown in Fig. 8.32 (b).

\(b\) If the centre line passes through the mid-section of the central span, the beam can be analysed by considering the stiffness of the central span as \(\frac{1}{3}\) of the stiffness of the memt and carrying out moment distribution for half of the beam. The equivalent beam is shown in Fig. 8.33 (b).