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The subject of this mathematical treatise is not pure mathematics but physics. The vocabulary of the physicist comprises a number of words such as length, time, energy, force, work, potential, current, etc., which we shall call briefly "physical quantities." Some of these terms occur in pure mathematics also; in that subject they may have a generalised meaning which does not concern us here. The pure mathematician deals with ideal quantities defined as having the properties which he deliberately assigns to them. But in an experimental science we have to discover properties not to assign them; and physical quantities are defined primarily according to the way in which we recognise them when confronted by them in our observation of the world around us.

Consider, for example, a length or distance between two points. It is a numerical quantity associated with the two points; and we all know the procedure followed in practice in assigning this numerical quantity to two points in nature. A definition of distance will be obtained by stating the exact procedure; that clearly must be the primary definition if we are to make sure of using the word in the sense familiar to everybody. The pure mathematician proceeds differently; he defines distance as an attribute of the two points, which obeys certain laws—the axioms of the geometry which he happens to have chosen—and he is not concerned with the question how this "distance" would exhibit itself in practical observation. So far as his own investigations are concerned, he takes care to use the word self-consistently; but it does not necessarily denote the thing which the rest of mankind are accustomed to recognise as the distance of the two points.

To find out any physical quantity we perform certain practical operations followed by calculations; the operations are called experiments or observations according as the conditions are more or less closely under our control. The physical quantity so discovered is primarily the result of the operations and calculations; it is, so to speak, a manufactured article—manufactured by our operations. But the physicist is not generally content to believe that the quantity he arrives at is something whose nature is inseparable from the kind of operations which led to it; he has an idea that if he could become a god contemplating the external world, he would see his manufactured physical quantity forming a distinct feature of the picture. By finding that he can lay a unit measuring-rod in a line between two points, he has manufactured the quantity which he calls the distance between the points; but he believes that that distance is something already existing in the picture of the world—a gulf which would be apprehended by a superior intelligence as existing in itself without reference to the notion of operations with measuring-rods.
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Yet he makes curious and apparently illogical discriminations. The parallax of a star is found by a well-known series of operations and calculations; the distance across the room is found by operations with a tape-measure. Both parallax and distance are quantities manufactured by our operations; but for some reason we do not expect parallax to appear as a distinct element in the true picture of nature in the same way that distance does. Or again, instead of cutting short the astronomical calculations when we reach the parallax, we might go on to take the cube of the result, and so obtain another manufactured quantity, a "cubic parallax." For some obscure reason we expect to see distance appearing plainly as a gulf in the true world-picture; parallax does not appear directly, though it can be exhibited as an angle by a comparatively simple construction; and cubic parallax is not in the picture at all. The physicist would say that he finds a length, and manufactures a cubic parallax; but it is only because he has inherited a preconceived theory of the world that he makes the distinction. We shall venture to challenge this distinction.

Distance, parallax and cubic parallax have the same kind of potential existence even when the operations of measurement are not actually made—if you will move sideways you will be able to determine the angular shift, if you will lay measuring-rods in a line to the object you will be able to count their number. Any one of the three is an indication to us of some existent condition or relation in the world outside us—a condition not created by our operations. But there seems no reason to conclude that this world-condition resembles distance any more closely than it resembles parallax or cubic parallax. Indeed any notion of "resemblance" between physical quantities and the world-conditions underlying them seems to be inappropriate. If the length $AB$ is double the length $CD$, the parallax of $B$ from $A$ is half the parallax of $E$ from $U$; there is undoubtedly some world-relation which is different for $AB$ and $CD$, but there is no reason to regard the world-relation of $AB$ as being better represented by double than by half the world-relation of $CD$.

The connection of manufactured physical quantities with the existent world-condition can be expressed by saying that the physical quantities are measure-numbers of the world-condition. Measure-numbers may be assigned according to any code, the only requirement being that the same measure-number always indicates the same world-condition and that different world-conditions receive different measure-numbers. Two or more physical quantities may thus be measure-numbers of the same world-condition, but in different codes, e.g. parallax and distance; mass and energy; stellar magnitude and luminosity. The constant formulae connecting these pairs of physical quantities give the relation between the respective codes. But in admitting that physical quantities can be used as measure-numbers of world-conditions existing independently of our operations, we do not alter their status as manufactured quantities. The same series of operations will naturally manufacture the
same result when world-conditions are the same, and different results when 
they are different. (Differences of world-conditions which do not influence 
the results of experiment and observation are ipso facto excluded from the 
domain of physical knowledge.) The size to which a crystal grows may be a 
measure-number of the temperature of the mother-liquor; but it is none the 
less a manufactured size, and we do not conclude that the true nature of size 
is caloric.

The study of physical quantities, although they are the results of our 
own operations (actual or potential), gives us some kind of knowledge of the 
world-conditions, since the same operations will give different results in 
different world-conditions. It seems that this indirect knowledge is all that 
we can ever attain, and that it is only through its influences on such opera-
tions that we can represent to ourselves a "condition of the world." Any 
attempt to describe a condition of the world otherwise is either mathematical 
symbolism or meaningless jargon. To grasp a condition of the world as 
completely as it is in our power to grasp it, we must have in our minds a 
symbol which comprehends at the same time its influence on the results of 
all possible kinds of operations. Or, what comes to the same thing, we must 
contemplate its measures according to all possible measure-codes—of course, 
without confusing the different codes. It might well seem impossible to 
realise so comprehensive an outlook; but we shall find that the mathematical 
calculus of tensors does represent and deal with world-conditions precisely in 
this way. A tensor expresses simultaneously the whole group of measure-
numbers associated with any world-condition; and machinery is provided for 
keeping the various codes distinct. For this reason the somewhat difficult 
tensor calculus is not to be regarded as an evil necessity in this subject, which 
ought if possible to be replaced by simpler analytical devices; our knowledge 
of conditions in the external world, as it comes to us through observation and 
experiment, is precisely of the kind which can be expressed by a tensor and 
not otherwise. And, just as in arithmetic we can deal freely with a billion 
objects without trying to visualise the enormous collection; so the tensor 
calculus enables us to deal with the world-condition in the totality of its 
aspects without attempting to picture it.

Having regard to this distinction between physical quantities and world-
conditions, we shall not define a physical quantity as though it were a feature 
in the world-picture which had to be sought out. A physical quantity is 
defined by the series of operations and calculations of which it is the result. 
The tendency to this kind of definition had progressed far even in pre-relativity 
physics. Force had become "mass x acceleration," and was no longer an in-
visible agent in the world-picture, at least so far as its definition was concerned. 
Mass is defined by experiments on inertial properties, no longer as "quantity 
of matter." But for some terms the older kind of definition (or lack of 
definition) has been obstinately adhered to; and for these the relativity
theory must find new definitions. In most cases there is no great difficulty in framing them. We do not need to ask the physicist what conception he attaches to "length"; we watch him measuring length, and frame our definition according to the operations he performs. There may sometimes be cases in which theory outruns experiment and requires us to decide between two definitions, either of which would be consistent with present experimental practice; but usually we can foresee which of them corresponds to the ideal which the experimentalist has set before himself. For example, until recently the practical man never confronted with problems of non-Euclidean space, and it might be suggested that he would be uncertain how to "construct a straight line when so confronted; but as a matter of fact he showed no hesitation, and the eclipse observers measured without ambiguity the bending of light from the "straight line." The appropriate practical definition was so obvious that there was never any danger of different people meaning different leis by this term. Our guiding rule will be that a physical quantity must be defined by prescribing operations and calculations which will lead to an unambiguous result, and that due heed must be paid to existing practice; the last clause should secure that everyone uses the term to denote the same quantity, however much disagreement there may be as to the conception attached to it.

When defined in this way, there can be no question as to whether the operations give us the real physical quantity or whether some theoretical correction (not mentioned in the definition) is needed. The physical quantity is the measure-number of a world-condition in some code; we cannot assert that a code is right or wrong, or that a measure-number is real or unreal; what we require is that the code should be the accepted code, and the measure-number the number in current use. For example, what is the real difference of time between two events at distant places? The operation of determining time has been entrusted to astronomers, who (perhaps for mistaken reasons) have elaborated a regular procedure. If the times of the two events are found in accordance with this procedure, the difference must be the real difference of time; the phrase has no other meaning. But there is a certain generalisation to be noticed. In cataloguing the operations of the astronomers, so as to obtain a definition of time, we remark that one condition is adhered to in practice evidently from necessity and not from design—the observer and his apparatus are placed on the earth and move with the earth. This condition is so accidental and parochial that we are reluctant to insist on it in our definition of time; yet it so happens that the motion of the apparatus makes an important difference in the measurement, and without this restriction the operations lead to no definite result and cannot define anything. We adopt what seems to be the commonsense solution of the difficulty. We decide that time is relative to an observer, that is to say, we admit that an observer on another star, who carries out all the rest of the operations and calculations
as specified in our definition, is also measuring time—not our time, but a
time relative to himself. The same relativity affects the great majority of
elementary physical quantities; the description of the operations is insuf-
ficient to lead to a unique answer unless we arbitrarily prescribe a particular
motion of the observer and his apparatus.

In this example we have had a typical illustration of "relativity," the
recognition of which has had far-reaching results revolutionising the outlook
of physics. Any operation of measurement involves a comparison between
a measuring-appliance and the thing measured. Both play an equal part in
the comparison and are theoretically, and indeed often practically, inter-
changeable; for example, the result of an observation with the meridian circle
gives the right ascension of the star or the error of the clock indifferently,
and we can regard either the clock or the star as the instrument or, the
object of measurement. Remembering that physical quantities are results of
comparisons of this kind, it is clear that they cannot be considered to belong
solely to one partner in the comparison. It is true that we standardise the
measuring appliance as far as possible (the method of standardisation being
explained or implied in the definition of the physical quantity) so that in
general the variability of the measurement can only indicate a variability of
the object measured. To that extent there is no great practical harm in
regarding the measurement as belonging solely to the second partner in
the relation. But even so we have often puzzled ourselves needlessly over
paradoxes, which disappear when we realise that the physical quantities are
not properties of certain external objects but are relations between these
objects and something else. Moreover, we have seen that the standardisation
of the measuring-appliance is usually left incomplete as regards the specifica-
tion of its motion; and rather than complete it in a way which would be
arbitrary and pernicious, we prefer to recognise explicitly that our physical
quantities belong not solely to the objects measured but have reference also
to the particular frame of motion that we choose.

The principle of relativity goes still further. Even if the measuring-
appliances were standardised completely, the physical quantities would still
involve the properties of the constant standard. We have seen that the
world-condition or object which is surveyed can only be apprehended in our
knowledge as the sum total of all the measurements in which it can be
concerned; any intrinsic property of the object must appear as a uniformity
of law in these measurements. When one partner in the comparison is fixed and
the other partner varies widely, whatever is common to all the measurements
may be ascribed exclusively to the first partner and regarded as an intrinsic
property of it. Let us apply this to the converse comparison; that is to say,
keep the measuring-appliance constant or standardised, and vary as widely
as possible the objects measured—or, in simpler terms, make a particular

* The most important exceptions are number (of discrete entities), action, and entropy.
kind of measurement in all parts of the field. Intrinsic properties of the measuring-appliance should appear as uniformities or laws in these measures. We are familiar with several such uniformities; but we have not generally recognised them as properties of the measuring-appliance. We have called them laws of nature.

The development of physics is progressive, and as the theories of the external world become crystallised, we often tend to replace the elementary physical quantities defined through operations of measurement by theoretical quantities believed to have a more fundamental significance in the external world. Thus the *vis viva* $mv^2$, which is immediately determinable by experiment, becomes replaced by a generalised energy, virtually defined by having the property of conservation; and our problem becomes inverted—we have not to discover the properties of a thing which we have recognised in nature, but to discover how to recognise in nature a thing whose properties we have assigned. This development seems to be inevitable; but it has grave drawbacks especially when theories have to be reconstructed. Fuller knowledge may show that there is nothing in nature having precisely the properties assigned; or it may turn out that the thing having these properties has entirely lost its importance when the new theoretical standpoint is adopted.

When we decide to throw the older theories into the melting-pot and make a clean start, it is best to relegate to the background terminology associated with special hypotheses of physics. Physical quantities defined by operations of measurement are independent of theory, and form the proper starting-point for any new theoretical development.

*Now that we have explained how physical quantities are to be defined, the reader may be surprised that we do not proceed to give the definitions of the leading physical quantities. But to catalogue all the precautions and provisos in the operation of determining even so simple a thing as length, is a task which we shirk. We might take refuge in the statement that the task though laborious is straightforward, and that the practical physicist knows the whole procedure without our writing it down for him. But it is better to be more cautious. I should be puzzled to say off-hand what is the series of operations and calculations involved in measuring a length of $10^{-10}$ cm.; nevertheless I shall refer to such a length when necessary as though it were a quantity of which the definition is obvious. We cannot be forever examining out foundations; we look particularly to those places where it is reported to us that they are insecure. I may be laying myself open to the charge that I am doing the very thing I criticise in the older physics—using terms that*

*We shall see in § 59 that this has happened in the case of energy. The dead hand of a superseded theory continues to embarrass us, because in this case the recognised terminology still has implicit reference to it. This, however, is only a slight drawback to set off against the many advantages obtained from the classical generalisation of energy as a step towards the more complete theory.*
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have no definite observational meaning, and mingling with my physical quantities things which are not the results of any conceivable experimental operation. I would reply—

By all means explore this criticism if you regard it as a promising field of inquiry. I here assume that you will probably find me a justification for my 10\textsuperscript{th} cnp.; but you may find that there is an insurmountable ambiguity in defining it. In the latter event you may be on the track of something which will give a new insight into the fundamental nature of the world. Indeed it has been suspected that the perplexities of quantum phenomena may arise from the tacit assumption that the notions of length and duration, acquired primarily from experiences in which the average effects of large numbers of quanta are involved, are applicable in the study of individual quanta. There may need to be much more excavation before we have brought to light all that is of value in this critical consideration of experimental knowledge. Meanwhile I want to set before you the treasure which has already been unearthed in this field.