DIFFERENTIAL CALCULUS:

CHAPTER I.

LIMITING VALUES. ELEMENTARY UNDETERMINED FORMS.

1. Object of the Differential Calculus. When an increasing or decreasing quantity is made the subject of mathematical treatment, it often becomes necessary to estimate its rate of growth. It is our principal object to describe the method to be employed and to exhibit applications of the processes described.

2. Explanation of Terms. The frequently recurring terms "Constant," "Variable," "Function," will be understood from the following example:

Let the student imagine a triangle of which two sides $x, y$ are unknown but of which the angle ($\Delta$) included between those sides is known. The area ($\Delta$) is expressed by

$$\Delta = \frac{1}{2} xy \sin \Delta.$$ 

The quantity $\Delta$ is a "constant" for by hypothesis it retains the same value, though the sides $x$ and $y$ may change in length while the triangle is under observation. The quantities $x, y$ and $\Delta$ are therefore called variables. $\Delta$, whose value depends upon those of $x$ and $y$, is called the dependent variable; $x$ and $y$, whose values may be any whatever, and may either or both take up any values which may be assigned to them, are called independent variables.

The quantity $\Delta$ whose value thus depends upon those of $x, y$ and $\Delta$ is said to be a function of $x, y$ and $\Delta$. 

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3. **Definitions.** We are thus led to the following definitions:

(a) A **Constant** is a quantity which, during any set of mathematical operations, retains the same value.

(b) A **Variable** is a quantity which, during any set of mathematical operations, does not retain the same value but is capable of assuming different values.

(c) An **Independent Variable** is one which may take up any arbitrary value that may be assigned to it.

(d) A **Dependent Variable** is one which assumes its value in consequence of some second variable or system of variables taking up any set of arbitrary values that may be assigned to them.

(e) When one quantity depends upon another or upon a system of others in such a manner as to assume a definite value when a system of definite values is given to the others it is called a **Function** of those others.

4. **Notation.** The usual notation to express that one variable is a function of another is

\[ y = f(x) \text{ or } y = F(x) \text{ or } y = \phi(x). \]

Occasionally the brackets are dispensed with when no confusion can thereby arise. Thus \( f(x) \) may be sometimes written for \( f(x) \). If \( u \) be an unknown function of several variables \( x, y, z \), we may express the fact by the equation \( u = f(x, y, z) \).

5. It has become conventional to use the letters \( a, b, c \ldots \alpha, \beta, \gamma \ldots \) from the beginning of the alphabet to denote constants and to retain later letters, such as \( u, v, w, x, y, z \) and the Greek letters \( \xi, \eta, \zeta \) for variables.

6. **Limiting Values.** The following illustrations will explain the meaning of the term "**Limiting Value**":
LIMITING VALUES.

(1) We say \( \epsilon = \frac{2}{3} \), by which we mean that by taking enough sixes we can make \( 0.666... \) differ by as little as we please from \( \frac{2}{3} \).

(2) The limit of \( \frac{2x + 3}{x + 1} \) when \( x \) is indefinitely diminished is 3.

For the difference between \( \frac{2x + 3}{x + 1} \) and 3 is \( \frac{x}{x + 1} \), and by diminishing \( x \) indefinitely this difference can be made less than any assignable quantity however small.

The expression can also be written \( \frac{2 + \frac{3}{x}}{1 + \frac{1}{x}} \), which shows that if \( x \) be increased indefinitely it can be made to continually approach and to differ by less than any assignable quantity from 2, which is therefore its limit in that case.

It is useful to adopt the notation \( \lim_{x \to a} \) to denote the words "the Limit when \( x = a \) of."

Thus \( \lim_{x \to 0} \frac{2x + 3}{x + 1} = 3 \); \( \lim_{x \to -\infty} \frac{2x + 3}{x + 1} = 2 \).

(3) If an equilateral polygon be inscribed in any closed curve and the sides of the polygon be decreased indefinitely, and at the same time their number be increased indefinitely, the polygon continually approximates to the form of the curve, and ultimately differs from it in area by less than any assignable magnitude, and the curve is said to be the limit of the polygon inscribed in it.

7. We thus arrive at the following general definition:

The LIMIT of a function for an assigned value of the independent variable is that value from which the function may be made to differ by less than any assignable quantity however small by making the independent variable approach sufficiently near its assigned value.

8. Undetermined forms. When a function involves the independent variable in such a manner that for a certain assigned value of that variable its value cannot be found by simply substituting that value of the variable, the function is said to take an undetermined form.
Differential Calculus.

One of the commonest cases occurring is that of a fraction whose numerator and denominator both vanish for the value of the variable referred to.

Let the student imagine a triangle whose sides are made of a material capable of shrinking indefinitely till they are smaller than any conceivable quantity. To fix the ideas suppose it to be originally a triangle whose sides are 3, 4 and 5 inches long, and suppose that the shrinkage is uniform. As the shrinkage proceeds the sides retain the same mutual ratio and may at any instant be written 3m, 4m, 5m and the angles remain unaltered. It thus appears that though each of these sides is ultimately immeasurably small, and to all practical purposes zero, they still retain the same mutual ratio 3 : 4 : 5 which they had before the shrinkage began.

These considerations should convince the student that the ultimate ratio of two vanishing quantities is not necessarily zero or unity.

9. Consider the fraction \( \frac{x^2 - a^2}{x - a} \); what is its value when \( x = a \)? Both numerator and denominator vanish when \( x \) is put \( = a \). But it would be incorrect to assume that the fraction therefore takes the value unity. It is equally incorrect to suppose the value to be zero for the reason that its numerator is evanescent; or that it is infinite since its denominator is evanescent, as the beginner is often fallaciously led to believe. If we wish to evaluate this expression we must never put \( x \) actually equal to \( a \). We may however put \( x = a + h \) where \( h \) is anything other than zero.

Thus
\[
\frac{x^2 - a^2}{x - a} = 2a + h,
\]

and it is now apparent that by making \( h \) indefinitely small (so that the value of \( x \) is made to approach indefinitely closely to its assigned value \( a \)) we may make the expression differ from \( 2a \) by less than any assignable
quantity. Therefore \( 2a \) is the limiting value of the given fraction.

10. Two functions of the same independent variable are said to be **ultimately equal** when as the independent variable approaches indefinitely near its assigned value the **limit of their ratio** is unity.

Thus \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \) by trigonometry,

and therefore when an angle is indefinitely diminished its sine and its circular measure are ultimately equal.

**EXAMPLES.**

1. Find the limit when \( x = 0 \) of \( \frac{y}{x^2} \),

   (a) when \( y = mx \),
   (b) when \( y = x^2/a \),
   (c) when \( y = ax^2 + b \).

2. Find \( \lim_{b \to a} \frac{ax + b}{bx + a} \), (i) when \( x = 0 \), (ii) when \( x = \infty \).

3. Find \( \lim_{x \to a} \frac{x^3 - \alpha^3}{x^2 - \alpha^2} \) \( \lim_{x \to a} \frac{x^5 - \alpha^5}{x^4 - \alpha^4} \).

4. Find the limit of \( \frac{cx + d}{x} \), (i) when \( x = 0 \), (ii) when \( x = \infty \).

5. Find \( \lim_{x \to 1} \frac{3x^2 - 4x + 1}{x^2 - 4x + 3} \).

6. The opposite angles of a cyclic quadrilateral are supplementary. What does this proposition become in the limit when two angular points coincide?

7. Evaluate the fraction \( \frac{x^3 - 6x^2 + 11x - 6}{x^3 - 6x^2 + 11x - 6} \) for the values \( x = \infty, 3, 2, 1, \frac{1}{2}, \frac{1}{3}, 0, -\infty \).

8. Evaluate \( \lim_{x \to 1} \frac{\sqrt{x^2 - 1}}{x - 1} \) and \( \lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x} \).
11. **Four Important Limits.** The following limits are important:

(I) \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \); \( \lim_{\theta \to 0} \cos \theta = \theta \),

(II) \( \lim_{x \to 1} \frac{x^n - 1}{x - 1} = n \),

(III) \( \lim_{x \to e} (1 + \frac{1}{x})^x = e \), where \( e \) is the base of the Napierian logarithms,

(IV) \( \lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a. \)

12. (I) The limits (I) can be found in any standard textbook on Plane Trigonometry.

13. (II) To prove \( \lim_{x \to 1} \frac{x^n - 1}{x - 1} = n \). Let \( x = 1 + z \). Then when \( x \) approaches unity \( z \) approaches zero. Hence we can consider \( z \) to be less than 1, and we may therefore apply the Binomial to the expansion of \( (1 + z)^n \) whatever \( n \) may be.

Thus \( \lim_{x \to 1} \frac{x^n - 1}{x - 1} = \lim_{z \to 0} \frac{(1 + z)^n - 1}{z} \).

\[
= \lim_{z \to 0} n z + \frac{n(n - 1)}{1 \cdot 2} z^2 + \ldots \\
= n + \frac{n(n - 1)}{1 \cdot 2} z + \ldots \]

14. (III) To prove \( \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e \).

Let \( y = \left(1 + \frac{1}{x}\right)^x \),

then \( \log_e y = x \log_e \left(1 + \frac{1}{x}\right) \).
FOUR IMPORTANT LIMITS.

Now $x$ is about to become infinitely large, and therefore $\frac{1}{x}$ may be throughout regarded as less than unity, and we may expand by the Logarithmic Theorem.

Thus \[ \log_e y = x \left( \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \ldots \right) \]

\[ = 1 - \frac{1}{2x} + \frac{1}{3x^2} - \ldots \cdot \]

\[ = 1 - \frac{1}{x} \times [a convergent series]. \]

Thus when $x$ becomes infinitely large

\[ \lim x \log_e y = 1, \]

and

\[ \lim y = e, \]

i.e.

\[ \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e. \]

\[ \text{Cor. } \lim_{x \to \infty} \left( 1 + \frac{a}{x} \right)^x = \lim_{a \to \infty} \left( 1 + \frac{a}{x} \right)^a = e^a. \]

15. (IV) To prove \( \lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a \)

Assume the expansion for $a^x$, viz.

\[ a^x = 1 + x \log_e a + \frac{x^2}{2!} (\log_e a)^2 + \ldots, \]

which is shown in Algebra to be a convergent series.

Hence \[ \frac{a^x - 1}{x} = \log_e a + x \frac{(\log_e a)^2}{2!} + \ldots \]

\[ = \log_e a + x \times [a convergent series]. \]

And the limit of the right-hand side, when $x$ is indefinitely diminished, is clearly $\log_e a$. 

16. Method of procedure. The rule for evaluating a function which takes the undetermined form \( \frac{0}{0} \) when the independent variable \( x \) ultimately coincides with its assigned value \( a \) is as follows:—

Put \( x = a + h \) and expand both numerator and denominator of the fraction. It will now become apparent that the reason why both numerator and denominator ultimately vanish is that some power of \( h \) is a common factor of each. This should now be divided out. Finally let \( h \) diminish indefinitely so that \( x \) becomes ultimately \( a \), and the true limiting value of the function will be clear.

In the particular case in which \( x \) is to become zero the expansion of numerator and denominator in powers of \( x \) should be at once proceeded with without any preliminary substitution for \( x \).

In the case in which \( x \) is to become infinite, put

\[
\frac{x}{y} = \frac{1}{y},
\]

so that when \( x \) becomes \( \infty \), \( y \) becomes 0.

Several other undetermined forms occur, viz. \( 0 \times \infty \), \( \frac{\infty}{\infty} \), \( \infty - \infty \), \( 0^0 \), \( \infty^0 \), \( 1^\infty \), but they may be made to depend upon the form \( \frac{0}{0} \) by special artifices.

The method thus indicated will be best understood by examining the mode of solution of the following examples:—

Ex. 1. Find

\[
\frac{x^7 - 2x^5 + 1}{x^8 - 3x^2 + 2}.
\]

This is of the form \( \frac{0}{0} \) if we put \( x = 1 \). Therefore we put \( x = 1 + h \) and expand. We thus obtain

\[
\frac{x^7 - 2x^5 + 1}{x^8 - 3x^2 + 2} = \frac{(1 + h)^7 - 2 (1 + h)^5 + 1}{(1 + h)^8 - 3 (1 + h)^2 + 2}
\]
UNDETERMINED FORMS.

\[ = L_{t = 0} \frac{(1 + 7h + 21h^2 + \ldots) - 2(1 + 5h + 10h^2 + \ldots) + 1}{(1 + 3h + 3h^2 + \ldots) - 3(1 + 2h + h^2) + 2} \]
\[ = L_{t = 0} \frac{-3h + h^2 + \ldots}{\cdots} \]
\[ = L_{t = 0} \frac{3 + h + \ldots}{\cdots} \]
\[ = \frac{-3}{-3} = 1. \]

It will be seen from this example that in the process of expansion it is only necessary in general to retain a few of the lowest powers of \( h \).

Ex. 2. Find \( L_{t_x = 0} \frac{a^x - b^x}{x} \).

Here numerator and denominator both vanish if \( x \) be put equal to 0. We therefore expand \( a^x \) and \( b^x \) by the exponential theorem. Hence

\[ = L_{t_x = 0} \frac{a^x - b^x}{x} \]
\[ = \frac{1 + x \log_e a + \frac{x^2}{2!} (\log_e a)^2 + \cdots}{x} \]
\[ - \frac{1 + x \log_e b + \frac{x^2}{2!} (\log_e b)^2 + \cdots}{x} \]
\[ = L_{t_x = 0} \left\{ \log_e a - \log_e b + \frac{x}{2!} (\log_e a - \log_e b)^2 + \cdots \right\} \]
\[ = \log_e a - \log_e b = \log_e \frac{a}{b}. \]

Ex. 3. Find \( L_{t_x = 0} \left( \frac{\tan x}{x} \right)^{\frac{1}{x^2}} \).

Since \( \tan x = \frac{\sin x}{\cos x} \), we have

\[ L_{t_x = 0} \frac{\tan x}{x} = 1. \]

Hence the form assumed by \( \left( \frac{\tan x}{x} \right)^{\frac{1}{x^2}} \) is an undetermined form \( 1^\infty \) when we put \( x = 0 \).

Expand \( \sin x \) and \( \cos x \) in powers of \( x \). This gives

\[ L_{t_x = 0} \left( \frac{\tan x}{x} \right)^{\frac{1}{x^2}} = L_{t_x = 0} \left( \frac{x - \frac{x^3}{3!} + \cdots}{\frac{x^3}{2!} + \cdots} \right)^{\frac{1}{x^2}}. \]
Differential Calculus.

\[
= \lim_{x \to 0} \left( 1 + \frac{x^2}{3} + \text{higher powers of } x \right)^{\frac{1}{x^2}}
\]

\[
\lim_{x \to 0} \left( \frac{\tan x}{x} \right)^{\frac{1}{x^2}} = \lim_{x \to 0} \left( \frac{1 + \frac{x^2}{3}}{1 + \frac{x^2}{8}} \right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}, \text{ by Art. 14.}
\]

Ex. 4. Find \( \lim_{x \to 1} x^{\frac{1}{1-x}} \).

This expression is of the undetermined form \(1^\infty\).

Put \(1 - x = y\),

and therefore, if \(x = 1\), \(y = 0\); therefore \(\text{Limit required} = \lim_{y \to 0} (1 - y)^{\frac{1}{y}} = e^{-1}\) (Art. 14).

Ex. 5. \( \lim_{x \to \infty} x (a^x - 1) \).

This is of the undetermined form \(\infty \times 0\).

Put \(x = \frac{1}{y}\),

therefore, if \(x = \infty\), \(y = 0\); hence \(\text{Limit required} = \lim_{y \to 0} a^y - 1\) (Art. 15).

17. The following Algebraical and Trigonometrical series are added, as they are wanted for immediate use. They should be learnt thoroughly.

\[
(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \ldots
\]

\[
(1 - x)^{-n} = 1 + nx + \frac{n(n+1)}{1 \cdot 2} x^2 + \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} x^3 + \ldots
\]

\[
a^x = 1 + x \log a + \frac{x^2 (\log a)^2}{2!} + \frac{x^3 (\log a)^3}{3!} + \ldots
\]

\[
\varepsilon^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\]
EXAMPLES.

\[
\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.
\]

\[
\log_e (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots.
\]

\[
\frac{1}{2} \log_e \left( \frac{1 + x}{1 - x} \right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots.
\]

\[
\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots.
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots.
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots.
\]

\[
\cosh x \left[ \text{which} \equiv \frac{e^x + e^{-x}}{2} \right] = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots.
\]

\[
\sinh x \left[ \text{which} \equiv \frac{e^x - e^{-x}}{2} \right] = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots.
\]

EXAMPLES.

Find the values of the following limits:

1. \( \lim_{x \to 0} \frac{a^x - 1}{b^x - 1} \).

2. \( \lim_{x \to 1} \frac{x^2 - 1}{x^8 - 1} \).

3. \( \lim_{x \to 1} \frac{x^m - 1}{x^n - 1} \).

4. \( \lim_{x \to 0} \frac{(1 + x)^n - 1}{x} \).

5. \( \lim_{x \to 1} \frac{x^4 + x^3 - x^2 - 5x - 4}{x^2 - 2x^3 + 9x - 4} \).

6. \( \lim_{x \to 0} \frac{x^8 - 2x^3 + 4x^2 - 9x - 4}{x^2} \).

7. \( \lim_{x \to 0} \frac{e^x - e^{-x}}{x} \).

8. \( \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{x^2} \).

9. \( \lim_{x \to 0} \frac{x \cos x - \log_e (1 + x)}{x^2} \).

10. \( \lim_{x \to 0} \frac{xe^x - \log_e (1 + x)}{x^3} \).

11. \( \lim_{x \to 0} \frac{x - \sin x \cos x}{x^3} \).

12. \( \lim_{x \to 0} \frac{\sin^{-1} x - x}{x^3 \cos x} \).

13. \( \lim_{x \to 0} \frac{\cosh x - \cos x}{x \sin x} \).

14. \( \lim_{x \to 0} \frac{x^2 - \sin x}{\tan^{-1} x} \).

15. \( \lim_{x \to 0} \frac{x^2 - \sinh x}{x^3} \).
Differential Calculus.

\[ x \cos^3 x - \log_e (1 + \nu) - \sin^{-1} \frac{\nu}{2} \]

16. \( \begin{align*}
&= 2 \sin x + \frac{1}{2} \log_e \frac{1 + x}{1 - x} - 3 \nu \\
&\quad \text{as } x = 0 \quad \frac{e^x \sin \nu - \nu - x^2}{x^2 + x \log_e (1 - \nu^2)} \\
&\text{(evaluated)}
\end{align*} \]

19. \( \begin{align*}
&= x^3 e^{\frac{\nu}{x}} - \sin^3 \frac{x^2}{2} \\
&\quad \text{as } x = 0 \\
&\text{(evaluated)}
\end{align*} \]

20. \( \begin{align*}
&= \left( \tan x \right)^\frac{1}{x} \\
&\quad \text{as } x = 0 \\
&\text{(evaluated)}
\end{align*} \]

21. \( \begin{align*}
&= \left( \frac{\tan x}{x} \right)^\frac{1}{x^2} \\
&\quad \text{as } x = 0 \\
&\text{(evaluated)}
\end{align*} \]

22. \( \begin{align*}
&= \left( \frac{\sin x}{x} \right)^\frac{1}{x} \\
&\quad \text{as } x = 0 \\
&\text{(evaluated)}
\end{align*} \]

23. \( \begin{align*}
&= \left( \frac{\sin x}{x} \right)^\frac{1}{x} \\
&\quad \text{as } x = 0 \\
&\text{(evaluated)}
\end{align*} \]

24. \( \begin{align*}
&= \left( \frac{\sin x}{x} \right)^\frac{1}{x^2} \\
&\quad \text{as } x = 0 \\
&\text{(evaluated)}
\end{align*} \]

25. \( \begin{align*}
&= \left( \cosec x \right)^\frac{1}{x^2} \\
&\quad \text{as } x = 0 \\
&\text{(evaluated)}
\end{align*} \]

26. \( \begin{align*}
&= \pi \left( \cosec x \right)^{\tan^2 x} \\
&\quad \text{as } x = 0 \\
&\text{(evaluated)}
\end{align*} \]

\[ \text{as } x = 0 \]
CHAPTER II.

DIFFERENTIATION FROM THE DEFINITION.

18. **Tangent of a Curve; Definition; Direction.**

Let $AB$ be an arc of a curve traced in the plane of the paper, $OX$ a fixed straight line in the same plane. Let

$P, Q$, be two points on the curve; $PM$, $QN$, perpendiculars on $OX$, and $PR$ the perpendicular from $P$ on $QN$. Join $P, Q$, and let $QP$ be produced to cut $OX$ at $T$.

- When $Q$, travelling along the curve, approaches indefinitely near to $P$, the limiting position of chord $QP$ is called the tangent at $P$. $QR$ and $PR$ both ultimately vanish, but the limit of their ratio is in general finite; for $\lim \frac{RQ}{PR} = \lim \tan RQP = \lim \tan XTP = \tan$ of the angle which the tangent at $P$ to the curve makes with $OX$. 
If \( y = \phi(x) \) be the equation of the curve and \( \dot{x} \), 
\( x + h \) the abscissae of the points \( P, Q \) respectively; 
then \( MP = \phi(x), \ NQ = \phi(x + h), \ RQ = \phi(x + h) - \phi(x) \) 
and \( PR = h \).

Thus 
\[
\frac{RQ}{PR} = \frac{\phi(x + h) - \phi(x)}{h}.
\]

Hence, to draw the tangent at any point \( (x, y) \) on the curve \( y = \phi(x) \), we must draw a line through that point, 
making with the axis of \( x \) an angle whose tangent is 
\[
\frac{\phi(x + h) - \phi(x)}{h}.
\]
and if this limit be called \( m \), the equation of the tangent 
at \( P(x, y) \) will be 
\[
Y - y = m (X - x),
\]
\( X, Y \) being the current co-ordinates of any point on the tangent; for the line represented by this equation goes 
through the point \( (x, y) \), and makes with the axis of \( x \) 
an angle whose tangent is \( m \).

19. Def.—Differential Coefficient.

Let \( \phi(x) \) denote any function of \( x \), and \( \phi(x + h) \) the same function of \( x + h \); then \( \frac{\phi(x + h) - \phi(x)}{h} \) 
is called the first derived function or differential coefficient of \( \phi(x) \) with respect to \( x \).

The operation of finding this limit is called differentiating \( \phi(x) \).

20. Geometrical meaning. The geometrical meaning of the above limit is indicated in the last article, 
where it is shown to be the tangent of the angle \( \gamma \) which 
the tangent at any definite point \( (x, y) \) on the curve 
\( y = \phi(x) \) makes with the axis of \( x \).

21. We can now find the differential coefficient of 
any proposed function by investigating the value of the
GEOMETRICAL MEANING.

above limit; but it will be seen later on that, by means of certain rules to be established in Chap. III., and a knowledge of the differential coefficients of certain standard forms to be investigated in Chap. IV., we can always avoid the labour of an ab initio evaluation.

Ex. 1. Find from the definition the differential coefficient of $\frac{x^2}{a}$, where $a$ is constant; and the equation of the tangent to the curve $ay = x^2$.

Here

$$\phi(x) = \frac{x^2}{a},$$

$$\phi(x + h) = \frac{(x + h)^2}{a},$$

therefore

$$\frac{\phi(x + h) - \phi(x)}{h} = \frac{(x + h)^2 - x^2}{ha} = \frac{2xh + h^2}{ha} = \frac{2x + h}{a}.$$

The geometrical interpretation of this result is that, if a tangent be drawn to the parabola $ay = x^2$ at the point $(x, y)$, it will be inclined to the axis of $x$ at the angle $\tan^{-1} \frac{2x}{a}$.

The equation of the tangent is therefore

$$Y - y = \frac{2x}{a} (X - x).$$

Ex. 2. Find from the definition the differential coefficient of $\log_{e} \sin \frac{x}{a}$, where $a$ is a constant.

Here

$$\phi(x) = \log_{e} \sin \frac{x}{a},$$

and

$$\frac{\phi(x + h) - \phi(x)}{h} = \frac{\log_{e} \sin \frac{x + h}{a} - \log_{e} \sin \frac{x}{a}}{h} = \frac{\frac{\sin \frac{x}{a} \cos \frac{h}{a} + \cos \frac{x}{a} \sin \frac{h}{a}}{a}}{h} = \frac{\frac{\sin \frac{x}{a}}{a}}{h}.$$

$$= \frac{1}{a} \log_{e} \left( 1 + \frac{h}{a} \cot \frac{x}{a} \right)^{\text{higher powers of } h}.$$
by substituting for \( \sin \frac{h}{a} \) and \( \cos \frac{h}{a} \) their expansions in powers of \( \frac{h}{a} \)

\[
- \frac{h}{a} \cot \frac{x}{a} = \text{higher powers of } h \]

\[
= \frac{1}{a} \cot \frac{x}{a}.
\]

Hence the tangent at any point on the curve \( y = \log a \sin \frac{x}{a} \) is inclined to the axis of \( x \) at an angle whose tangent is \( \cot \frac{x}{a} \), that is at an angle \( \frac{\pi}{2} \), and the equation of the tangent at the point \( x, y \) is

\[
1 - y = \cot \frac{x}{a} (x - r)
\]

**EXAMPLES.**

Find the equation of the tangent at the point \( (r, y) \) on each of the following curves

1. \( y = x^3 \)
2. \( y = r^4 \)
3. \( y = \sqrt{x} \)
4. \( y = x^2 + x^4 \)
5. \( y = \sin x \)
6. \( y = e^x \)
7. \( y = \log x \)
8. \( y = \tan x \)
9. \( x^2 + y^2 = r^2 \)
10. \( r^2/\pi^2 + \theta^2/\beta^2 = 1 \)

22. **Notation.** It is convenient to use the notation \( \delta x \) for the same quantity which we have denoted by \( h \), viz. a small but finite increase in the value of \( x \). We may similarly denote by \( \delta y \) the consequent change in the value of \( y \). Thus if \( (x, y), (x + \delta x, y + \delta y) \) be contiguous points upon a given curve \( y = \phi (x) \), we have

\[
y + \delta y = \phi (x + \delta x),
\]

and

\[
\delta y = \phi (x + \delta x) - \phi (x).
\]

Thus the differential coefficient

\[
\lim_{\delta x \to 0} \frac{\phi (x + \delta x) - \phi (x)}{\delta x}
\]
may be written

\[ \text{Lt}_{x=0} \frac{\delta y}{\delta x}, \]

which more directly indicates the geometrical meaning

\[ \text{Lt}_{x=0} \frac{RQ}{PR}. \]

pointed out in Art. 18.

The result of the operation expressed by

\[ \text{Lt}_{h=0} \frac{\phi(x + h) - \phi(x)}{h}, \]

or by

\[ \text{Lt}_{x=0} \frac{\delta y}{\delta x}, \]

is denoted by

\[ \frac{d}{dx} y \text{ or } \frac{dy}{dx}. \]

The student must guard against the fallacious notion that \( dx \) and \( dy \) are separate small quantities, as \( \delta x \) and \( \delta y \) are. He must remember that \( \frac{d}{dx} \) is a symbol of operation which when applied to any function \( \phi(x) \) means that we are

1. to increase \( x \) to \( x + h \),
2. to subtract the original value of the function,
3. to divide the remainder by \( h \),
4. to evaluate the limit when \( h \) ultimately vanishes.

Other notations expressing the same thing are

\[ \frac{d\phi(x)}{dx}, \frac{d\phi}{dx}, \phi'(x), \phi, \phi_x, y', y, y. \]

**EXAMPLES.**

Find \( \frac{dy}{dx} \) in the following cases.

1. \( y = 2x \).
2. \( y = 2 + x \).
3. \( y = 2 + 3x \).

* E. D. C.
4. \( y = 2 + 3x^2 \). 
5. \( y = \frac{1}{x} \).
6. \( y = \frac{1}{x} + \alpha \).
7. \( y = \frac{1}{x^2} + \alpha \).
8. \( y = a \sqrt{x} \).
9. \( y = \sqrt{x^2 + \alpha^2} \).
10. \( y = e^{\sqrt{x}} \).
11. \( y = e^{\sin x} \).
12. \( y = \log x \sec x \).
13. \( y = x \sin x \).
14. \( y = \frac{\sin x}{x} \).
15. \( y = x^x \).

\( \sqrt{23}. \) **Aspect of the Differential Coefficient as a Rate-Measurer.** When a particle is in motion in a given manner the space described is a function of the time of describing it. We may consider the time as an independent variable, and the space described in that time as the dependent variable.

The rate of change of position of the particle is called its velocity.

If **uniform** the velocity is measured by the space described in one second; if **variable**, the velocity at any instant is measured by the space which would be described in one second if, for that second, the velocity remained unchanged.

Suppose a space \( s \) to have been described in time \( t \) with varying velocity, and an additional space \( \delta s \) to be described in the additional time \( \delta t \). Let \( v_1 \) and \( v_2 \) be the greatest and least values of the velocity during the interval \( \delta t \); then the spaces which would have been described with uniform velocities \( v_1 \), \( v_2 \), in time \( \delta t \) are \( v_1 \delta t \) and \( v_2 \delta t \), and are respectively greater and less than the actual space \( \delta s \).

Hence \( v_1 \frac{\delta s}{\delta t} \) and \( v_2 \) are in descending order of magnitude.

If then \( \delta t \) be diminished indefinitely, we have in the limit \( v_1 = v_2 \) = the velocity at the instant considered, which is therefore represented by \( Lt \frac{\delta s}{\delta t} \), i.e. by \( \frac{ds}{dt} \).
24. It appears therefore that we may give another interpretation to a differential coefficient, viz. that \( \frac{ds}{dt} \) means the rate of increase of \( s \) in point of time. Similarly \( \frac{dx}{dt} \), \( \frac{dy}{dt} \), mean the rates of change of \( x \) and \( y \) respectively in point of time, and measure the velocities, resolved parallel to the axes, of a moving particle whose coordinates at the instant under consideration are \( x, y \). If \( x \) and \( y \) be given functions of \( t \), and therefore the path of the particle defined, and if \( \delta x, \delta y, \delta t \) be simultaneous infinitesimal increments of \( x, y, t \), then

\[
\frac{dy}{dx} = \frac{\delta y}{\delta x} \quad \frac{dy}{dt} = \frac{\delta y}{\delta t} \quad \frac{dy}{dt} = \frac{\delta y}{\delta t} \quad \frac{\delta y}{\delta x} \quad \frac{\delta y}{\delta t}
\]

and therefore represents the ratio of the rate of change of \( y \) to that of \( x \). The rate of change of \( x \) is arbitrary, and if we choose it to be unit velocity, then

\[
\frac{dy}{dx} = \frac{dy}{dt} = \text{absolute rate of change of } y.
\]

If \( x \) be increasing with \( t \), the \( x \)-velocity is positive, whilst, if \( x \) be decreasing while \( t \) increases, that velocity is negative. Similarly for \( y \).

Moreover, since \( \frac{dy}{dx} = \frac{dt}{dx} \), \( \frac{dy}{dx} \) is positive when \( x \) and \( y \) increase or decrease together, but negative when one increases as the other decreases.

This is obvious also from the geometrical interpretation of \( \frac{dy}{dx} \). For, if \( x \) and \( y \) are increasing together,
\[ \frac{dy}{dx} \] is the tangent of an acute angle and therefore positive, while if, as \( x \) increases \( y \) decreases, \( \frac{dy}{dx} \) represents the tangent of an obtuse angle and is negative.

26. The above article frequently affords important information with regard to the sign of a given expression. For if, for instance, \( \phi(x) \) be a continuous function which is positive when \( x = a \) and when \( x = b \), and if \( \phi'(x) \) be of one sign for all values of \( x \) lying between \( a \) and \( b \) so that it is known that \( \phi(x) \) is always increasing or always decreasing from the one value \( \phi(a) \) to the other \( \phi(b) \), it will follow that \( \phi(x) \) must be positive for all intermediate values of \( x \).

Ex. Let \( \phi(x) = (x - 1) e^x + 1. \)

Here \( \phi(0) = 0 \) and \( \phi'(x) = I t h = 0 \frac{(x + h - 1) e^{x+h} - (x - 1) e^x}{h} \)

\[ = I t h = 0 \frac{(x + h - 1)(1 + h + ...) - (x - 1) e^x}{h} \]

\[ = I t h = 0 h e^x + \text{higher powers of } h \]

So that \( \phi'(x) \) is positive for all positive values of \( x \). Therefore as \( x \) increases from 0 to \( \infty \), \( \phi(x) \) is always increasing. Hence since its initial value is zero the expression is positive for all positive values of \( x \).

EXAMPLES.

1. Differentiate the following expressions, and shew that they are each positive for all positive values of \( x \):

   (i) \( (x - 2) e^x + x + 2 \),
   (ii) \( (x - 3) e^x + \frac{x^2}{2} + 2x + 3 \),
   (iii) \( x - \log_e (1 + x) \).

2. In the curve \( y = e^x \), if \( \psi \) be the angle which the tangent at any point makes with the axis of \( x \), prove \( y = e \tan \psi \).

3. In the curve \( y = e \cosh \frac{x}{e} \), prove \( y = e \sec \psi \).
4. In the curve $3b^2y = x^3 - 3ax^2$ find the points at which the tangent is parallel to the axis of $x$.

[N.B.—This requires that $\tan \psi = 0$.]

5. Find at what points of the ellipse $x^2/a^2 + y^2/b^2 = 1$ the tangent cuts off equal intercepts from the axes.

[N.B.—This requires that $\tan \psi = \pm 1$.]

6. Prove that if a particle moves so that the space described is proportional to the square of the time of description, the velocity will be proportional to the time, and the rate of increase of the velocity will be constant.

7. Shew that if a particle moves so that the space described is given by $s \propto \sin \mu t$, where $\mu$ is a constant, the rate of increase of the velocity is proportional to the distance of the particle measured along its path from a fixed position.

8. Shew that the function

$$x \sin x + \cos x + \cos^2 x$$

continually diminishes as $x$ increases from $0$ to $\pi/2$.

9. If

$$y = 2x - \tan^{-1} x - \log_e(x + \sqrt{1 + x^2}),$$

shew that $y$ continually increases as $x$ changes from zero to positive infinity.

10. A triangle has two of its angular points at $(a, 0)$, $(0, b)$, and the third $(x, y)$ is movable along the line $y = x$. Shew that if $A$ be its area

$$2 \frac{dA}{dx} = a + b,$$

and interpret this result geometrically.

11. If $A$ be the area of a circle of radius $x$, shew that the circumference is $\frac{dA}{dx}$. Interpret this geometrically.

12. $O$ is a given point and $NP$ a given straight line upon which $ON$ is the perpendicular. The radius $OP$ rotates about $O$ with the constant angular velocity $\omega$. Shew that $NP$ increases at the rate

$$\omega \cdot ON \sec^2 NOP.$$
CHAPTER III.

FUNDAMENTAL PROPOSITIONS.

27. It will often be convenient in proving standard results to denote by a small letter the function of \( x \) considered, and by the corresponding capital the same function of \( x + h \), e.g. if \( u = \phi (x) \), then \( U = \phi (x + h) \), or if \( u = ax \), then \( U = ax + h \).

Accordingly we shall have

\[
\frac{du}{dx} = L_{t=0} \frac{U - u}{h}, \quad \frac{dv}{dx} = L_{t=0} \frac{V - v}{h}.
\]

etc.

We now proceed to the consideration of several important propositions.

\checkmark 28. Prop. I. **The Differential Coefficient of any Constant is zero.** This proposition will be obvious when we refer to the definition of a constant quantity. A constant is essentially a quantity of which there is no variation, so that if \( y = c \), \( \delta y = \) absolute zero whatever may be the value of \( \delta x \). Hence \( \frac{\delta y}{\delta x} = 0 \) and

\[
\frac{dy}{dx} = 0 \text{ when the limit is taken.}
\]

Or geometrically: \( y = c \) is the equation of a straight line parallel to the \( x \)-axis. At each point of its length it is its own tangent and makes an angle whose tangent is zero with the \( x \)-axis.
\textbf{29. Prop. II. Product of Constant and Function.} \hfill (23)

The differential coefficient of a product of a constant and a function of $x$ is equal to the product of the constant and the differential coefficient of the function, or, stated algebraically,

\[
\frac{d}{dx} (cu) = c \frac{du}{dx}.
\]

For

\[
\frac{d}{dx} (cu) = \lim_{h \to 0} \frac{cU - cu}{h} = c \lim_{h \to 0} \frac{U - u}{h} = c \frac{du}{dx}.
\]

\textbf{30. Prop. III. Differential Coefficient of a Sum.} \hfill (23)

The differential coefficient of the sum of a set of functions of $x$ is the sum of the differential coefficients of the several functions.

Let $u, v, w, \ldots$, be the functions of $x$, and $y$ their sum.

Let $U, V, W, \ldots, Y$ be what these expressions severally become when $x$ is changed to $x + h$.

Then

\[
y = u + v + w + \ldots
\]

\[
Y = U + V + W + \ldots,
\]

and therefore

\[
Y - y = (U - u) + (V - v) + (W - w) + \ldots;
\]

dividing by $h$,

\[
\frac{Y - y}{h} = \frac{U - u}{h} + \frac{V - v}{h} + \frac{W - w}{h} + \ldots
\]

and taking the limit

\[
\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \ldots.
\]

If some of the connecting signs had been $-$ instead of $+$ a corresponding result would immediately follow, e.g. if

\[
y = u - v - w + \ldots.
\]
then \( \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx} + \ldots \).

31. Prop. IV. The Differential Coefficient of the product of two functions is

\[
(\text{First Function}) \times (\text{Diff. Coeff. of Second}) + (\text{Second Function}) \times (\text{Diff. Coeff. of First}),
\]

or, stated algebraically,

\[
\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}.
\]

With the same notation as before, let

\[ y = uv, \text{ and therefore } Y = UV; \]

whence

\[
Y - y = UV - uv
\]

\[
= u(V - v) + V(U - u);
\]

therefore

\[
\frac{Y - y}{h} = u \frac{V - v}{h} + V \frac{U - u}{h};
\]

and taking the limit

\[
\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}.
\]

32. On division by \( uv \) the above result may be written

\[
\frac{1}{y} \frac{dy}{dx} = \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx}.
\]

Hence it is clear that the rule may be extended to products of more functions than two.

For example, if \( y = uvw; \) let \( vw = z, \) then \( y = uz. \)

Whence

\[
\frac{1}{y} \frac{dy}{dx} = \frac{1}{u} \frac{du}{dx} + \frac{1}{z} \frac{dz}{dx},
\]

but

\[
\frac{1}{z} \frac{dz}{dx} = \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx},
\]
whence by substitution

\[
\frac{1}{y} \frac{dy}{dx} = \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx}.
\]

Generally, if

\[
\frac{1}{y} \frac{dy}{dx} = \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx} + \frac{1}{t} \frac{dt}{dx} + \ldots,
\]

and if we multiply by \(uv\ldots\) we obtain

\[
\frac{dy}{dx} = (uv\ldots) \frac{du}{dx} + (uv\ldots) \frac{dv}{dx} + (uv\ldots) \frac{dw}{dx} + \ldots,
\]

i.e. multiply the differential coefficient of each separate function by the product of all the remaining functions and add up all the results; the sum will be the differential coefficient of the product of all the functions.

33. Prop. V. The Differential Coefficient of a quotient of two functions is

\[
\left(\frac{\text{Diff. Coeff. of Num.}}{\text{Den.}}\right) - \left(\frac{\text{Diff. Coeff. of Den.}}{\text{Num.}}\right) \text{ Square of Denominator}
\]

or, stated algebraically,

\[
\frac{d}{dx} \left(\frac{u}{v}\right) = \frac{du}{dx} \frac{v}{v} - \frac{dv}{dx} \frac{u}{v}.
\]

With the same notation as before, let

\[
y = \frac{u}{v}, \text{ and therefore } Y = \frac{U}{V},
\]

whence

\[
Y - y = \frac{U}{V} - \frac{u}{v} = \frac{UV - Vu}{Vv};
\]
Therefore \( \frac{V - y}{h} = \frac{U - u}{h} \frac{V - v}{h} \),
and taking the limit
\[
\frac{dy}{dx} = \frac{du}{dx} \frac{v - dv}{v^2} - \frac{dx}{u}.
\]

34. To illustrate these rules let the student recall to memory the
differential coefficients of \( x^2 \) and \( a \log_e \sin \frac{x}{a} \) established in Art. 21, viz.

\( 2x \) and \( \cot \frac{x}{a} \) respectively.

Ex. 1. Thus if \( y = x^2 + a \log_e \sin \frac{x}{a} \),
we have by Prop. III.
\[
\frac{dy}{dx} = 2x + \cot \frac{x}{a}.
\]

Ex. 2. If \( y = x^2 \times a \log_e \sin \frac{x}{a} \),
we have by Prop. IV.
\[
\frac{dy}{dx} = 2x \times a \log_e \sin \frac{x}{a} + x^2 \cot \frac{x}{a}.
\]

Ex. 3. If
we have by Prop. V.
\[
\frac{dy}{dx} = \frac{x^3 \cot \frac{x}{a} - 2x \cdot a \log_e \sin \frac{x}{a}}{x^3}.
\]

**EXAMPLES.**

[The following differential coefficients obtained as results of preceding examples may for present purposes be assumed:

\( y = x^3 \), \( y_1 = 3x^2 \).

\( y = e^x \), \( y_1 = e^x \).

\( y = x^4 \), \( y_1 = 4x^3 \).

\( y = \log_e x \), \( y_1 = \frac{1}{x} \).

\( y = \sqrt{x} \), \( y_1 = \frac{1}{2 \sqrt{x}} \).

\( y = \tan x \), \( y_1 = \sec^2 x \).

\( y = \sin x \), \( y_1 = \cos x \).

\( y = \log_e \sin x \), \( y_1 = -\cot x \).]
FUNCTION OF A FUNCTION.

Differentiate the following expressions by aid of the foregoing rules:

1. \( x^3 \sin x, \quad x^3 e^x, \quad x^3 \log_e x, \quad x^3 \tan x, \quad x^3 \log_e \sin x \).
2. \( x^4 / \sin x, \quad e^x \sin x / x^4, \quad \sin x / x^4, \quad e^x / \sin x. \)
3. \( \tan x \cdot \log_e \sin x, \quad e^x \log_e x, \quad \sin^2 x / \cos x. \)
4. \( x^3 e^x \sin x, \quad x \tan x \log_e x. \)
5. \( x^3 \sin x / e^x, \quad x^3 / e^x \sin x, \quad 1 / x^3 e^x \sin x. \)
6. \( 2 \sqrt{x} \cdot \sin x, \quad 3 \tan x / \sqrt{x}, \quad 5 + 4e^x / \sqrt{x}. \)
7. \( e^x (x^3 + \sqrt{x}), \quad (x^3 + x^4) (e^x \log_e x). \)

35. Function of a function.

Suppose

\[ u = f(v) \] ..............................(1),

where \[ v = \phi(x) \] ..............................(2).

If \( x \) be changed to \( x + \delta x \), \( v \) will become \( v + \delta v \), and in consequence \( u \) will become \( u + \delta u \).

Now if \( v \) had been first eliminated between equations (1) and (2) we should have a result of the form

\[ u = F(x) \] ..............................(3).

This equation will be satisfied by the same simultaneous values \( x + \delta x, \ u + \delta u \), which satisfy equations (1) and (2). Also

\[ \frac{\delta u}{\delta x} = \frac{\delta u}{\delta v} \cdot \frac{\delta v}{\delta x}, \]

and proceeding to the limit

- \( Lt_{\delta x=0} \frac{\delta u}{\delta x} = \frac{du}{dx} \) as obtained from equation (3),
- \( Lt_{\delta v=0} \frac{\delta u}{\delta v} = \frac{du}{dv} \) as obtained from equation (1),
- \( Lt_{\delta x=0} \frac{\delta v}{\delta x} = \frac{dv}{dx} \) as obtained from equation (2).

Thus

\[ \frac{dv}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx}. \]
36. For instance, the diff. coeff. of $x^2$ is $2x$, \[ \text{Art. 21.} \]
and of $\log_{e}\sin x$ is $\cot x$.

Suppose $u = (\log_{e}\sin x)^2$, i.e. $v^2$ where $v = \log_{e}\sin x$, then
\[
\frac{du}{dx} = \frac{dv}{dx} \cdot \frac{du}{dv} = 2v \cdot \cot x = 2 \cot x \cdot \log_{e}\sin x.
\]

37. It is obvious that the above result may be extended. For, if $u = \phi(v)$, $v = \psi(w)$, $w = f(x)$, we have
\[
\frac{du}{dx} = \frac{dv}{dx} \cdot \frac{dw}{dv} \cdot \frac{dw}{dx},
\]

but
\[
\frac{dv}{dx} = \frac{dw}{dx} \cdot \frac{dv}{dw};
\]

and therefore
\[
\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx};
\]

and a similar result holds however many functions there may be.

The rule may be expressed thus:
\[
\frac{d(1st\ Func.)}{dx} = \frac{d(1st\ Func.)}{d(2nd\ Func.)} \cdot \frac{d(2nd\ Func.)}{d(3rd\ Func.)} \cdots \frac{d(3rd\ Func.)}{dx}
\]
or if $u = \phi [\psi [F'(fx)]]$,
\[
\frac{du}{dx} = \phi' [\psi [F'(fx)]] \times \psi' [F'(fx)] \times F''(fx) \times f'(x).
\]

Thus in the preceding Example
\[
\frac{d}{dx} (\log_{e}\sin x)^2 = \frac{d}{dx} (\log_{e}\sin x)^2 \cdot \frac{d}{dx} \log_{e}\sin x \cdot \frac{d}{dx} = 2 \log_{e}\sin x \cdot \cot x.
\]

Again,
\[
\frac{d}{dx} (\log_{e}\sin x^2) = \frac{d}{dx} (\log_{e}\sin x^2) \cdot \frac{d}{dx} \sin x^2 \cdot \frac{dx^2}{dx} \cdot \frac{dx}{dx} \cdot \frac{dx}{dx} = \frac{1}{\sin x^2} \cdot \cos x^2 \cdot 2x = 2x \cdot \cot x^2.
\]
\textbf{38. Interchange of the dependent and independent variable.} If in the theorem
\[
\frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx}
\]
we put
\[u = x,
\]
then
\[
\frac{du}{dx} = \frac{dx}{dx} = \frac{d}{dx} \left( x + h \right) - \frac{x}{h} = 1,
\]
and we obtain the result
\[
\frac{dy}{dx} \cdot \frac{dx}{dy} = 1,
\]
or
\[
\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.
\]

\textbf{39. The truth of this is also manifest geometrically,}

for \( \frac{dy}{dx} \) and \( \frac{dx}{dy} \) are respectively the tangent and the co-tangent of the angle \( \psi \) which the tangent to the curve \( y = f(x) \) makes with the \( x \)-axis.

\textbf{40. This formula is very useful in the differentiation of an inverse function.}

Thus if we have \( y = f^{-1}(x) \),
\[x = f(y),
\]
and
\[
\frac{dx}{dy} = f'(y),
\]
a form which we are supposing to have been investigated.

Thus
\[
\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f' \left( f^{-1}(x) \right)}.
\]
EXAMPLES.

Assuming as before for present purposes the following differential coefficients,

\[
\frac{d}{dx} x^3 = 3x^2, \quad \frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}, \quad \frac{d}{dx} \sin x = \cos x,
\]

\[
\frac{d}{dx} e^x = e^x, \quad \frac{d}{dx} \log_e x = \frac{1}{x}, \quad \frac{d}{dx} \tan x = \sec^2 x.
\]

Write down the differential coefficients of the following combinations:

1. \(e^{3x}, e^{-x}, \sin^3 x, \sqrt{\sin x}, \sqrt{\log_e x}, \sqrt{\tan x}, \sin \sqrt{x}\).

2. \(e^{\sin x}, e^{\tan x}, e^{x^2}, e^{\sqrt{x}}, e^{\log_e x}\).

3. \(\log_e \sin x, \log_e \tan x, \log_e \sqrt{x}, \log_e e^{x}\).

4. \(\sin \log_e x, \tan \log_e x, \sqrt{\sin \log_e x}, \sqrt{\sin \sqrt{x}}, \log_e \sin \sqrt{x}\).

5. \(\log_e \sqrt{\sin \sqrt{x}}, \tan \log_e \sin e^{\sqrt{x}}\).
CHAPTER IV.

STANDARD FORMS.

41. It is the object of the present Chapter to investigate and tabulate the results of differentiating the several standard forms referred to in Art. 21.

We shall always consider angles to be measured in circular measure, and all logarithms to be Napierian, unless the contrary is expressly stated.

It will be remembered that if \( u = \phi(x) \), then, by the definition of a differential coefficient,

\[
\frac{du}{dx} = \lim_{h \to 0} \frac{\phi(x + h) - \phi(x)}{h}.
\]

42. Differential Coefficient of \( x^n \).

If \( u = \phi(x) = x^n \), then \( \phi(x + h) = (x + h)^n \), and

\[
\frac{du}{dx} = \lim_{h \to 0} \frac{(x + h)^n - x^n}{h} = \lim_{h \to 0} \left( x^n + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \cdots + h^n \right) / h
\]

\[
= \lim_{h \to 0} \left( 1 + \frac{\binom{n}{1} x^{n-1}}{1} h + \frac{\binom{n}{2} x^{n-2}}{2} h^2 + \cdots \right)
\]

\[
= \lim_{h \to 0} \left( x^n + \frac{n}{1} x^{n-1} h + \frac{n(n-1)}{2} x^{n-2} h^2 + \cdots \right)
\]

\[
= \lim_{h \to 0} \left( x^n + \frac{n}{1} x^{n-1} h + \frac{n(n-1)}{2} x^{n-2} h^2 + \cdots \right) = x^n
\]

\( n \), since \( h \) is to be ultimately zero, we may consider it to be less than unity, and we can therefore apply the
BINOMIAL THEOREM to expand \((1 + \frac{h}{x})^n\), whatever be the value of \(n\); hence

\[
d\frac{u}{x} = L_{h=0} x^n \left\{ n \frac{h}{x} + \frac{n(n-1)}{2} \frac{h^2}{x^2} + \cdots \right\} + \frac{n(n-1)(n-2)}{3} \frac{h^3}{x^3} + \cdots \}
\]

\[= L_{h=0} nx^{n-1} \left(1 + \frac{h}{x} \times (a \text{ convergent series})\right)
\]

\[= nx^{n-1}.
\]

43. It follows by Art. 35 that if \(u =(x)\)^n then

\[
d\frac{u}{x} = u \left[ \phi(x) \right]^{n-1} \phi'(x).
\]

EXAMPLES.

Write down the differential coefficients of

1. \(x, \ x^1, \ x^{-1}, \ x^{10}, \ x^{\frac{3}{2}}, \ x^{\frac{1}{2}}, \ x^{3}, \ x^{-3}, \ a\ x^{-n}.
\)

2. \((x+a)^n, \ x^n + an, \ x^{\frac{3}{2}} + a^{\frac{1}{2}}, \ \frac{1}{x+a}, \ \frac{1}{\sqrt{x+a}}.
\)

3. \((ax+b)^n, \ a\ x^n + b, \ (ax)^n + b, \ a \ x^{\frac{1}{2}}, \ a^n \ (x+b).
\)

4. \(x + x^2 + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots.
\)

5. \(a + b \sqrt{x}, \ a + \sqrt{x}, \ a + bx, \ \sqrt{a + x}, \ \sqrt{3/\sqrt{a-x}}.
\)

6. \(\frac{a x^2 + bx + c}{cx^3 + bx + a}, \ (x+a)^n \ (x+b)^q, \ (x+a)^p/(x+b)^q.
\)

44. Differential Coefficient of \(u^x\).

If \(u = \phi(x) = a^x\),

\(\phi(x + h) = a^{x+h}\),
and
\[ \frac{du}{dx} = Lt_{h=0} \frac{a^{x+h} - a^x}{h} = a^x Lt_{h=0} \frac{a^h - 1}{h} = a^x \log_a u. \quad \text{[Art. 15.]} \]

Cor. 1. If \( u = e^x \), \( \frac{du}{dx} = e^x \log_e e = e^x \).

Cor. 2. It follows by Art. 35 that if \( u = e^{\phi(x)} \), then
\[ \frac{du}{dx} = e^{\phi(x)} \cdot \phi'(x). \]

45. **Differential Coefficient of** \( \log_a a \).

If
\[ u = \phi(x) = \log_a x, \]
\[ \phi(x + h) = \log_a (x + h), \]
and
\[ \frac{du}{dx} = Lt_{h=0} \frac{\log_a (x + h) - \log_a x}{h} = Lt_{h=0} \frac{1}{h} \log_a \left( 1 + \frac{h}{x} \right). \]

Let \( \frac{a}{h} = z \), so that if \( h \to 0 \), \( z = \infty \); therefore
\[ \frac{du}{dx} = Lt_{z=\infty} \frac{z}{x} \log_a \left( 1 + \frac{1}{z} \right) = \frac{1}{x} \log_a e. \quad \text{[Art. 14(1)]} \]

Cor. 1. If \( u = \log_a x \), \( \frac{du}{dx} = \frac{1}{x} \log_e e = \frac{1}{x} \).

Cor. 2. And it follows as before that if
\[ u = \log_a \phi(x), \]
then
\[ \frac{du}{dx} = \frac{\phi'(x)}{\phi(x)}. \]

E. D. C.
EXAMPLES.

Write down the differential coefficients of

1. \( e^{2x}, \ e^{-x}, \ e^{ax}, \ \cosh x, \ \sinh x, \ \frac{e^{2x} + e^{3x}}{1 + e^{-x}} \).

2. \( \log \sqrt{x}, \ \log (x + a), \ \log (ax + b), \ \log (ax^2 + bx + c), \ \log \frac{1 + x}{1 - x}, \ \log \frac{1 + x^2}{1 - x^2}, \ \log x a. \)

3. \( \phi (e^x), \ \phi (\log x), [\phi (x)]^\frac{1}{2}, \ [\phi (a+x)]^n, \ \phi [(a+x)^n]. \)

4. \( e^x \log (x + a), \ a^{x^2}, \ a^x, \ e^x, \ 2^x, \ x^\circ \) (degrees).

5. \( \log (x + e^x), \ e^x + \log x, \ \frac{e^x}{\log x}. \)

6. \( e^{\log x}, \ \log (x e^x), \ \log x^a. \)

46. Differential Coefficient of \( \sin x. \)

If \( u = \phi (x) = \sin x, \)

\( \phi (x + h) = \sin (x + h), \)

and

\[ \frac{du}{dx} = \lim_{h \to 0} \frac{\sin (x + h) - \sin x}{h} \]

\[ = \lim_{h \to 0} \frac{2 \sin \frac{h}{2} \cos \left(x + \frac{h}{2}\right)}{h} \]

\[ = \lim_{h \to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \cos \left(x + \frac{h}{2}\right) \]

\[ = \cos x. \quad \text{[Art. 11; I.]} \]

47. Differential Coefficient of \( \cos x. \)

If \( u = \phi (x) = \cos x, \)

\( \phi (x + h) = \cos (x + h), \)

and

\[ \frac{du}{dx} = \lim_{h \to 0} \frac{\cos (x + h) - \cos x}{h} \]
\[ \sin \frac{h}{2} = - \lim_{h \to 0} \frac{\sin (x + \frac{h}{2}) - \sin x}{h} \]

\[ = - \sin x. \]

**Cor.** And as in previous cases the differential coefficients of \( \sin \phi(x) \) and \( \cos \phi(x) \) are respectively

\[ \cos \phi(x) \cdot \phi'(x), \]

and

\[ - \sin \phi(x) \cdot \phi'(x). \]

**EXAMPLES.**

Write down the differential coefficients of

1. \( \sin 2x, \sin nx, \sin^n x, \sin x^n, \sin \sqrt{x} \)
2. \( \sqrt{\sin \sqrt{x}}, \log \sin x, \log \sin \sqrt{x}, e^{\sin x}, e^{\sqrt{\sin x}} \)
3. \( \sin^m x \cos^n x, \sin^m x \cos^n x, \sin^n (x^n), e^{\pi \sin h \pi} \)
4. \( \sin x \sin 2x \sin 3x, \sin x \cdot \sin 2x / \sin 3x \)
5. \( \cos x \cos 2x \cos 3x, \cos x \cos 2x \cos 3x \)

48. The remaining circular functions can be differentiated from the definition in the same way. It is a little quicker however to proceed thus after obtaining the above results.

(i) If \( y = \tan x = \frac{\sin x}{\cos x} \);

\[ \frac{dy}{dx} = \frac{d}{dx}(\sin x) \cdot \cos x - \frac{d}{dx}(\cos x) \sin x \]

\[ \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x. \]
(ii) If \( y = \cot x = \frac{\cos x}{\sin x} \),
\[
\frac{dy}{dx} = \frac{(\sin x) \sin x - \cos x (\cos x)}{\sin^2 x} = -\csc^2 x.
\]

(iii) If \( y = \text{sec } x = (\cos x)^{-1} \),
\[
\frac{dy}{dx} = (-1)(\cos x)^{-2} \frac{d}{dx}(\cos x) = \frac{\sin x}{\cos^2 x} = \text{sec } x \tan x.
\]

(iv) If \( y = \cosec x = (\sin x)^{-1} \),
\[
\frac{dy}{dx} = (-1)(\sin x)^{-2} \frac{d}{dx}(\sin x) = -\frac{\cos x}{\sin^2 x} = -\cosec x \cot x.
\]

(v) If \( y = \text{vers } x = 1 - \cos x \),
\[
\frac{dy}{dx} = \sin x.
\]

(vi) If \( y = \text{covers } x = 1 - \sin x \),
\[
\frac{dy}{dx} = -\cos x.
\]

49. Differentiation of the inverse functions.

We may deduce the differential coefficients of all the inverse functions directly from the definition as shewn below.

For this method it seems useful to recur to the notation of Art. 27 and to denote \( \phi (x + h) \) by \( U \).

50. Then if \( u = \phi (x) = \sin^{-1} x \),
\[
U = \phi (x + h) = \sin^{-1} (x + h).
\]

Hence \( x = \sin u \), and \( x + h = \sin U \);
therefore \( h = \sin U - \sin u \),
and
\[
\frac{du}{dx} = Lt_{h=0} \frac{U - u}{h} = Lt_{U=1} \frac{U - u}{\sin U - \sin u}.
\]
\[ u = \sin^{-1} x, \quad x = \sin u; \]

whence \[ \frac{dx}{du} = \cos u; \]

and therefore \[ \frac{du}{dx} = \frac{1}{\cos u} = \frac{1}{\sqrt{1 - \sin^2 u}} = \frac{1}{\sqrt{1 - x^2}}, \]

and since \[ \cos^{-1} x = \frac{u}{2} - \sin^{-1} x, \]

we have \[ \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}. \]

(i) If \[ u = \tan^{-1} x, \]

we have \[ x = \tan u; \]

whence \[ \frac{dx}{du} = \sec^2 u; \]

and therefore \[ \frac{du}{dx} = \frac{1}{\sec^2 u} - \frac{1}{1 + \tan^2 u} - \frac{1}{1 + x^2}, \]

and since \[ \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x, \]
we have \[ \frac{d}{dx} \cot^{-1} x = \frac{1}{1 + x^2}. \]

(iii) If \( u = \sec^{-1} x, \)
we have \( x = \sec u; \)
whence \( \frac{dx}{du} = \sec u \tan u; \)
and therefore \( \frac{du}{dx} = \frac{\cos^2 u}{\sin u} = \frac{1}{1 - \frac{1}{x^2}} - \frac{1}{x} \sqrt{x^2 - 1}. \)

and since \( \csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x; \)
we have \( \frac{d}{dx} \csc^{-1} x = \frac{1}{x} \sqrt{x^2 - 1}. \)

(iv) If \( u = \vers^{-1} x, \)
we have \( x = \vers u = 1 - \cos u; \)
whence \( \frac{dx}{du} = \sin u; \)
and therefore \( \frac{du}{dx} = \frac{1}{\sin u} \cdot \frac{1}{\sqrt{1 - \cos^2 u}} = \frac{1}{\sqrt{2x - x^2}}; \)
whence also \( \frac{d}{dx} \vers^{-1} x = \frac{1}{\sqrt{2x - x^2}}. \)

**EXAMPLES.**

Write down the differential coefficients of each of the following expressions:

1. \( \sec x^2, \sec^{-1} x^2, \tan x^2, \tan^{-1} x^2, \vers x^2, \vers^{-1} x^2. \)
2. \( \tan^{-1} e^x, \tan e^x, \log \tan x, \log \tan^{-1} x, \log (\tan x)^{-1}. \)
3. \( \vers^{-1} \frac{x}{a}, \vers^{-1} (x + a), \tan \frac{x^2}{2}, \cos^{-1} \frac{1 - x^2}{1 + x^2}. \)
4. \( \sqrt{\vers x}, \tan^n x, (\tan^{-1} x)^q, \cdot \log \tan^{-1} x. \)
5. \( \tan x \cdot \sin^{-1} x, \sec^{-1} \tan x, \tan^{-1} \sec x, e^x \sin \omega x. \)
### Table of Results to be Committed to Memory

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = x^n$</td>
<td>$\frac{du}{dx} = nx^{n-1}$</td>
</tr>
<tr>
<td>$u = a^x$</td>
<td>$\frac{du}{dx} = a^x \log a$</td>
</tr>
<tr>
<td>$u = e^x$</td>
<td>$\frac{du}{dx} = e^x$</td>
</tr>
<tr>
<td>$u = \log_a x$</td>
<td>$\frac{du}{dx} = \frac{1}{x \log a e}$</td>
</tr>
<tr>
<td>$u = \log_e x$</td>
<td>$\frac{du}{dx} = \frac{1}{x}$</td>
</tr>
<tr>
<td>$u = \sin x$</td>
<td>$\frac{du}{dx} = \cos x$</td>
</tr>
<tr>
<td>$u = \cos x$</td>
<td>$\frac{du}{dx} = -\sin x$</td>
</tr>
<tr>
<td>$u = \tan x$</td>
<td>$\frac{du}{dx} = \sec^2 x$</td>
</tr>
<tr>
<td>$u = \cot x$</td>
<td>$\frac{du}{dx} = -\csc^2 x$</td>
</tr>
<tr>
<td>$u = \sec x$</td>
<td>$\frac{du}{dx} = \frac{\sin x}{\cos^2 x}$</td>
</tr>
<tr>
<td>$u = \csc x$</td>
<td>$\frac{du}{dx} = -\csc x \cot x$</td>
</tr>
<tr>
<td>$u = \sin^{-1} x$</td>
<td>$\frac{du}{dx} = \frac{1}{\sqrt{1 - x^2}}$</td>
</tr>
<tr>
<td>$u = \cos^{-1} x$</td>
<td>$\frac{du}{dx} = -\frac{1}{\sqrt{1 - x^2}}$</td>
</tr>
<tr>
<td>$u = \tan^{-1} x$</td>
<td>$\frac{du}{dx} = \frac{1}{1 + x^2}$</td>
</tr>
<tr>
<td>$u = \cot^{-1} x$</td>
<td>$\frac{du}{dx} = -\frac{1}{1 + x^2}$</td>
</tr>
</tbody>
</table>
Differential Calculus.

\[ u = \sec^{-1} x, \quad \frac{du}{dx} = \frac{1}{x \sqrt{x^2 - 1}} \]

\[ u = \csc^{-1} x, \quad \frac{du}{dx} = -\frac{1}{x \sqrt{x^2 - 1}} \]

\[ u = \text{vers}^{-1} x, \quad \frac{du}{dx} = \frac{1}{\sqrt{2x - x^2}} \]

\[ u = \text{covers}^{-1} x, \quad \frac{du}{dx} = -\frac{1}{\sqrt{2x - x^2}} \]

53. The Form \( w^v \). Logarithmic Differentiation.

In functions of the form \( w^v \), where both \( u \) and \( v \) are functions of \( x \), it is generally advisable to take logarithms before proceeding to differentiate.

Let \[ y = w^v, \]
then \[ \log_e y = v \log_e u; \]
therefore \[ \frac{1}{y} \frac{dy}{dx} = \frac{dv}{dx} \cdot \log_e u + v \cdot \frac{1}{u} \frac{du}{dx}, \text{ Arts. 31, 45,} \]
or \[ \frac{dy}{dx} = w^v \left( \log_e u \cdot \frac{dv}{dx} + \frac{v}{u} \frac{du}{dx} \right) ! \]

Three cases of this proposition present themselves.

1. If \( v \) be a constant and \( u \) a function of \( x \), \( \frac{dv}{dx} = 0 \) and the above reduces to

\[ \frac{dy}{dx} = v \cdot w^v \cdot \frac{du}{dx}, \]

as might be expected from Art. 43.
II. If \( u \) be a constant and \( v \) a function of \( x \), \( \frac{du}{dx} = 0 \) and the general form proved above reduces to
\[
\frac{dy}{dx} = u^v \log_e u \cdot \frac{dv}{dx},
\]
as might be expected from Art. 44.

III. If \( u \) and \( v \) be both functions of \( x \), it appears that the general formula
\[
\frac{dy}{dx} = u^v \log e u \cdot \frac{dv}{dx} + vu^{v-1} \frac{du}{dx}
\]
is the sum of the two special forms in I. and II., and therefore we may, instead of taking logarithms in any particular example, consider first \( u \) constant and then \( v \) constant and add the results obtained on these suppositions.

**Ex. 1.** Thus if \( y = (\sin x)^x \),
\[
\log y = x \log \sin x;
\]
therefore
\[
\frac{1}{y} \cdot \frac{dy}{dx} = \log \sin x + x \cot x,
\]
and
\[
\frac{dy}{dx} = (\sin x)^x \{\log \sin x + x \cot x\}.
\]

**Ex. 2.** In cases such as \( y = x^x + (\sin x)^x \), we cannot take logarithms directly. Let \( u = x^x \) and \( v = (\sin x)^x \).

Then
\[
\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx},
\]
but
\[
\log u = x \log x,
\]
and
\[
\log v = x \log \sin x;
\]
whence
\[
\frac{du}{dx} = x^x \{1 + \log x\},
\]
and
\[
\frac{dv}{dx} = (\sin x)^x \{\log \sin x + x \cot x\},
\]
and
\[
\frac{dy}{dx} = x^x \{1 + \log x\} + (\sin x)^x \{\log \sin x + x \cot x\}.
\]

The above compound process is called **Logarithmic differentiation** and is useful whenever variables occur
as an index or when the expression to be differentiated consists of a product of several involved factors.

**EXAMPLES:**

1. Differentiate \( v^{\sin x}, (\sin^{-1} x)^x, x^2, x^2 x. \)
2. Differentiate \((\sin x)^{\cos x} + (\cos x)^{\sin x}, (\tan x)^x + x^{\tan x}.\)
3. Differentiate \(\tan x \log x \times e^x \times x^x \times \sqrt{x}\).

54. **Transformations.** Occasionally an Algebraic or Trigonometrical transformation before beginning to differentiate will much shorten the work.

(i) For instance, suppose

\[ y = \tan^{-1} \frac{2x}{1-x^2}. \]

We observe that \(y = 2 \tan^{-1} x\);

whence

\[ dy = \frac{2}{1+x^2} \cdot dx. \]

(ii) Suppose \(y = \tan^{-1} \frac{1+x}{1-x}\).

Here, \(y = \tan^{-1} x + \tan^{-1} 1\),

and therefore

\[ dy = \frac{1}{1+x^2} \cdot dx. \]

(iii) If

\[ y = \tan^{-1} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}, \]

we have \(y = \tan^{-1} \frac{1}{4} \cdot \frac{1}{1+x^2} = \frac{\pi}{4} - \tan^{-1} \frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4} - 2 \cos^{-1} x^2; \)

\[ \therefore \quad \frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}}. \]

**EXAMPLES.**

1. \(\tan^{-1} \frac{3x}{1-3x^2}\)
2. \(\tan^{-1} \frac{p-q}{q+px}\)
3. \(\tan^{-1} \frac{\sqrt{1+x^2} - 1}{x}\)
4. \(\tan^{-1} \frac{x}{\sqrt{1-x^2}}\)
5. \(e^{\log x}\)
6. \(\sec^{-1} \frac{1}{1-2x^2}\).
7. \sec \tan^{-1} x.
8. \cos^{-1} \frac{y - x^{-1}}{x + y^{-1}}.
9. \sin^{-1} (3x - 4x^3).
10. \tan^{-1} \frac{\sqrt{x} - x}{1 + x^{2/3}}.
11. \cos^{-1} (1 - 2x^2).
12. \log \left\{ e^{x} \left( \frac{x - 2}{x + 2} \right)^{\frac{3}{2}} \right\}.

55. Examples of Differentiation.

Ex. 1. Let \( y = \sqrt{z} \), where \( z \) is a known function of \( x \).

Here
\[
y = z^{\frac{1}{2}},
\]
and
\[
\frac{dy}{dx} = \frac{1}{2} z^{-\frac{1}{2}} = \frac{1}{2} \sqrt{z},
\]
whence
\[
\frac{dy}{dx} = \frac{dz}{dx} \cdot \frac{1}{\sqrt{z}}, \quad \text{(Art. 35)}
\]
\[
= 2 \sqrt{z} \cdot \frac{dz}{dx}.
\]
This form occurs so often that it will be found convenient to commit it to memory.

Ex. 2. Let \( y = e^{\sqrt{\cot x}} \).

Here
\[
\frac{d}{dx} (e^{\sqrt{\cot x}}) = \frac{d}{dx} (\sqrt{\cot x}) \cdot \frac{d}{dx} (\cot x)
\]
\[
= e^{\sqrt{\cot x}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\cot x}} \cdot (\csc^2 x).
\]

Ex. 3. Let \( y = (\sin x)^{\ln x} \cot \{e^x (a + bx)\} \).

Taking logarithms
\[
\log y = \log x \cdot \log \sin x + \log \cot \{e^x (a + bx)\}.
\]
The differential coefficient of \( \log y \) is
\[
\frac{1}{y} \frac{dy}{dx}.
\]
Again, \( \log x \cdot \log \sin x \) is a product, and when differentiated becomes (Art. 31)
\[
\frac{1}{x} \log \sin x + \log x \cdot \cot x.
\]

Also, \( \log \cot \{e^x (a + bx)\} \) becomes when differentiated
\[
\frac{1}{\cot \{e^x (a + bx)\}} \cdot [\csc \{e^x (a + bx)\}] \cdot \{e^x (a + bx) + be^x\};
\]
\[
\therefore \quad \frac{dy}{dx} = (\sin x)^{\ln x} \cot \{e^x (a + bx)\} \left[ \frac{1}{x} \log \sin x + \cot x \cdot \log x \right.
\]
\[
\left. - 2e^{x} (a + b + bx) \csc 2(e^x (a + bx)) \right].
\]
Ex. 4. Let \( y = \sqrt{a^2 - b^2 \cos^2 (\log x)} \). Then

\[
\frac{dy}{dx} = \frac{d}{dx} \sqrt{a^2 - b^2 \cos^2 (\log x)} \times \frac{d}{d} \left\{ a^2 - b^2 \cos^2 (\log x) \right\} \times \frac{d}{dx} \left( \cos (\log x) \right) \times \frac{d (\log x)}{dx}
\]

\[
= \frac{1}{2} \left\{ a^2 - b^2 \cos^2 (\log x) \right\}^{-\frac{1}{2}} \times \left\{ -2b^2 \cos (\log x) \right\} \times \left\{ -\sin (\log x) \right\} \times \frac{1}{x}
\]

\[
= \frac{b^2 \sin 2 (\log x)}{2x \sqrt{a^2 - b^2 \cos^2 (\log x)}}.
\]

Ex. 5. Differentiate \( x^6 \) with regard to \( x^2 \).

Let \( x^2 = z \).

Then

\[
\frac{dx^6}{dz} = \frac{dx^5}{dx} \cdot \frac{dx}{dz} = \frac{dx^5}{dx} \cdot \frac{5x^4}{2x} = \frac{5x^4}{2x} = \frac{5x^3}{2}.
\]

56. **Implicit relation of \( x \) and \( y \).** So far we have been concerned with the case in which \( y \) is expressed explicitly, i.e. directly in terms of \( x \).

Cases however are of frequent occurrence in which \( y \) is not expressed directly in terms of \( x \), but its functionality is implied by an algebraic relation connecting \( x \) and \( y \).

In the case of such an **implicit relation**, we proceed as follows:—

Suppose for instance

\[ x^3 + y^3 = 3axy, \]

then

\[ 3x^2 + 3y^2 \frac{dy}{dx} = 3a \left( y + x \frac{dy}{dx} \right), \]

i.e.

\[ 3 \left( x^2 - ay \right) + 3 \left( y^2 - ax \right) \frac{dy}{dx} = 0, \]

giving

\[ \frac{dy}{dx} = \frac{ay}{y^2 - ax}. \]
57. **Partial Differentiation.** It will be perceived in the foregoing example that the expressions $3 (x^2 - ay)$ and $3 (y^3 - ax)$ occurring are algebraically the same as would be given by differentiating the expression $x^3 + y^3 - 3axy$ first with regard to $x$, keeping $y$ a constant, and second with regard to $y$, keeping $x$ a constant.

When such processes are applied to a function $f(x, y)$ of two or more variables the results are denoted by the symbols $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$. Thus in the above example

\[
\frac{\partial f}{\partial x} = 3 (x^2 - ay),
\]

and

\[
\frac{\partial f}{\partial y} = 3 (y^3 - ax).
\]

This is termed *partial* differentiation, and the results are called partial differential coefficients.

58. **A general proposition.** It appears that in the preceding example

\[
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0,
\]

\[
\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.
\]

This proposition is true for all implicit relations between two variables, such as $f(x, y) = 0$.

Suppose the function capable of expansion by any means in powers of $x$ and $y$, so that any general term may be denoted by $A x^p y^q$.

Then

\[
f(x, y) = \sum A x^p y^q = 0.
\]
DIFFERENTIAL CALCULUS.

Then differentiating

$$\Sigma \left( A_p x^{p-1} y^p + A x^p y y^{-1} \frac{dy}{dx} \right) = 0,$$

or

$$\Sigma A_p x^{p-1} y^p + (\Sigma A q x^p y^{q-1}) \frac{dy}{dx} = 0,$$

or

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0.$$

Ex. If 

$$f(x, y) = x^5 + x^4 y + y^3 = 0,$$

we have

$$\frac{\partial f}{\partial x} = 5x^4 + 4x^3 y,$$

$$\frac{\partial f}{\partial y} = x^4 + 3y^2;$$

$$\frac{dy}{dx} = \frac{5x^4 + 4x^3 y}{x^4 + 3y^2}.$$

EXAMPLES.

Find \( \frac{dy}{dx} \) in the following cases:

1. \( x^3 + y^3 = \alpha^3 \).  
2. \( x^n + y^n = \alpha^n \).  
3. \( e^v = xy \).
4. \( \log xy = x^2 + y^2 \).  
5. \( x^v, y^v = 1 \).  
6. \( x^v + y^v = 1 \).

59. Euler's Theorem.

If \( u = A x^a y^b + B x^c y^d + \ldots = \Sigma A x^a y^b, \) say, where

$$\alpha + \beta = \alpha' + \beta' = \ldots = n,$$

to show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu.$$

By partial differentiation we obtain

$$\frac{\partial u}{\partial x} = \Sigma A ax^{a-1} y^b,$$

$$\frac{\partial u}{\partial y} = \Sigma A \beta x^a y^{b-1},$$
EULER'S THEOREM.

Then
\[ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sum \alpha x^\alpha y^\beta + \sum \beta x^\alpha y^\beta \]

\[ = \sum (\alpha + \beta) x^\alpha y^\beta \]

\[ = \sum A x^\alpha y^\beta = nu. \]

It is clear that this theorem can be extended to the case of three or of any number of independent variables, and that if, for example,

\[ u = A x^\alpha y^\beta z^\gamma + B x^\alpha y^\beta z^\gamma + \ldots, \]

where \[ \alpha + \beta + \gamma = \alpha' + \beta' + \gamma' = \ldots = n, \]

then will
\[ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu. \]

The functions thus described are called homogeneous functions of the \( n \)th degree, and the above result is known as Euler's Theorem on homogeneous functions.

EXAMPLES.

Verify Euler's theorem for the expressions:

\( (x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^m + y^m), \quad \frac{1}{x^2 + xy + y^2}, \quad x^n \sin^2 y. \)

EXAMPLES.

Find \( \frac{dy}{dx} \) in the following cases:

1. \( y = \frac{2 + x^2}{1 + x} \)
2. \( y = \sqrt{\alpha + x} \)
3. \( y = \sqrt{\alpha^2 + x^2} \)
4. \( y = \sqrt{\frac{1 - x}{1 + x}} \)
5. \( y = \frac{1 - x^2}{\sqrt{1 + x^2}} \)
6. \( y = \frac{x\sqrt{x^2 - 4\alpha^2}}{\sqrt{x^2 - \alpha^2}} \)
7. \( y = \sqrt{\frac{1 - x}{1 + x + x^2}} \)
8. \( y = \log \frac{x^2 + x + 1}{x^2 - x + 1} \)
9. \( y = \tan^{-1} (\log x) \)
10. \( y = \sin x^2 \)
11. \( y = \sin (e^x) \log x \)
12. \( y = \tan^{-1} (e^x) \log \cot x \)
13. \( y = \log \cosh x \)
14. \( y = \text{vers}^{-1} \log (\cot x) \).
15. \[ y = \cot^{-1} (\csc x) \]  
16. \[ y = \sin^{-1} \frac{1}{\sqrt{1 + x^2}} \]

17. \[ y = \tan^{-1} \sqrt{\frac{1}{x^2 - 1}} \]

18. \[ y = (\tan^{-1} x)^n (\cos^{-1} x)^m \]

19. \[ y = \sin (\omega \log x) \sqrt{1 - (\log x)^2} \]

20. \[ y = \left( \frac{x}{n} \right)^n \left( 1 + \log \frac{3}{n} \right) \]

21. \[ y = h \tan^{-1} \left( \frac{x}{a} \cot \frac{x}{a} \right) \]

22. \[ y = \frac{x \cos^{-1} x}{\sqrt{1 - x^2}} \]

23. \[ y = \cos \left( x \sin^{-1} \frac{1}{x} \right) \]

24. \[ y = \sin^{-1} \frac{a + b \cos x}{b + a \cos x} \]

25. \[ y = e^{\tan^{-1} x \log (\sec^2 x)} \]

26. \[ y = e^{\tan^{-1} x} \cos (h \tan^{-1} x) \]

27. \[ y = \tan^{-1} \left( e^{c_1 x} \cdot x^2 \right) \]

28. \[ y = \sec \left( \log_a \sqrt{a^2 + x^2} \right) \]

29. \[ y = \tan^{-1} x + \frac{1}{2} \log \frac{1 + x}{1 - x} \]

30. \[ y = \log (\log x) \]

31. \[ y = \log^n (x) \text{ where } \log^n \text{ means } \log \log \log \ldots (\text{repeated } n \text{ times}) \]

32. \[ y = \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\sqrt{b + a + \sqrt{b - a \tan x}}}{\sqrt{b + a - \sqrt{b - a \tan x}}} \]

33. \[ y = \sin^{-1} (x \sqrt{1 - x} - \sqrt{x} \sqrt{1 - x^2}) \]

34. \[ y = 10^{10^x} \]

35. \[ y = e^{\sqrt{x}} \]

36. \[ y = e^{-x} \]

37. \[ y = x^x \]

38. \[ y = \sin x \]

39. \[ y = x^x + x^x \]

40. \[ y = (\cot x)^{\cosh x} + (\cosh x)^{\cot x} \]

41. \[ y = \tan^{-1} \left( \tan x \cdot x \cdot \sin \frac{1}{x} \right) \]

42. \[ y = \sin^{-1} (x \tan^{-1} x) \]

43. \[ y = \sqrt{\left( \frac{1 + \cos \frac{m}{x}}{x} \right) \left( 1 - \sin \frac{m}{x} \right)} \]

44. \[ y = \tan^{-1} \sqrt{x + \cos^{-1} x} \]

45. \[ y = \left( \frac{1 + \sqrt{x}}{1 + 2 \sqrt{x}} \right)^{\sin^{-1} x} \]

46. \[ y = (\cos x)^{\cosh x} \]

47. \[ y = (\cot^{-1} x)^{e^x} \]

48. \[ y = \left( 1 + \frac{1}{x} \right)^{\frac{1}{2} \cdot x^{1 + \frac{1}{2}}} \]

49. \[ y = b \tan^{-1} \left( \frac{x}{a + \tan^{-1} x} \right) \]
EXAMPLES.

50. \( \tan y = e^{\cos^2 x \sin x} \).

51. \( ax^2 + 2hxy + by^2 = 1 \).

52. \( e^y = \frac{(a + bx)^{\frac{1}{3}} - a^{\frac{1}{3}}}{(bx)^{\frac{1}{3}}} \).

53. \( (\cos x)^y = (\sin y)^x \).

54. \( x = e^{\tan^{-1} \frac{y}{x^2}} \).

55. \( x = y \log xy \).

56. \( y = x^y \).

57. \( y = x^y \).

58. \( y = x \log \frac{y}{a + bx} \).

59. \( ax^2 + 2hxy + by^2 + 2yv + cy + e = 0 \).

60. \( x^m y^n = (x + y)^{m + n} \).

61. \( y = e^{\tan^{-1} y \log \sec^2 x^3} \).

62. If \( y = \frac{1}{2} \left( \frac{a^2 - b^2}{a^2 + b^2} \right)^x \left( \frac{p + \sqrt{x}}{p + 1} + \frac{q + \sqrt{x}}{q + 1} \right) \), shew that when

\[
\left( \frac{a + b}{a - b} \right)^{2m} \text{ then } \frac{dy}{dx} = \left( \frac{a + b}{a - b} \right)^{2m}.
\]

63. Differentiate \( \log_{10} x \) with regard to \( x^2 \).

64. Differentiate \( (x^2 + ax + a^2)^n \log \cot \frac{x}{2} \) with regard to \( \tan^{-1} (a \cos bx) \).

65. Differentiate \( x^{2m-1} x \) with regard to \( \sin^{-1} x \).

66. Differentiate \( \tan^{-1} \sqrt{1 - x^2} \) with regard to \( \tan^{-1} x \).

67. Differentiate \( \frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}} \) with regard to \( \sqrt{1 - x^2} \).

68. Differentiate \( \frac{1}{2x^2 - 1} \) with regard to \( \sqrt{1 - x^2} \).

69. Differentiate \( \tan^{-1} \frac{x}{\sqrt{1 - x^2}} \) with regard to \( \sec^{-1} \frac{1}{2x^3 - 1} \).

70. Differentiate \( \tan^{-1} \frac{2x}{1 - x^2} \) with regard to \( \sin^{-1} \frac{2x}{1 + x^2} \).

71. Differentiate \( x^a \log \tan^{-1} x \) with regard to \( \frac{\sin \sqrt{x}}{x^4} \).

72. If \( y = x^y \) prove \( x \frac{dy}{dx} = \frac{y^2}{1 - y \log e} \).

E. D. C.
73. If \( y = \frac{x}{1 + x} \), prove \( \frac{dy}{dx} = \frac{1}{1 + 2x} \).

74. If \( y = x + \frac{1}{x} \), prove \( \frac{dy}{dx} = \frac{1}{x + 1} \).

75. If \( y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \sqrt{\text{etc. to } \infty}}}} \),

\[ y_1 = \cos x/(2y - 1). \]

76. If \( S_n = \) the sum of a g. r. to \( n \) terms of which \( r \) is the common ratio, prove that

\[ (r - 1) \frac{dS_n}{dr} = (n - 1) S_n - n S_{n-1}. \]

77. If \( P = a + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots + 1}}} \), prove \( \frac{d}{dx} \left( \frac{1}{P} \right) = \pm \frac{1}{P^2}. \)

78. Given \( C = 1 + r \cos \theta + \frac{r^2 \cos 2\theta}{2!} + \frac{r^3 \cos 3\theta}{3!} + \ldots \)

and \( S = r \sin \theta + \frac{r^2 \sin 2\theta}{2!} + \frac{r^3 \sin 3\theta}{3!} + \ldots \)

show that \( C \frac{dC}{dr} + S \frac{dS}{dr} = (C^2 + S^2) \cos \theta; \)

\( C \frac{dS}{dr} - S \frac{dC}{dr} = (C^2 + S^2) \sin \theta. \)

79. If \( y = \sec 4x \), prove that \( \frac{dy}{dx} = \frac{16t}{(1 - t^4)^2} \), where \( t = \tan x. \)

80. If \( y = e^{-x^2} \sec^{-1} \left( x \sqrt[4]{z} \right) \) and \( x^4 + x^2 z = x^2, \) find \( \frac{dy}{dx} \) in terms of \( x \) and \( z. \)

81. Prove that if \( x \) be less than unity

\[ \frac{1}{1 + x} + \frac{2x}{1 + x^2} + \frac{4x^2}{1 + x^4} + \frac{8x^4}{1 + x^8} + \ldots \text{ ad inf.} = \frac{1}{1 - x}. \]
EXAMPLES.

82. Prove that if \( x \) be less than unity
\[
\frac{1 - 2x}{1 - x^3 + \frac{2x - 4x^3}{1 - x^4 + x^8} + \frac{4x^3 - 8x^7}{1 - x^4 + x^8} + \ldots} = \frac{1 + 2x}{1 + x + x^2}.
\]

83. Given Euler's Theorem that
\[
M_n \to \cos \frac{x}{2^n} \cos \frac{x}{2^{2n}} \cos \frac{x}{2^{3n}} \ldots \cos \frac{x}{2^n} = \frac{\sin x}{x},
\]
prove \( \frac{1}{2} \tan \frac{x}{2} + \frac{1}{2^2} \tan \frac{x}{2^2} + \frac{1}{2^3} \tan \frac{x}{2^3} + \ldots \) ad inf. \( \frac{1}{x} - \cot x, \)
and \( \frac{1}{2^2} \sec^2 \frac{x}{2} + \frac{1}{2^4} \sec^2 \frac{x}{2^2} + \frac{1}{2^6} \sec^2 \frac{x}{2^3} + \ldots \) ad inf. \( \csc x \cdot \frac{1}{x}, \)

84. Differentiate logarithmically the expressions for \( \sin \theta \) and \( \cos \theta \) in factors, and deduce the sums to infinity of the following series

\[
\begin{align*}
(a) & \quad \frac{1}{\theta^2 - \pi^2} + \frac{1}{\theta^2 - 2^2\pi^2} + \frac{1}{\theta^2 - 3^2\pi^2} + \frac{1}{\theta^2 - 4^2\pi^2} + \ldots \\
(b) & \quad \frac{1}{1^2 + x^2} + \frac{1}{2^2 + x^2} + \frac{1}{3^2 + x^2} + \frac{1}{4^2 + x^2} + \ldots \\
(c) & \quad \frac{1}{1^2 + x^2} + \frac{1}{2^2 + x^2} + \frac{1}{3^2 + x^2} + \frac{1}{4^2 + x^2} + \ldots \\
(d) & \quad 1 + \frac{2}{1 + 2^2} + \frac{2}{1 + 3^2} + \ldots
\end{align*}
\]

85. Sum to infinity the series
\[
\frac{1}{1 + x} + \frac{1}{2} - \frac{1}{1 + x} + \frac{1}{4} - \frac{1}{1 + x} + \frac{1}{8} - \frac{1}{1 + x} + \ldots
\]

86. If \( H_n \) represent the sum of the homogeneous products of \( n \) dimensions of \( x, y, z \), prove

\[
\begin{align*}
(a) & \quad x \frac{\partial H_n}{\partial x} + y \frac{\partial H_n}{\partial y} + z \frac{\partial H_n}{\partial z} = nH_n^*; \\
(b) & \quad \frac{\partial H_n}{\partial x} + \frac{\partial H_n}{\partial y} + \frac{\partial H_n}{\partial z} = (n + 2) H_{n-1}.
\end{align*}
\]
CHAPTER V.

SUCCESSIVE DIFFERENTIATION.

60. When \( y \) is a given function of \( x \), and \( \frac{dy}{dx} \) has been found, we may proceed to differentiate a second time obtaining \( \frac{d}{dx} \left( \frac{dy}{dx} \right) \). This expression is called the second differential coefficient of \( y \) with respect to \( x \). We may then differentiate again and obtain the third differential coefficient and so on.

The expression \( \frac{d}{dx} \left( \frac{dy}{dx} \right) \) is abbreviated into \( \left( \frac{d}{dx} \right)^2 y \)
or \( \frac{d^2y}{dx^2} \); \( \frac{d}{dx} \left( \frac{d^2y}{dx^2} \right) \) is written \( \frac{d^3y}{dx^3} \); and so on.

Thus the several differential coefficients of \( y \) are written

\[
\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}, \ldots
\]

They are often further abbreviated into

\[ y_1, y_2, y_3, \ldots, y_n, \ldots \]

Ex. 1. Thus if \( y = x^n \), we have

\[
y_1 = nx^{n-1},
\]

\[
y_2 = n(n - 1)x^{n-2},
\]

\[
y_3 = n(n - 1)(n - 2)x^{n-3},
\]

and generally \( y_r = n(n - 1)(n - 2) \ldots (n - r + 1)x^{n-r} \)

\[
y_{n+1} = y_{n+2} = y_{n+3} = \ldots = 0.
\]
Ex. 2. If \( y = \tan x \),
\[
y_1 = \sec^2 x = 1 + y^2,
\]
\[
y_n = 2y_1 = y (y + y^3),
\]
\[
y_3 = 2 (1 + 3y^2) y_1 = 2 (1 + 4y^2 + 3y^4),
\]
\[
y_4 = 2 (8y + 12y^3) y_1 = 8 (2y + 5y^3 + 3y^5).
\]
&c.

Ex. 3. If \( y = (\sin^{-1} x)^2 \),
\[
y_1 = 2 (\sin^{-1} x) \sqrt{1 - x^2},
\]
\[
\therefore \text{ squaring, } (1 - x^2) y_1^2 = 4y.
\]
Hence differentiating, \( (1 - x^2) 2y_1 y_2 - 2x y_1^2 = 4y_1 \),
and dividing by \( 2y_1 \), \( (1 - x^2) y_2 - x y_1 = 2 \).

61. Standard results and processes.

The \( n \)th differential coefficient of some functions are easy to find.

- Ex. 1. If \( y = e^{ax} \) we have \( y_1 = a e^{ax}, y_2 = a^2 e^{ax}, \ldots \)
\[
y_n = a^n e^{ax}.
\]

Cor. i. If \( a = 1 \),
\[
y = e^x, \ y_1 = e^x, \ y_2 = e^x, \ldots \ y_n = e^x.
\]

Cor. ii.
\[
y = a^x = e^{x \log_e a}, \]
\[
y_1 = (\log_e a) e^{x \log_e a} = (\log_e a) a^x;
\]
\[
y_2 = (\log_e a)^2 e^{x \log_e a} = (\log_e a)^2 a^x;
\]

dtc. = etc.
\[
y_n = (\log_e a)^n e^{x \log_e a} = (\log_e a)^n a^x.
\]

Ex. 2. If \( y = \log_e (x + a) \);
\[
y_1 = \frac{1}{x + a}; \ y_2 = -\frac{1}{(x + a)^2}; \ y_3 = \frac{-1}{(x + a)^3} \ldots \;
\]
\[
\int \frac{(-1)(-2)(-3) \ldots (-n + 1)}{(x + a)^n} \int = (-1)^{n-1}(n - 1)!
\]
\[
(x + a)^n.
\]

Cor. If \( y = \frac{1}{x + a}, \ y_n = \frac{(-1)^n n!}{(x + a)^{n+1}} \).
Ex. 3. If \( y = \sin (a x + b) \);
\[ y_1 = a \cos (a x + b) = a \sin \left( a x + b + \frac{\pi}{2} \right); \]
\[ y_2 = a^2 \sin \left( a x + b + \frac{2\pi}{2} \right); \]
\[ y_3 = a^3 \sin \left( a x + b + \frac{3\pi}{2} \right); \]
\[ \ldots \ldots \ldots \ldots \ldots \]
\[ y_n = a^n \sin \left( a x + b + \frac{n\pi}{2} \right). \]

Similarly, if \( y = \cos (a x + b) \),
\[ y_n = a^n \cos \left( a x + b + \frac{n\pi}{2} \right). \]

Con. If \( a = 1 \) and \( b = 0 \);
then, when \( y = \sin x, \ y_n = \sin \left( x + \frac{n\pi}{2} \right) \);
and, when \( y = \cos x, \ y_n = \cos \left( x + \frac{n\pi}{2} \right) \).

Ex. 4. If \( y = e^{ax} \sin (bx + c) \);
\[ y_1 = ae^{ax} \sin (bx + c) + be^{ax} \cos (bx + c). \]

Let \( a = r \cos \phi \) and \( b = r \sin \phi \),
so that \( a^2 + b^2 \) and \( \tan \phi = \frac{b}{a} \);
and therefore \( y_1 = re^{ax} \sin (bx + c + \phi) \).

Thus the operation of differentiating this expression is equivalent to multiplying by \( r \) and adding \( \phi \) to the angle.

Thus \( y_2 = r^2 e^{ax} \sin (bx + c + 2\phi) \),
and generally \( y_n = r^n e^{ax} \sin (bx + c + n\phi) \).

Similarly, if \( y = e^{ax} \cos (bx + c) \),
\[ y_n = r^n e^{ax} \cos (bx + c + n\phi). \]

These results are often wanted and the student should be able to obtain them immediately.

Ex. 5. Find the \( n \)th differential coefficient of \( \sin^3 x \).

We have \( y = \sin^3 x = \frac{1}{4} (3 \sin x - \sin 3x) \).

Hence \( y_n = \frac{1}{4} \left( 3 \sin \left( x + \frac{n\pi}{2} \right) - 3^n \sin \left( 3x + \frac{n\pi}{2} \right) \right) \).
Ex. 6. If \( y = \sin^2 x \cos^2 x \), find \( y_n \).

Here
\[
y = \frac{1}{4} \sin^2 2x \cos x = \frac{1}{8} (1 - \cos 4x) \cos x
\]
\[
e = \frac{1}{16} (2 \cos x - \cos 3x - \cos 5x),
\]
and
\[
y_n = \frac{1}{16} \left\{ 2 \cos \left( x + \frac{n\pi}{2} \right) - 3^n \cos \left( 3x + \frac{n\pi}{2} \right) - 5^n \cos \left( 5x + \frac{n\pi}{2} \right) \right\}.
\]

**EXAMPLES.**

Find \( y_n \) in the following cases:

1. \( \frac{1}{ax + b} \)
2. \( \frac{1}{a - x} \)
3. \( \frac{1}{a - bx} \)
4. \( \frac{x}{a + bx} \)
5. \( \frac{ax + b}{cx + d} \)
6. \( \frac{x^2}{x - a} \)
7. \( (x + a)^i \)
8. \( \sqrt{x + a} \)
9. \( (x + a)^{-\frac{3}{2}} \)
10. \( \log (ax + b)^p \)
11. \( y = \sin x \sin 2x \)
12. \( y = e^x \sin x \sin 2x \)
13. \( y = e^x \sin^2 x \)
14. \( y = e^{ax} \cos^2 bx \)
15. \( y = \sin x \sin 2x \sin 3x \)
16. \( y = e^{3x} \sin^2 x \cos^3 x \)
17. \( y = \sin^2 x \sin 2x \)
18. \( y = e^{2x} \sin^2 x \sin 2x \)

**62. Use of Partial Fractions.**

Fractional expressions whose numerators and denominators are both rational algebraic expressions are differentiated \( n \) times by first putting them into partial fractions.

Ex. 1. \( y = \frac{x^2}{(x - a)(x - b)(x - c)} = \frac{a^2}{(a - b)(a - c)} \frac{1}{x - a} \)
\[
+ \frac{b^2}{(b - c)(b - a)} \frac{1}{x - b} + \frac{c^2}{(c - a)(c - b)} \frac{1}{x - c}
\]
(see note on partial fractions Art. 66);

therefore
\[
y_n = \frac{a^2}{(a - b)(a - c)} (-1)^n n! \frac{1}{(x - a)^{n+1}} + \frac{b^2}{(b - c)(b - a)} (-1)^n n! \frac{1}{(x - b)^{n+1}}
\]
\[
+ \frac{c^2}{(c - a)(c - b)} (-1)^n n! \frac{1}{(x - c)^{n+1}}.
\]
63. **Application of De Moivre's Theorem.**

When quadratic factors which are not resolvable into real linear factors occur in the denominator, it is often convenient to make use of De Moivre’s Theorem as in the following example.

Let
\[
y = \frac{1}{x^2 + a^2} = \frac{1}{(x + ia)(x - ia)}
\]

Then
\[
y_n = \frac{(-1)^n n!}{2ia} \left( \frac{1}{(x - ia)^{n+1}} - \frac{1}{(x + ia)^{n+1}} \right).
\]

Let \(x = r \cos \theta\) and \(a = r \sin \theta\),

whence
\[r^2 = x^2 + a^2\] and \(\tan \theta = \frac{a}{x}\).

Hence
\[
y_n = \frac{(-1)^n n!}{2iar^{n+1}} \left\{ (\cos \theta - i\sin \theta)^{-n-1} - (\cos \theta + i\sin \theta)^{-n-1} \right\}
\]

\[
= \frac{(-1)^n n!}{2iar^{n+1}} \cdot 2i \sin (n + 1) \theta
\]

\[
= (-1)^n \frac{n!}{r^{n+2}} \sin (n + 1) \theta \sin n^2 \theta,
\]

where \(\theta = \tan^{-1} \frac{a}{x}\).
Cor. 1. Similarly if \( y = \frac{1}{(c + b)^2 + a^2} \),
\[ y_n = \frac{(-1)^n n!}{a^{n+2}} \sin (n+1) \theta \sin^{n+1} \theta, \]
where \( \theta = \tan^{-1} \frac{a}{b + c} \).

Cor. 2. If \( y = \tan^{-1} \frac{t}{a} \), \( y_1 = \frac{a}{x^2 + a^2} \),
and
\[ y_n = \frac{(-1)^n (n-1)!}{a^n} \sin n\theta \sin^n \theta, \]
where \( \tan \theta = \frac{a}{x} = \cot y \).

**EXAMPLES.**

Find the \( n \)th differential coefficients of \( y \) with respect to \( x \) in the following cases:

1. \( y = \frac{1}{4a^2 - 1} \).
2. \( y = \frac{1}{4x^2 + 1} \).
3. \( y = \frac{1}{2} \log \frac{x + a}{x - a} \).
4. \( y = \frac{1}{x^2 - a^2} \).
5. \( y = \frac{1}{(x^2 - a^2)(x^2 + b^2)} \).
6. \( y = \frac{1}{(x^2 + a^2)(x^2 + b^2)} \).
7. \( y = \tan^{-1} \frac{2x}{1 - x^2} \).
8. \( y = x^2 + x + 1 \).
9. \( y = x^4 + x^2 + 1 \).
10. \( y = x^4 + x^3 + 2x^2 + x + 1 \).

64. **Leibnitz's Theorem.**

* [Lemma. If \( \binom{n}{r} \) denote the number of combinations of \( n \) things
  at a time then will

\[ n \binom{n}{r} + n \binom{n-1}{r+1} = n+1 \binom{n+1}{r+1}. \]

For
\[ \binom{n}{r} = \binom{n-1}{r+1} \frac{1}{r+1} \frac{1}{n-r-1} \left( \frac{1}{n-r} + \frac{1}{r+1} \right), \]

\[ \binom{n+1}{r+1} = \frac{1}{r+1} \frac{1}{n-r} \binom{n-1}{r+1}. \]
Let \( y = uv \), and let suffixes denote differentiations with regard to \( x \). Then

\[
y_1 = u_1v + uv_1,
\]

\[
y_2 = u_2v + 2uv_1 + uv_2, \text{ by differentiation.}
\]

Assume generally that

\[
y_n = u_nv + nC_1u_{n-1}v_1 + nC_2u_{n-2}v_2 + \ldots
\]

\[
+ nC_{r+1}u_{n-r-1}v_{r+1} + \ldots + uv_n \ldots (\alpha).
\]

Therefore differentiating

\[
y_{n+1} = u_{n+1}v + u_nv_1 \left( nC_1 \right) + u_{n-1}v_2 \left( +1 \right) + \ldots
\]

\[
+ u_{n-r}v_{r+1} \left( +C_{r+1} \right) + \ldots + uv_{n+1}
\]

\[
= u_{n+1}v + nC_1u_nv_1 + nC_2u_{n-1}v_2 + nC_{r+1}u_{n-r-1}v_{r+1} + \ldots + uv_{n+1}, \text{ by the Lemma;}
\]

therefore if the law \((\alpha)\) hold for \( n \) differentiations it holds for \( n+1 \).

But it was proved to hold for two differentiations, and therefore it holds for three; therefore for four; and so on; and therefore it is generally true, i.e.,

\[
(\alpha^n)_n = u_nv + nC_1u_{n-1}v_1 + nC_2u_{n-2}v_2 + \ldots
\]

\[
+ nC_{r+1}u_{n-r-1}v_{r+1} + \ldots + uv_n.
\]

65. Applications.

Ex. 1. \( y = x^2 \sin ax \).

Here we take \( \sin ax \) as \( u \) and \( x^2 \) as \( v \).

Now \( v_1 = 2x, v_2 = 2, v_3 = 3, v_4 = 0, \) and \( v_4 \) &c. are all zero.

Also \( u_n = a^n \sin \left( ax + \frac{n\pi}{2} \right) \), etc.

Hence by Leibnitz's Theorem we have

\[
y_n = x^2a^n \sin \left( ax + \frac{n\pi}{2} \right) + n3x^2a^{n-1} \sin \left( ax + \frac{n-1}{2} \pi \right)
\]
\[ + \frac{n(n-1)}{2!} 3 \cdot 2n x^n \sin \left( ax + \frac{n-2}{2} \pi \right) \]
\[ + \frac{n(n-1)(n-2)}{3!} 3 \cdot 2^n n-3 \sin \left( ax + \frac{n-3}{2} \right) \]

The student will note that if one of the factors be a power of \( x \)
it will be advisable to take that factor as \( v \).

Ex. 2. Let \( y = x^4 e^{ax} \); find \( y_5 \).

Here \( v = x^4, \ u = e^{ax} \),
so that \( v_1 = 4x^3, v_2 = 12x^2, v_3 = 24x, v_4 = 24, \) and \( v_n \) etc. all vanish.

Also \( u_n = u^n e^{ax} \), etc.

whence
\[ y_5 = n^5 e^{ax} x^4 + 5n^4 e^{ax} x^3 + 10 \cdot a e^{ax} x^2 + 2(2n^2 + 10a^2 x^n) \cdot 21x + 5a e^{ax} \cdot 24 \]
\[ = a e^{ax} \left( x^4 + 20a^3 x^3 + 120a^5 x^2 + 240a^7 x + 120 \right). \]

Ex. 3. Differentiate \( n \) times the equation
\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0. \]

\[ \frac{d^n}{dx^n} (x^2 y_2) = x^2 y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n(n-1)}{2!} 2y_n. \]

\[ \frac{d^n}{dx^n} (xy_1) = xy_{n+1} + ny_n, \]
\[ \frac{d^n y}{dx^n} = y_n; \]

therefore by addition
\[ x^2 y_{n+2} + (2n+1) xy_{n+1} + (n^2 + 1) y_n = 0, \]
or
\[ x^2 \frac{d^{n+2} y}{dx^{n+2}} + (2n+1)x \frac{d^{n+1} y}{dx^{n+1}} + (n^2 + 1) \frac{d^n y}{dx^n} = 0. \]

Ex. 4. Even when the general value of \( y_n \) cannot be obtained we
may sometimes find its value for \( x = 0 \) as follows.

Suppose \( y = \left[ \log (x + \sqrt{1 + x^2}) \right]^2 \),
then \( y_1 = 2 \log (x + \sqrt{1 + x^2})/\sqrt{1 + x^2}, \) \( \ldots \) \( (1), \)
and \( (1+x^2)y_1^2 = 4y, \)
whence differentiating and dividing by \( 2y_1 \),
\[ (1+x^2) y_2 + xy_1 = 2 \quad \ldots \quad \ldots \quad \ldots \quad (2). \]
Differential Calculus.

Differentiating \( n \) times by Leibnitz's Theorem

\[
(1 + x^2) y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n = 0
\]

or

\[
(1 + x^2) y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0.
\]

Putting \( x = 0 \) we have

\[
(y_{n+2})_0 = -n^2(y_n)_0, \ldots \quad \text{.....} \quad \ldots \quad \text{(3)},
\]

indicating by suffix zero the value attained upon the vanishing of \( x \).

Now, when \( x = 0 \) we have from the value of \( y \) and equations (1) and (2)

\[
(y)_0 = 0, \quad (y_1)_0 = 0, \quad (y_2)_0 = 2.
\]

Hence equation (3) gives

\[
(y_3)_0 = (y_0)_0 - (y_1)_0 = (y_2)_0 = 0
\]

and

\[
(y_4)_0 = -2^2 \cdot 2,
\]

\[
(y_5)_0 = 4^2 \cdot 2^2 \cdot 2,
\]

\[
(y_6)_0 = -6^2 \cdot 4^2 \cdot 2^2 \cdot 2,
\]

etc.,

\[
(y_{2k})_0 = (1)^{k-1} 2 \cdot 2^2 \cdot 4^2 \cdot 6^2 \ldots (2k-2)^2
\]

\[
= (-1)^k 1^{2k-1} ((k-1)!)^2.
\]

EXAMPLES.

Applying Leibnitz's Theorem to find \( y_n \) in the following cases:

1. \( y = x^2x^2 \).
2. \( y = x^2 \sin x \).
3. \( y = x^3 \log x \).
4. \( y = x^2 \sin x \).
5. \( y = e^{ax} \sin bx \).
6. \( y = \frac{a}{x} \).
7. \( y = x \tan^{-1} x \).
8. \( y = x^2 \tan^{-1} x \).

66. NOTE ON PARTIAL FRACTIONS.

Since a number of examples on successive differentiation and on integration depend on the ability of the student to put certain fractional forms into partial fractions, we give the methods to be pursued in a short note.

Let \( \phi(x) \) be the fraction which is to be resolved into its partial fractions.
1. If \( f(x) \) be not already of lower degree than the denominator, we can divide out until the numerator of the remaining fraction is of lower degree; e.g.

\[
\frac{j^2}{(x-1)(x-2)} = 1 + \frac{3x-2}{(x-1)(x-2)}.
\]

Hence we shall consider only the case in which \( f(x) \) is of lower degree than \( \phi(x) \).

2. If \( \phi(x) \) contain a single factor \((x-a)\), not repeated, we proceed thus: suppose

\[
\phi(x) = (x-a) \psi(x),
\]

and let

\[
f(x) \quad A \quad \frac{\chi(x)}{\psi(x)}
\]

\( (x-a) \psi(x) \quad \frac{x-a}{\psi(x)} \),

\( A \) being independent of \( x \).

Hence

\[
f(x) \quad A \quad \frac{\chi(x)}{\psi(x)}
\]

\( \psi(x) \)

This is an identity and therefore true for all values of the variable \( x \); put \( x = a \). Then, since \( \psi(\tilde{a}) \) does not vanish when \( x = a \) (for by hypothesis \( \psi(x) \) does not contain \( x - a \) as a factor), we have

\[
A = \frac{f(a)}{\psi(a)}.
\]

Hence the rule to find \( A \) is, "Put \( x = a \) in every portion of the fraction except in the factor \( x - a \) itself."

Ex. (i)

\[
\frac{x-c}{(x-a)(x-b)} = \frac{a^2 - c}{a - b} \cdot \frac{1}{x-a} + \frac{b - c}{a - b} \cdot \frac{1}{x-b}.
\]

Ex. (ii)

\[
\frac{x^2 + px + q}{(x-a)(x-b)(x-c)} = \frac{a^2 + p\tilde{a} + q}{(a-b)(a-c)} \cdot \frac{1}{x-a} + \frac{b^2 + p\tilde{b} + q}{(b-c)(a-b)} \cdot \frac{1}{x-b} + \frac{c^2 + p\tilde{c} + q}{(c-a)(c-b)} \cdot \frac{1}{x-c}.
\]

Ex. (iii)

\[
(x-1)(x-2)(x-3) = \frac{2}{x-1} - \frac{2}{x-2} + \frac{3}{x-3}.
\]

Ex. (iv)

\[
(x-a)(x-b).
\]

Here the numerator not being of lower degree than the denominator, we divide the numerator by the denominator. The result will then be expressible in the form \( 1 + \frac{A}{x-a} + \frac{B}{x-b} \), where \( A \) and \( B \) are found as before and are respectively \( \frac{a^2}{a-b} \) and \( \frac{b^3}{b-a} \).
3. Suppose the factor \((x - a)\) in the denominator to be repeated \(r\) times so that

\[
\phi(x) = (x - a)^r \psi(x).
\]

Put

\[
x - a = y.
\]

Then

\[
f(x) = \frac{f(a + y)}{(x - a)^r \psi(a + y)},
\]
or expanding each function by any means in ascending powers of \(y\),

\[
\frac{A_0 + A_1 y + A_2 y^2 + \ldots}{y^r (B_0 + B_1 y + B_2 y^2 + \ldots)}\]

Divide out thus:

\[
B_0 + B_1 y + \ldots | A_0 + A_1 y + \ldots | C_0 + C_1 y + C_2 y^2 + \ldots,
\]
and let the division be continued until \(y^r\) is a factor of the remainder.

Let the remainder be \(y^r \chi(y)\).

Hence the fraction

\[
\frac{C_0}{y^r} + \frac{C_1}{y^{r-1}} + \frac{C_2}{y^{r-2}} + \ldots + \frac{C_r}{y} + \chi(y)
\]

\[
= \frac{C_0}{(x - a)^r} + \frac{C_1}{(x - a)^{r-1}} + \frac{C_2}{(x - a)^{r-2}} + \ldots
\]

\[
+ C_r + \chi(x - a),
\]

Hence the partial fractions corresponding to the factor \((x - a)^r\) are determined by a long division sum.

Ex. Take

\[(x - 1)^3(x + 1)\]

Put

\[x - 1 = y.
\]

Hence the fraction

\[
\frac{(1 + y)^2}{y^3(2 + y)}
\]

\[
= \frac{1}{1 + \frac{3}{2}y + \frac{1}{2}y^2} \left( \frac{1}{y^2} + \frac{1}{2}y + \frac{1}{2}y^2 - \frac{1}{2}y - \frac{y^2}{2 + y} \right)
\]

\[
= \frac{\frac{1}{2}y^2 + \frac{1}{2}y'}{-\frac{1}{3}y^3}
\]

Therefore the fraction

\[
\frac{1}{2y^2 + 3y^2 + 8y - 8(2 + y)}
\]

\[
= \frac{1}{2(x - 1)^3} + \frac{3}{4(x - 1)^2} + \frac{1}{8(x - 1)} - \frac{1}{8(x + 1)}
\]
4. If a factor, such as \( x^2 + ax + b \), which is not resolvable into real linear factors occur in the denominator, the form of the corresponding partial fraction is \( \frac{Ax+B}{x^2+ax+b} \). For instance, if the expression be

\[
\frac{(x-a)}{(x-b)^2} \frac{(x^2+a^2)}{(x^2+b^2)^2}
\]

the proper assumption for the form in partial fractions would be

\[
\frac{B}{x-a} + \frac{Dx+E}{x-b} + \frac{Fx+G}{(x-b)^2} + \frac{Hx+K}{x^2+a^2} + \frac{m^2}{x^2+b^2} + \frac{n^2}{(x^2+b^2)^2},
\]

where \( A, B, \) and \( C \) can be found according to the preceding methods, and on reduction to a common denominator we can, by equating coefficients of like powers in the two numerators, find the remaining letters \( D, E, F, G, H, K \). Variations upon these methods will suggest themselves to the student.

**EXAMPLES.**

1. Given \( y = \sin x^2 \), find \( y_2, y_3, y_4 \).
2. Given \( y = x \sin x \), find \( y_2, y_3, y_4 \).
3. Given \( y = e^x \sin x \), find \( y_2, \ldots, y_6 \).
4. Given \( y = x^3 e^{ax} \), find \( y_3 \) and \( y_n \).
5. If \( y = Axe^{mx} + Be^{-mx} \), prove \( y_2 = m^2 y \).
6. If \( y = A \sin mx + B \cos mx \), prove \( y_2 = -m^2 y \).
7. If \( y = a \sin \log x \), prove \( y_2 + xy_1 + y = 0 \).
8. If \( y = \log \left( \frac{x^2 + bx}{c + bx} \right) \), prove \( x^2 y_2 = (y - xy_1)^2 \).
9. If \( y = A (x + \sqrt{x^2 - 1})^n + B (x - \sqrt{x^2 - 1})^n \), prove \( (x^2 - 1) y_2 + xy_1 - n^2 y = 0 \).
10. If \( y = \frac{(x-a)(x-b)}{(x-c)(x-d)} \), find \( y_n \).
11. If \( y = \frac{1}{(x-1)^2 (x-2)} \), find \( y_n \).
12. If \( y = e^{ax} \log x \), find \( y_2, y_3, y_n, y_{n+1} \).
13. If \( x = \cosh \left( \frac{x}{m} \log y \right) \), prove \( (x^2 - 1) y_2 + xy_1 - m^2 y = 0 \), and \( (x^2 - 1) y_{n+2} + (2n+1) xy_{n+1} + \frac{1}{2} (n^2 - m^2) y_n = 0 \).
14. Find \( y_n \) if 
\[
y = \frac{1}{x^3 + 1}
\]

15. Find \( y_n \) if 
\[
y = \frac{1}{(x - 1)^2 (x + 1)}
\]

16. Find \( y_n \) if 
\[
y = \frac{1}{(x - 1)^3 (x + 1)}
\]

17. Prove that if 
\[
y = \sin (m \sin^{-1} x),
\]

\[
(1 - x^2) y_{n+2} - (2n + 1) x y_{n+1} + (n^2 - m^2) y_n
\]

and 
\[
(1 - x^2) y_{n+2} = (2n + 1) x y_{n+1} + (n^2 - m^2) y_n.
\]

Hence show that 
\[
L y_n = \frac{y_{n+2} - (2n + 1) x y_{n+1} + (n^2 - m^2) y_n}{y_n}.
\]

18. If 
\[
y = e^{a x},
\]
prove that 
\[
(1 + x^2) y_{n+2} + (2(n + 1)x - 1) y_{n+1} + (n^2 + 1) y_n = 0.
\]

19. If 
\[
y = e^{n \sin^{-1} x},
\]
prove that 
\[
(1 - x^2) y_{n+2} - (2n + 1)x y_{n+1} + (n^2 + 1) y_n = 0,
\]

and 
\[
L y_n = \frac{y_{n+2} - (2n + 1)x y_{n+1} + (n^2 + 1) y_n}{y_n}.
\]

20. If 
\[
y = \sin \theta + \cos \theta
\]

21. If 
\[
y = e^{a x} \sin^2 x - 2n \cos x + (n + 1),
\]

\[y_n = a^{n+2} x^{n+2a^2},\]

22. If 
\[
x \cos \theta + y \sin \theta = a,
\]

\[x \sin \theta - y \cos \theta = b,
\]

prove that 
\[
\frac{d}{dx} y = \frac{d}{dy} x - \frac{d}{d\theta} \frac{d}{d\theta}
\]
is constant.

23. Prove that 
\[
d^n \left( \frac{\sin x}{x} \right) = \sum P \sin \left( x + \frac{m\pi}{n} \right) + Q \cos \left( x + \frac{m\pi}{n} \right)
\]

where 
\[
P = n^2 - n (n-1), x^{n-2} + n (n-1) (n-2) (n-3) x^{n-4} + \ldots,
\]

and 
\[
Q = n^2 - n (n-1) (n-2) x^{n-3} + \ldots,
\]

and 
\[
\sum P \sin \left( x + \frac{m\pi}{n} \right) + Q \cos \left( x + \frac{m\pi}{n} \right)
\]
EXAMPLES.

24. Prove
\[ \frac{d^n}{dx^n} \left( \frac{\cos x}{x} \right) = \left[ P \cos \left( x + \frac{n\pi}{2} \right) - Q \sin \left( x + \frac{n\pi}{2} \right) \right] / x^{n+1}, \]
where \( P \) and \( Q \) have the same values as in 23.

25. Prove that
\[ \frac{d^n}{dx^n} \left( \frac{e^{ax} \sin bx}{x} \right) = e^{ax} \left\{ P \sin (bx + n\phi) + Q \cos (bx + n\phi) \right\} / x^{n+1}, \]
where
\[ P = (rx)^n - n (rx)^{n-1} \cos \phi + n (n-1) (rx)^{n-2} \cos 2\phi - \ldots, \]
\[ Q = n (rx)^{n-1} \sin \phi - n (n-1) (rx)^{n-2} \sin 2\phi + \ldots, \]
\[ r^2 = a^2 + b^2, \text{ and } \tan \phi = b/a. \]

26. Prove that
\[ \frac{d^n}{dx^n} (x^n \sin x) = n! (P \sin x + Q \cos x), \]
where \[ P = 1 - nC_2 \frac{x^2}{2!} + nC_4 \frac{x^4}{4!} - \ldots \]
and \[ Q = nC_1 x - nC_3 \frac{x^3}{3!} + nC_5 \frac{x^5}{5!} - \ldots. \]

*27. Shew that
\[ \frac{d^n}{dx^n} \left( \frac{\log x}{x^m} \right) \]
\[ = \frac{(-1)^n n!}{(m-1)! x^{m-n}} \left[ \frac{(m+n-1)!}{n!} \log x - \sum_{r=0}^{r=n-1} \frac{(m+r-1)!}{n! (n-r)} \right]. \]

* [I. C. B., 1892.]
CHAPTER VI.

EXPANSIONS.

67. The student will have already met with several expansions of given explicit functions in ascending integral powers of the independent variable; for example, those tabulated on pages 10 and 11, which occur in ordinary Algebra and Trigonometry.

The principal methods of development in common use may be briefly classified as follows:

I. By purely Algebraical or Trigonometrical processes.

II. By Taylor’s or Maclaurin’s Theorems.

III. By the use of a differential equation.

IV. By Differentiation of a known series, or a converse process.

These methods we proceed to explain and exemplify.


Ex. 1. Find the first three terms of the expansion of log sec \( x \) in ascending powers of \( x \).

By Trigonometry

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots
\]

Hence

\[
\log \sec x = -\log \cos x = -\log (1 - z),
\]

where

\[
z = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots
\]
and expanding $\log (1 - z)$ by the logarithmic theorem we obtain

$$
\log \sec x = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots
$$

$$
= \left[ \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots \right] + \frac{1}{2} \left[ \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots \right] + \frac{1}{3} \left[ \frac{x^2}{2!} - \ldots \right]^3 + \ldots
$$

$$
= \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} + \frac{x^4}{8} - \frac{x^6}{48} + \ldots
$$

$$
+ \frac{x^6}{24} + \ldots
$$

hence

$$
\log \sec x = \frac{x}{2} + \frac{x^3}{12} + \frac{x^5}{45} + \ldots
$$

Ex. 2. Expand $\cos^3 x$ in powers of $x$.

Since

$$
4 \cos^3 x = \cos 3x + 3 \cos x
$$

$$
= 1 - \frac{3^2 x^2}{2!} + \frac{3^4 x^4}{4!} - \ldots + (-1)^n \frac{3^{2n} x^{2n}}{(2n)!} + \ldots
$$

$$
+ 3 \left[ \frac{1 - x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \right],
$$

we obtain

$$
\cos^3 x = \frac{1}{4} \left\{ (1 + 3) - (3^2 + 3) \frac{x^2}{2!} + (3^4 + 3) \frac{x^4}{4!} - \ldots + (-1)^n \frac{(3^{2n} + 3)}{(2n)!} \frac{x^{2n}}{(2n)!} + \ldots \right\}.
$$

Similarly

$$
\sin^4 x = \frac{1}{4} \left\{ (3^3 - 3) \frac{x^3}{3!} - (3^5 - 3) \frac{x^5}{5!} + (3^7 - 3) \frac{x^7}{7!} - \ldots + (-1)^n \frac{3^{2n-1} - 3}{(2n-1)!} \frac{x^{2n-1}}{(2n-1)!} + \ldots \right\}.
$$

Ex. 3. Expand $\tan x$ in powers of $x$ as far as the term involving $x^5$.

Since

$$
\tan x = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots}
$$

we may by actual division show that

$$
\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \ldots
$$
Ex. 4. Expand \( \frac{1}{2} \{ \log (1 + x) \}^2 \) in powers of \( x \).

Since

\[
(1 + x)^y = e^{y \log(1 + x)},
\]

we have, by expanding each side of this identity,

\[
1 + y + y \frac{(y - 1)}{2!} x + y \frac{(y - 1)(y - 2)}{3!} x^2 + y \frac{(y - 1)(y - 2)(y - 3)}{4!} x^3 + \ldots
\]

\[
\equiv 1 + y \log (1 + x) + \frac{y^2}{2!} \{ \log (1 + x) \}^2 + \ldots
\]

Hence, equating coefficients of \( y^2 \),

\[
\frac{1}{2} \{ \log (1 + x) \}^2 = \frac{x^2}{2!} - \frac{1 + 2}{3!} x^3 + \frac{1 + 2 + 2}{4!} x^4 + \ldots
\]

a series which may be written in the form

\[
\frac{x^2}{2} - \left(1 + \frac{1}{2}\right) \frac{x^3}{3} + \left(1 + \frac{1}{2} + \frac{1}{3}\right) \frac{x^4}{4} - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) \frac{x^5}{5} + \ldots
\]

**EXAMPLES.**

1. Prove \( e^{x \sin x} = 1 + x^2 + \frac{1}{3} x^4 + \frac{1}{120} x^6 + \ldots \)

2. Prove \( \cosh^n x = 1 + \frac{n x^2}{2!} + \frac{n (3n - 2)}{4!} x^4 + \ldots \)

3. Prove \( \log \frac{\sin x}{x} = -\frac{x^2}{6} - \frac{x^4}{180} + \ldots \)

4. Prove \( \log \frac{\sinh x}{x} = \frac{x^2}{6} - \frac{x^4}{180} + \ldots \)

5. Prove \( \log x \cot x = -\frac{x^2}{3} - \frac{1}{90} x^4 + \ldots \)

6. Prove \( \log \frac{\tan^{-1} x}{x} = -\frac{x^2}{3} + \frac{13}{90} x^4 - \frac{251}{5.7.9^3} x^6 + \ldots \)

7. Prove \( \log (1 - x + x^2) = -x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{3} - \frac{x^7}{7} + \frac{x^8}{8} \ldots \)

8. Expand \( \log (1 + x^3 e^x) \) as far as the term containing \( x^5 \).
9. Expand in powers of \( x \),

(a) \( \tan \frac{q + px}{1 + x^2} \),  
(b) \( \tan^{-1} \frac{3x - x^3}{1 - 3x^2} \),  
(c) \( \sin^{-1} \frac{2x}{1 + x^2} \),  
(d) \( \cos^{-1} \frac{x - x^3}{x + x^3} \)

69. Method II. Taylor's and Maclaurin's Theorems.

It has been discovered that the Binomial, Exponential, and other well-known expansions are all particular cases of one general theorem, which has for its object the expansion of \( f(x + h) \) in ascending integral positive powers of \( h \), \( f(x) \) being a function of \( x \) of any form whatever. It is found that such an expansion is not always possible, but the student is referred to a later chapter for a rigorous discussion of the limitations of the Theorem.

70. Taylor's Theorem.

- The theorem referred to is that under certain circumstances

\[
 f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots \\
 + \frac{h^n}{n!} f^n(x) + \ldots \
\]

an expansion of \( f(x + h) \) in powers of \( h \).
- This is known as Taylor's Theorem.

Assuming the possibility of expanding \( f(x + h) \) in a convergent series of positive integral powers of \( h \), let

\[
 f(x + h) = A_0 + A_1 h + A_2 \frac{h^2}{2!} + A_3 \frac{h^3}{3!} + \ldots \quad (1),
\]

where \( A_0, A_1, A_2, \ldots \) are functions of \( x \) alone which are to be determined.
Now \( \frac{df(x+h)}{dh} = \frac{df(x+h)}{d(x+h)} \cdot \frac{d(x+h)}{dh} = f'(x+h) \),

for \( x \) and \( h \) are independent quantities and therefore \( x \) may be considered constant in differentiating with regard to \( h \), so that \( \frac{d(x+h)}{dh} = 1 \).

Similarly

\[ \frac{d^2f(x+h)}{dh^2} = f''(x+h) ; \text{ and so on.} \]

Differentiating (1) then with regard to \( h \), we have

\[ f'(x+h) = \frac{df(x+h)}{dh} = A_1 + A_2 h + A_3 \frac{h^2}{2!} + A_4 \frac{h^3}{3!} + \ldots (2), \]

\[ f''(x+h) = \frac{df'(x+h)}{dh} = A_2 + A_3 h + A_4 \frac{h^2}{2!} + \ldots (3), \]

\[ f'''(x+h) = \frac{df''(x+h)}{dh} = A_3 + A_4 h + \ldots (4), \]

etc. = etc.

Putting \( h = 0 \), we have at once from (1), (2), etc.

\[ A_0 = f(x), \quad A_1 = f'(x), \quad A_2 = f''(x), \quad A_3 = f'''(x), \text{ etc.,} \]

where \( f'(x), f''(x), f'''(x), \ldots \) are the several differential coefficients of \( f(x) \) with respect to \( x \). Substituting these values in (1),

\[ f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots. \]

Ex. 1. Let \( f(x) = x^n \).

Then \( f'(x) = nx^{n-1}, \ f''(x) = n(n-1) x^{n-2}, \text{ etc., and} \)

\[ f(x+h) = (x+h)^n. \]

Thus Taylor's Theorem gives the Binomial expansion

\[ (x+h)^n = x^n + nhx^{n-1} + \frac{n(n-1)}{2!} h^2 x^{n-2} + \ldots \]
**Ex. 2.** Let \( f(x) = \sin x \).

Then \( f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \) etc., and

\[
f(x + h) = \sin (x + h).
\]

Thus we obtain

\[
\sin (x + h) = \sin x + h \cos x - \frac{h^3}{2} \sin x - \frac{h^3}{6} \cos x + \ldots.
\]

**EXAMPLES.**

Prove the following results :

1. \( e^{x + h} = e^x + he^x + \frac{h^2}{2!} e^x + \frac{h^3}{3!} e^x + \ldots \)

2. \( \tan^{-1}(x + h) \)

\[
= \tan^{-1} x + \frac{h}{1 + x^2} - \frac{hx^2}{(1 + x^2)^2} - \frac{1 - 3x^2}{3!} \frac{h^3}{(1 + x^2)^3} 3! + \ldots.
\]

3. \( \sin^{-1}(x + h) \)

\[
= \sin^{-1} x + \frac{h}{\sqrt{1 - x^2}} + \frac{x - x^2}{(1 - x^2)^{3/2} 2!} + \frac{1 - 3x^2}{3!} \frac{h^3}{(1 - x^2)^3} 3! + \ldots.
\]

4. \( \sec^{-1}(x + h) \)

\[
= \sec^{-1} x + \frac{h}{r^2 (r^2 - 1)^{3/2} 2!} - \frac{2r^2 - 1}{r^2 (r^2 - 1)^{5/2} 2!} \frac{h^3}{r^4} + \ldots.
\]

5. \( \log \sin (x + h) \)

\[
= \log \sin x + h \cot x - \frac{h^4}{2} \csc^2 x + \frac{h^3}{3} \cos x + \ldots.
\]

**71. Stirling's or Maclaurin's Theorem.**

If in Taylor's expansion

\[
f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots
\]

we put 0 for \( x \), and \( x \) for \( h \), we arrive at the result

\[
f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots
\]

\[
+ \frac{x^n}{n!} f^{(n)}(0) + \ldots
\]
the meaning of \( f^r(0) \) being that \( f(x) \) is to be differentiated \( r \) times with respect to \( x \), and then \( x \) is to be put zero in the result.

This result is generally known as Maclaurin's Theorem. Being a form of Taylor's Theorem it is subject to similar limitations.

Ex. 1. Expand \( \sin x \) in powers of \( x \).

Here

\[
\begin{align*}
  f(x) &= \sin x, \\
  f'(x) &= \cos x, \\
  f''(x) &= -\sin x, \\
  f'''(x) &= -\cos x, \\
  & \quad \text{&c.} \\
  f^n(x) &= \sin \left( x + \frac{n\pi}{2} \right). \\
  f^n(0) &= \sin \frac{n\pi}{2}
\end{align*}
\]

Hence \( f(0) = 0 \), \( f'(0) = 1 \), \( f''(0) = 0 \), \( f'''(0) = -1 \), \( f^n(0) = \sin \frac{n\pi}{2} \),

Thus \( \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \).

Ex. 2. Expand \( \log \cos x \) in powers of \( x \).

Here

\[
\begin{align*}
  f(x) &= \log \cos x, \\
  f'(x) &= -\tan x = -t, \text{ say,} \\
  f''(x) &= -\sec^2 x = -(1 + t^2), \\
  f'''(x) &= -2 \tan x \sec^2 x = -2t(1 + t^2), \\
  f^{(4)}(x) &= -2(1 + 3t^2)(1 + t^2) = -2(1 + 4t^2 + 3t^4), \\
  f^{(5)}(x) &= -2(8t + 12t^3)(1 + t^2) = -2(8t + 20t^3 + 12t^5), \\
  f^{(6)}(x) &= -2(8 + 60t^2 + 60t^4)(1 + t^2) = -2(8 + 68t^2 + 120t^4 + 60t^6), \\
  & \quad \text{etc.}
\end{align*}
\]

Whence \( f(0) = \log \cos 0 = \log 1 = 0 \),

and \( f'(0) = f^{(3)}(0) = f^{(5)}(0) = \cdots = 0 \),

also \( f''(0) = -1 \), \( f^{(4)}(0) = -2 \), \( f^{(6)}(0) = -16 \), etc.

Hence

\[
\log \cos x = -\frac{x^2}{2} - 2\frac{x^4}{4!} - 16\frac{x^6}{6!} - \text{etc.}
\]
EXAMPLES.

Apply Maclaurin's Theorem to prove

1. \[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \ldots. \]

2. \[ \log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^n \frac{x^n}{n} + \ldots. \]

3. \[ \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^n \frac{x^{2n-1}}{2n-1} + \ldots. \]

4. \[ e^{\sin x} = 1 + x + \frac{1}{2} x^2 - \frac{1}{8} x^4 - \ldots. \]

5. \[ \log (1 + e^x) = \log 2 + \frac{1}{2} x + \frac{1}{8} x^2 - \frac{x^4}{192} \ldots. \]

6. \[ e^{ax} \cos bx = 1 + ax + \frac{a^2 - b^2}{2} x^2 + \frac{a(a^2 - 3b^2)}{3!} x^3 + \ldots \]

\[ + \left( \frac{a^2 + b^2}{n!} \right)^n x^n \cos \left( n \tan^{-1} \frac{b}{a} \right) + \ldots. \]

72. METHOD III. By the formation of a Differential Equation. First form a differential equation as in Ex. 3, Art. 60, etc., and assume the series

\[ a_0 + a_1 x + a_2 x^2 + \ldots \]

for the expansion.

Substitute the series for \( y \) in the differential equation and equate coefficients of like powers of \( x \) in the resulting identity. We thus obtain sufficient equations to find all the coefficients except one or two of the first which may easily be obtained from the values of \( f(0), f'(0), \) etc.

Ex. 1. To apply this method to the expansion of \( (1 + x)^n \).

Let \[ y = (1 + x)^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \ldots \ldots (1). \]

Then \[ y_1 = n (1 + x)^{n-1} \text{ or } (2 + x) y_1 = ny \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2). \]

But \[ y_1 = a_1 + 2a_2 x + 3a_3 x^2 + \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3). \]
Therefore substituting from (1) and (3) in the differential equation (2)

\[(1 + x)(a_1 + 2a_2x + 3a_3x^2 + \ldots) = n(a_0 + a_1x + a_2x^2 + \ldots).\]

Hence, comparing coefficients

\[a_1 = na_0,\]
\[2a_2 + a_1 = na_1,\]
\[3a_3 + 2a_2 = na_2, \text{ etc.},\]

and by putting \(x = 0\) in equation (1),

\[a_1 = n,\]
\[a_2 = \frac{n - 1}{2} a_1 = \frac{n(n - 1)}{2!},\]
\[a_3 = \frac{n - 2}{3} a_2 = \frac{n(n - 1)(n - 2)}{3!}, \text{ etc.},\]
\[a_r = \frac{n - r + 1}{r} a_{r-1} = \frac{n(n - 1) \ldots (n - r + 1)}{r!},\]

whence

\[(1 + x)^n = 1 + nx + \frac{n(n - 1)}{2!} x^2 + \ldots\]

**Ex. 2.** Let \(y = f(x) = (\sin^{-1} x)^2.\)
\[y_1 = 2 \sin^{-1} x \cdot \frac{1}{\sqrt{1 - x^2}},\]
\[\therefore (1 - x^2)y_1^2 = 4y.\]

Differentiating, and dividing by \(2y_1,\) we have

\[(1 - x^2)y_2 = xy_1 + 2 \ldots \ldots \ldots (1).\]

Now, let \(y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots + a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \ldots,\)

therefore

\[y_1 = a_1 + 2a_2x + \ldots + na_nx^n + (n + 1) a_{n+1}x^n + (n + 2) a_{n+2}x^{n+1} + \ldots,\]

and

\[y_2 = 2a_2 + \ldots + n(n - 1)a_{n-1}x^{n-1} + (n + 1) na_nx^n + (n + 2)(n + 1)a_{n+1}x^{n+1} + \ldots.\]

Picking out the coefficient of \(x^n\) in the equation (which may be done without actual substitution) we have

\[(n + 2)(n + 1) a_{n+2} - n(n - 1) a_n = n(a_n);\]

therefore

\[a_{n+2} = \frac{n^2}{(n + 1)(n + 2)} a_n \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2).\]
Now, \[ a_0 = f(0) = (\sin^{-1} 0)^9, \]
and if we consider \( \sin^{-1} x \) to be the smallest positive angle whose sine is \( x \),
\[ \sin^{-1} 0 = 0. \]

Hence
\[ a_0 = 0. \]
Again,
\[ a_1 = f'(0) = 2 \sin^{-1} 0 \cdot \frac{1}{\sqrt{1 - 0^2}} = 0, \]
and
\[ a_2 = \frac{1}{2} f''(0) = \frac{1}{2} \left( \frac{2}{1} - 0 + 0 \right) = 1. \]

Hence, from equation (2), \( a_3, a_5, a_7, \ldots \), are each \( 0 \),
and
\[ a_4 = \frac{2^2}{3 \cdot 4}, \quad a_2 = \frac{2^4}{3 \cdot 4}, \quad a_4 = \frac{2^6}{5 \cdot 6}, \quad a_6 = \frac{2^8}{5 \cdot 6}, \quad \text{etc.} \]
therefore
\[ (\sin^{-1} x)^2 = \frac{2x^2}{2!} + \frac{2^2}{4!} 2x^4 + \frac{2^4}{6!} 4x^6 + \frac{2^6}{8!} 4x^8 + \ldots \]

A different method of proceeding is indicated in the following example:—

Ex. 3. Let
\[ y = \sin (m \sin^{-1} x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \quad \ldots \quad (1). \]
Then
\[ y_1 = \cos (m \sin^{-1} x) \cdot \frac{m}{\sqrt{1 - x^2}}, \]
whence
\[ (1 - x^2) y_1^2 = m^2 (1 - y^2). \]
Differentiating again, and dividing by \( 2y_1 \), we have
\[ (1 - x^2) y_2 - xy_1 + m^2 y = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2). \]
Differentiating this \( n \) times by Leibnitz's Theorem
\[ (1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} + (m^2 - n^2) y_n = 0. \quad \ldots \quad \ldots \quad (3). \]
Now
\[ a_0 = (y)_x = 0 = \sin (m \sin^{-1} 0) = 0, \]
(assuming that \( \sin^{-1} x \) is the smallest positive angle whose sine is \( x \))
\[ a_1 = (y_1)_x = m, \]
\[ a_2 = (y_2)_x = 0, \]
\[ \text{etc.} \]
\[ a_n = (y_n)_x = 0. \]
Hence, putting \( x = 0 \) in equation (3),
\[
a_{n+2} = -(m^2 - n^2)a_n.
\]
Hence \( a_4, a_6, a_8, \ldots \), each = 0,
and
\[
\begin{align*}
a_9 &= -(m^2 - 1^2)a_1 = -m(m^2 - 1^2), \\
a_7 &= -(m^2 - 3^2)a_3 = m(m^2 - 1^2)(m^2 - 3^2), \\
a_5 &= -(m^2 - 5^2)a_5 = -m(m^2 - 1^2)(m^2 - 3^2)(m^2 - 5^2), \\
\end{align*}
\]
etc.

Whence
\[
\sin (m \sin^{-1} x) = mx - \frac{m(m^2 - 1^2)}{3!} x^3 + \frac{m(m^2 - 1^2)(m^2 - 3^2)}{5!} x^5 \\
- \frac{m(m^2 - 1^2)(m^2 - 3^2)(m^2 - 5^2)}{7!} x^7 + \ldots.
\]

The corresponding series for \( \cos (m \sin^{-1} x) \) is
\[
\cos (m \sin^{-1} x) = 1 - \frac{m^2 x^2}{2!} + \frac{m^2(m^2 - 2^2)}{4!} x^4 \\
- \frac{m^2(m^2 - 2^2)(m^2 - 4^2)}{6!} x^6 + \ldots.
\]

If we write \( x = \sin \theta \) these series become
\[
\begin{align*}
\sin m\theta &= m \sin \theta - \frac{m(m^2 - 1^2)}{3!} \sin^3 \theta \\
&\quad + \frac{m(m^2 - 1^2)(m^2 - 3^2)}{5!} \sin^5 \theta - \text{etc.}, \\
\cos m\theta &= 1 - \frac{m^2}{2!} \sin^2 \theta + \frac{m^2(m^2 - 2^2)}{4!} \sin^4 \theta \\
&\quad - \frac{m^2(m^2 - 2^2)(m^2 - 4^2)}{6!} \sin^6 \theta + \text{etc.}
\end{align*}
\]

**EXAMPLES.**

1. Apply this method to find the known expansions of
   \( a^2 \), \( \log (1 + x) \), \( \sin x \), \( \tan^{-1} x \).

2. If \( y = \sin^{-1} x = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \),
   prove
   \[
   \begin{align*}
   (1) & \quad (1 - x^2) y_2 = xy_1, \\
   (2) & \quad (n + 1)(n + 2)a_{n+2} = n^2a_n, \\
   (3) & \quad \sin^{-1} x = x + \frac{x^3}{3} + \frac{1}{3} \cdot \frac{x^5}{5} + \ldots.
   \end{align*}
   \]
3. If \( y = e^{a \sin^{-1} x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \),
prove

\[ (1 - x^2)^n y_2 = x y_1 + a^2 y, \]
\[ (n+1) (n+2) a_{n+2} = (n^2 + a^2) a_n, \]
\[ e^{a \sin^{-1} x} = 1 + a x + \frac{a^2 x^2}{2!} + \frac{a (a^2 + 1)}{3!} x^3 + \frac{a^2 (a^2 + 2^2)}{4!} x^4 + \frac{a (a^2 + 1) (a^2 + 3^2)}{5!} x^5 + \ldots, \]

(4) Deduce from (3) by expanding the left side by the exponential theorem and equating coefficients of \( a, a^2, a^3 \ldots \) the series for \( \sin^{-1} x \), \( (\sin^{-1} x)^2 \), \( (\sin^{-1} x)^3 \).

4. Prove that

\[ (\tan^{-1} x)^2 \]
\[ = \frac{x^2}{2} - \left( 1 + \frac{1}{3} \right) \frac{x^4}{4} + \left( 1 + \frac{1}{3} + \frac{1}{5} \right) \frac{x^6}{6} - \left( 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \right) \frac{x^8}{8} + \ldots. \]

5. Prove that

\[ (a) \quad \frac{1}{2} \left[ \log (x + \sqrt{1 + x^2}) \right]^2 = \frac{x^2}{2} - \frac{2}{3} \frac{x^4}{4} + \frac{2}{5} \frac{x^6}{6} - \ldots, \]
\[ \log (x + \sqrt{1 + x^2}) = \frac{x}{3} x^3 + \frac{2}{3} \frac{x^5}{5} - \ldots. \]

73. Method IV. Differentiation or integration of a known series. The method of treatment is best indicated by examples.

Ex. 1. If we differentiate the series

\[ \sin^{-1} x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1.8}{2.4.6} \frac{x^5}{5} + \frac{1.3.5}{2.4.6.7} x^7 + \ldots. \]

we obtain the binomial expansion

\[ \frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2} x^2 + \frac{1.8}{2.4} x^4 + \frac{1.3.5}{2.4.6.8} x^6 + \ldots, \]

and it is clear that we must be able by a reverse process (integration) to infer the first series from the second.

The student unacquainted with integration may obtain the expansion of \( \sin^{-1} x \) from that of \( (1 - x^2)^{-1/2} \) as follows:
Let \( \sin^{-1} x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \),
then differentiating
\[
\frac{1}{\sqrt{1 - x^2}} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots.
\]
But
\[
\frac{1}{\sqrt{1 - x^2}} = 1 + \frac{1}{2} x^2 + \frac{1}{2 \cdot 4} x^4 + \ldots.
\]
Hence
\[ a_1 = 1, \quad 2a_2 = 0, \quad 3a_3 = \frac{1}{2}, \quad 4a_4 = 0, \quad 5a_5 = \frac{1}{2 \cdot 4}, \text{ etc.} \]
Also
\[ a_0 = \sin^{-1} 0 = 0 \]
(if we take the smallest positive value of the inverse function).

Hence substituting the values of these coefficients
\[
\sin^{-1} x = x + \frac{1}{2} x^3 + \frac{1}{2 \cdot 4} x^5 + \ldots.
\]

Ex. 2. We have proved in Ex. 2 Art. 72 that
\[
\frac{(\sin^{-1} x)^2}{2!} = x^2 + \frac{2^2 x^4}{2!} + \frac{2^2 \cdot 4^2 x^6}{4!} + \ldots.
\]
Hence differentiating we arrive at a new series
\[
\frac{\sin^{-1} x}{\sqrt{1 - x^2}} = x + \frac{2^2}{3!} x^3 + \frac{2^2 \cdot 4^2}{5!} x^5 + \ldots.
\]
If we put \( x = \sin \theta \) we may write this as
\[
\frac{2\theta}{\sin 2\theta} = 1 + \frac{2^2}{3!} \sin^2 \theta + \frac{2^2 \cdot 4^2}{5!} \sin^4 \theta + \frac{2^2 \cdot 4^2 \cdot 6^2}{7!} \sin^6 \theta + \ldots.
\]
or
\[
= 1 + \frac{2^2}{3!} \sin^2 \theta + \frac{2 \cdot 4}{3 \cdot 5} \sin^4 \theta + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \sin^6 \theta + \ldots.
\]

**EXAMPLES.**

1. Obtain in this manner the expansion of
\[
\log (1 + x), \quad \tan^{-1} x, \quad \log \frac{1 + x}{1 - x}.
\]

2. Prove
\[
\log (x + \sqrt{1 + x^2}) = x - \frac{1}{2} x^3 + \frac{1}{2 \cdot 3} x^5 + \frac{1}{2 \cdot 4} x^7 - \ldots.
\]
3. Expand
\[ \sin^{-1} \frac{2x}{1+x^2}, \tan^{-1} \frac{x}{\sqrt{1-x^2}}, \tan^{-1} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} \]
in powers of \( x \).

4. Prove
\[
\left( \frac{\theta}{\sin \theta} \right)^2
= 1 + \frac{2^2}{3 \cdot 4} \sin^2 \theta + \frac{2^2 \cdot 4^2}{3 \cdot 4 \cdot 5 \cdot 6} \sin^4 \theta + \frac{2^2 \cdot 4^2 \cdot 6^2}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \sin^6 \theta + \ldots.
\]

5. Prove that
\[
e^{a \sin^{-1} x} \frac{1}{\sqrt{1-x^2}}
= 1 + \frac{ax}{1!} + \frac{(a^2+1^2) x^2}{2!} + \frac{a(a^2+2^2) x^3}{3!} + \frac{(a^2+1^2)(a^2+3^2) x^4}{4!} + \ldots,
\]
\[
\frac{\sin \theta}{\cos \theta}
= 1 + \frac{\sin \theta}{1!} + \frac{(1+1^2) \sin^2 \theta}{2!} + \frac{(1+2^2) \sin^3 \theta}{3!} + \frac{(1+1^2)(1+3^2) \sin^4 \theta}{4!} + \ldots.
\]

EXAMPLES.

1. Prove
\[ \log (1 + \tan x) = x - \frac{1}{2} x^2 + \frac{2}{3} x^3 + \ldots. \]

2. Prove
\[ e^{x \cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11x^4}{24} - \frac{x^5}{5} \ldots. \]

3. Prove
\[
\log \left( \frac{1}{x} e^x \log (1+x) \right) = \frac{5x^2}{24} - \frac{x^3}{8} + \frac{251}{2880} x^4 + \ldots.
\]

4. Prove
\[ \log (1 - x + x^2) = -x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{3} - \frac{x^7}{7} + \frac{x^8}{8} + \ldots. \]
5. Prove \[ \cosh (x \cos x) = 1 + \frac{x^2}{2} - \frac{11x^4}{24} \ldots, \]
\[ \sinh (x \cos x) = x - \frac{x^3}{3} - \frac{x^5}{5} \ldots. \]

6. Prove \[ e \log \frac{\tan x}{x} = \frac{x^3}{3} + \frac{7}{50} x^5 \ldots. \]

7. Prove \[ \cos^{-1} (\tanh \log x) = -\pi - 2 \left\{ x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \right\}. \]

8. Prove \[ \tan^{-1} \sqrt{\frac{1 + x^2}{x^2} - 1} = \frac{x}{2} - \frac{x^3}{6} + \frac{x^5}{10} - \frac{x^7}{14} + \ldots. \]

9. Prove \[ \log (3x^2 + 4x^3 + \sqrt{1 + 8x^2 + 16x^4}) \]
\[ = 3 \left\{ x - \frac{1}{2} \frac{x^3}{3} + \frac{3}{2} \frac{x^5}{5} - \ldots \right\}. \]

10. Prove that

(a) \[ (1 - x^2)^{3/2} \sin^{-1} x = x - \frac{x^3}{3} - \frac{2}{3} \frac{x^5}{5} - \frac{2}{3} \frac{4}{5} \frac{x^7}{7} - \ldots, \]

(b) \[ \cot \theta = 1 - \frac{\sin^2 \theta}{3} - \frac{2}{3} \frac{\sin^4 \theta}{5} - \frac{2.4}{3.5} \frac{\sin^6 \theta}{7} - \ldots, \]

(c) \[ \frac{\pi}{4} = 1 - \frac{1}{3} \left( \frac{1}{2} \right) - \frac{2}{3} \frac{1}{5} \left( \frac{1}{2} \right)^2 - \frac{2.4}{3.5} \frac{1}{7} \left( \frac{1}{2} \right)^3 - \ldots. \]

11. Prove that

\[ (r + \sqrt{1 + r^2})^n = 1 + nr + \frac{n^2 r^2}{2!} + \frac{n(n^2 - 1^2)}{3!} r^3 + \frac{n^2(n^2 - 2^2)}{4!} r^4 + \frac{n(n^2 - 1^2)(n^2 - 3^2)}{5!} r^5 + \ldots, \]

and deduce the expansions of

\[ \log (x + \sqrt{1 + x^2}), \quad \frac{1}{2!} \{\log (x + \sqrt{1 + x^2})\}^2, \quad \frac{1}{3!} \{\log (x + \sqrt{1 + x^2})\}^3. \]

12. If \[ y = e^{ax} \cos bx, \]
prove that \[ y_x - 2ay_x + (a^2 + b^2)y = 0, \]
and hence that

\[ e^{ax} \cos bx = 1 + ax + \frac{a^2 - b^2}{2!} x^2 + \frac{a(a^2 - 3b^2)}{3!} x^3 + \ldots. \]