13  Prove

(i) \( \sin (m \tan^{-1} \phi) (1 + \phi^2)^m \)
\[ = m(m-1)(m-2) \ldots \frac{1}{3} \]

(ii) \( \cos (m \tan^{-1} \phi) (1 + \phi^2)^m \)
\[ = 1 - m(m-1)2^m + m(m-1)(m-2)4^m \ldots \]

14. Deduce from 13 (i)

\( \tan^{-1} \alpha \log \sqrt{1 + \phi^2} \)
\[ = \frac{1}{2} \alpha^3 - \left( \frac{1}{3} + \frac{1}{2} + \frac{1}{1} \right) \alpha^4 \]

15  Prove

(a) \( \cosh \theta = 1 + \frac{\sin \theta}{2} \cdot (1 + \frac{\sin \theta}{3}) \cdot (1 + \frac{\sin \theta}{4}) \cdot \ldots \)
(b) \( \sinh \theta = \frac{\cosh \theta - 1}{\cosh \theta + 1} \cdot \sinh \theta \cdot \cosh \theta + \sinh \theta \cdot \cosh \theta \cdot \sinh \theta \cdot \cosh \theta \cdot \ldots \)

16  Prove

\( \tan^{-1} \alpha + 1 = \tan^{-1} (1 + h \sin \theta) \sin \theta \cdot \frac{h \sin \theta}{2} \sin 2\theta \cdot \frac{(\sin \theta)^3}{3} \sin 3\theta \cdot \frac{(\sin \theta)^4}{4} \sin 4\theta \cdot \ldots \)

where \( h = \cot \theta \)

17. Deduce from 16

(i) \( \frac{\pi}{2} - \theta + \cos \theta \sin \theta + \frac{\cos^2 \theta}{2} \sin 2\theta + \frac{\cos^3 \theta}{3} \sin 3\theta + \ldots \)

by putting \( h = 1 + \cot \theta \)

(ii) \( \frac{\pi}{2} - \theta - \sin \theta + \frac{\sin \theta}{2} \sin 2\theta + \frac{\sin^2 \theta}{3} \sin 3\theta + \frac{1}{4} \sin 4\theta + \ldots \)

by putting \( h = \sqrt{1 + \phi} \)

(iii) \( \frac{\pi}{2} - \sin \theta + \frac{1}{2} \sin 2\theta + \frac{1}{3} \sin 3\theta + \frac{1}{4} \sin 4\theta + \ldots \)

by putting \( h = 1 + \tan \theta \)
18. Show that

\[ \frac{1}{2!} \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}} = \frac{x^2}{2!} + 1^2 \cdot 3^2 \left( \frac{1}{1^2 + \frac{1}{3^2}} \right) \frac{x^4}{4!} + 1^2 \cdot 3^2 \cdot 5^2 \left( \frac{1}{1^2 + \frac{1}{3^2} + \frac{1}{5^2}} \right) \frac{x^6}{6!} + \ldots, \]

\[ \frac{\theta^2}{\sin 2\theta} = \frac{\sin \theta}{2!} + 1^2 \cdot 3^2 \left( \frac{1}{1^2 + \frac{1}{3^2}} \right) \frac{\sin^3 \theta}{4!} + 1^2 \cdot 3^2 \cdot 5^2 \left( \frac{1}{1^2 + \frac{1}{3^2} + \frac{1}{5^2}} \right) \frac{\sin^5 \theta}{6!} + \ldots. \]

19. Prove

\[ \tan^{-1}x = \frac{1}{2} x^3 - \frac{1}{3} \left( \frac{1}{2} + \frac{1}{4} \left( 1 + \frac{1}{3} \right) \right) \frac{x^5}{5} + \frac{1}{6} \left( \frac{1}{2} + \frac{1}{4} \left( 1 + \frac{1}{3} \right) \right) \frac{x^7}{7} + \ldots. \]

20. Prove

\[ (a) \sqrt{2x} = 1 + \frac{1}{3} x^2 + \frac{1}{3} \cdot 5 \frac{x^4}{2!} + \frac{1}{3} \cdot 5 \cdot 7 \frac{x^6}{3!} + \ldots, \]

\[ (b) \left( \frac{\text{vers}^{-1}x}{2} \right)^2 = x + \frac{1}{3} \frac{x^2}{2} + \frac{1}{3} \cdot 5 \frac{x^3}{3} + \frac{1}{3} \cdot 5 \cdot 7 \frac{x^4}{4} + \ldots. \]

21. Prove that

\[ \frac{f(x + h) + f(x - h)}{2} = f(x) + \frac{h^2}{2!} f''(x) + \frac{h^4}{4!} f'''(x) + \ldots. \]

22. Prove that

(a) \( f(mx) \)

\[ f(x) + (m - 1) x f'(x) + (m - 1)^2 \frac{x^2}{2!} f''(x) + (m - 1)^3 \frac{x^3}{3!} f'''(x) + \ldots, \]

(b) \( f \left( \frac{x^2}{1+x} \right) \)

\[ = f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{2!} f''(x) - \frac{x^3}{3!} f'''(x) + \ldots, \]

(c) \( f(x) = f(0) + x f'(x) - \frac{x^2}{2!} f''(x) + \frac{x^3}{3!} f'''(x) - \text{etc.} \)
CHAPTER VII.

INFINITESIMALS.

74. Orders of Smallness.

If we conceive any magnitude $A$ divided into any large number of equal parts, say a billion ($10^{12}$), then each part $\frac{A}{10^{12}}$ is extremely small, and for all practical purposes negligible, in comparison with $A$. If this part be again subdivided into a billion equal parts, each $= \frac{A}{10^{24}}$, each of these last is extremely small in comparison with $\frac{A}{10^{12}}$, and so on. We thus obtain a series of magnitudes, $A, \frac{A}{10^{12}}, \frac{A}{10^{24}}, \frac{A}{10^{36}}, \ldots$, each of which is excessively small in comparison with the one which precedes it, but very large compared with the one which follows it. This furnishes us with what we may designate a scale of smallness.

75. More generally, if we agree to consider any given fraction $f$ as being small in comparison with unity, then $fA$ will be small in comparison with $A$, and we may term the expressions $fA, f^2A, f^3A, \ldots$, small quantities of the first, second, third, etc., orders; and the numerical quantities $f, f^2, f^3, \ldots$, may be called small fractions of the first, second, third, etc., orders.
Thus, supposing $A$ to be any given finite magnitude, any given fraction of $A$ is at our choice to designate a small quantity of the first order in comparison with $A$. When this is chosen, any quantity which has to this small quantity of the first order a ratio which is a small fraction of the first order, is itself a small quantity of the second order. Similarly, any quantity whose ratio to a small quantity of the second order is a small fraction of the first order is a small quantity of the third order, and so on. So that generally, if a small quantity be such that its ratio to a small quantity of the $p$th order be a small fraction of the $q$th order, it is itself termed a small quantity of the $(p+q)$th order.

76. *Infinitesimals.*

If these small quantities $Af, Af^2, Af^3, \ldots$, be all quantities whose limits are zero, then supposing $f$ made smaller than any assignable quantity by sufficiently increasing its denominator, these small quantities of the first, second, third, etc., orders are termed infinitesimals of the first, second, third, etc., orders.

From the nature of an infinitesimal it is clear that, if any equation contain finite quantities and infinitesimals, the infinitesimals may be rejected.

77. Prop. In any equation between infinitesimals of different orders, none but those of the lowest order need be retained.

Suppose, for instance the equation to be

$$A_1 + B_1 + C_1 + D_2 + E_3 + F_4 + \ldots = 0 \ldots \ldots (i),$$

each letter denoting an infinitesimal of the order indicated by the suffix.

Then, dividing by $A_1$,

$$1 + \frac{B_1}{A_1} + \frac{C_1}{A_1} + \frac{D_2}{A_1} + \frac{E_3}{A_1} + \frac{F_4}{A_1} + \ldots = 0 \ldots \ldots (ii),$$
the limiting ratios \( \frac{B_1}{A_1} \) and \( \frac{C_1}{A_1} \) are finite, while \( \frac{D_2}{A_1}, \frac{E_2}{A_1} \) are infinitesimals of the first order, \( \frac{F_3}{A_1} \) is an infinitesimal of the second order, and so on. Hence, by Art. 76, equation (ii) may be replaced by

\[
1 + \frac{B_1}{A_1} + \frac{C_1}{A_1} = 0,
\]

and therefore equation (i) by

\[
A_1 + B_1 + C_1 = 0,
\]

which proves the statement.

78. Prop. In any equation connecting infinitesimals we may substitute for any one of the quantities involved any other which differs from it by a quantity of higher order.

For if \( A_1 + B_1 + C_1 + D_2 + \ldots = 0 \)

be the equation, and if \( A_1 = F_1 + f_2, \)

\( f_2 \) denoting an infinitesimal of higher order than \( F_1 \), we have

\[
F_1 + B_1 + C_1 + f_2 + D_2 + \ldots = 0,
\]

i.e. by the last proposition we may write

\[
F_1 + B_1 + C_1 = 0,
\]

which may therefore, if desirable, replace the equation

\[
A_1 + B_1 + C_1 = 0.
\]

79. Illustrations.

(1) Since

\[
\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots.
\]

and

\[
\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots.
\]

\( \sin \theta, 1 - \cos \theta, \theta - \sin \theta \) are respectively of the first, second, and third orders of small quantities, when \( \theta \) is of the first order; also, \( \theta \) may be written instead of \( \cos \theta \) if second order quantities are to be rejected, and \( \theta \) for \( \sin \theta \) when cubes and higher powers are rejected.
Again, suppose $AP$ the arc of a circle of centre $O$ and radius $a$. Suppose the angle $AOP (=\theta)$ to be a small quantity of the first order. Let $PN$ be the perpendicular from $P$ upon $OA$ and $AQ$ the tangent at $A$, meeting $OP$ produced in $Q$. Join $P$, $A$.

Then $\text{arc } AP = a\theta$ and is of the first order,

\[
NP = a \sin \theta \quad \text{do. do.,}
\]

\[
AQ = a \tan \theta \quad \text{do. do.,}
\]

chord $AP = 2a \sin \frac{\theta}{2} \quad \text{do. do.,}$

\[
N.A = a (1 - \cos \theta) \text{ and is of the second order.}
\]

So that $OP - ON$ is a small quantity of the second order.

\[
\begin{align*}
\text{Again, arc } AP - \text{chord } AP & = a\theta - 2a \sin \frac{\theta}{2} \\
& = a\theta - 2a \left( \frac{\theta}{2} - \frac{\theta^3}{8} + \ldots \right) \\
& = a\theta - 4.3! - \text{etc.,}
\end{align*}
\]

and is of the third order.

\[
PQ - NA = NA \text{ (sec } \theta - 1) \\
\Rightarrow 2\sin^2 \frac{\theta}{2} = \overline{NA} \\
= \text{(second order)(second order)} \\
= \text{fourth order of small quantities,}
\]

and similarly for others.

80. The base angles of a triangle being given to be small quantities of the first order, to find the order of the difference between the base and the sum of the sides.

By what has gone before, (Art. 79 (2)), if $APB$ be the
triangle and \( PM \) the perpendicular on \( AB \), \( AP - AM \)

and \( BP - BM \) are both small quantities of the second order as compared with \( AB \).

Hence \( AP + PB - AB \) is of the second order compared with \( AB \).

If \( AB \) itself be of the first order of small quantities, then \( AP + PB - AB \) is of the third order.

**81. Degree of approximation in taking a small chord for a small arc in any curve.**

Let \( AB \) be an arc of a curve supposed continuous between \( A \) and \( B \), and so small as to be concave at each point throughout its length to the foot of the perpendicular from that point upon the chord. Let \( AP, BP \) be the tangents at \( A \) and \( B \). Then, when \( A \) and \( B \) are taken sufficiently near together, the chord \( AB \) and the angles at \( A \) and \( B \) may each be considered small quantities of at least the first order, and therefore, by what has gone before, \( AP + PB - AB \) will be at least of the third order. Now we may take as an axiom that the length of the arc \( AB \) is intermediate between the length of the chord \( AB \) and the sum of the tangents \( AP, BP \). Hence the difference of the arc \( AB \) and the chord \( AB \), which is less than that between \( AP + PB \) and the chord \( AB \), must be at least of the third order.
Differential Calculus.

Examples.

1. In the figure on page 86 suppose $PM$ drawn at right angles to $AQ$, and prove
   
   \((a)\) Segment cut off by $AP$ is of the third order of small quantities,
   
   \((b)\) Triangle $PM$ is of the third order,
   
   \((c)\) Triangle $PQM$ is of the fifth order.

2. $OA_1B$ is a triangle right-angled at $A_1$ and of which the angle at $O$ is small and of the first order. $A_1B_1$ is drawn perpendicular to $OB$, $B_1A_2$ to $A_1B$, $A_2B_2$ to $OB$, and so on.

Prove
   
   \((a)\) $A_nB_n$ is a small quantity of the $(2n - 1)^{th}$ order,
   
   \((b)\) $B_nA_{n+1}$ is of the $2n^{th}$ order,
   
   \((c)\) $B_nB$ is of the $2n^{th}$ order,
   
   \((d)\) triangle $B_1A_nB_n$ is of the $(2m + 2n - 1)^{th}$ order.

3. A straight line of constant length slides between two straight lines at right angles, viz. $C_1A_1$, $CB$; $AB$, $ab$ are two positions of the line, and $P$ their point of intersection. Show that, in the limit, when the two positions coincide, we have

\[
\frac{AA}{CB} = \frac{PA}{CB^2} \quad \text{and} \quad \frac{BB}{CB} = \frac{PB}{CB^2}.
\]

4. From a point $T$ in a radius of a circle, produced, a tangent $TP$ is drawn to the circle touching it in $P$. $PN$ is drawn perpendicular to the radius $OA$. Show that, in the limit when $P$ moves up to $A$,

\[
\frac{AA}{TA} = \frac{AT}{TA}.
\]

5. Tangents are drawn to a circular arc at its middle point and at its extremities; show that the area of the triangle formed by the chord of the arc and the two tangents at the extremities is ultimately four times that of the triangle formed by the three tangents.

6. A regular polygon of $n$ sides is inscribed in a circle. Show that when $n$ is very great the ratio of the difference of the circumferences to the circumference of the circle is approximately $\pi^2/6n^2$.

7. Show that the difference between the perimeters of the earth and that of an inscribed regular polygon of ten thousand sides is less than a yard (rad. of Earth = 4000 miles).
8. The sides of a triangle are 5 and 6 feet and the included angle exceeds 60° by 10°. Calculating the third side for an angle of 60°, find the correction to be applied for the extra 10°.

9. A person at a distance $q$ from a tower of height $p$ observes that a flag-pole upon the top of it subtends an angle $\theta$ at his eye. Neglecting his height, show that if the observed angle be subject to a small error $a$, the corresponding error in the length of the pole has to the calculated length the ratio

$$\frac{qa}{\csc \theta (q \cos \theta - p \sin \theta)}.$$

10. If in the equation $\sin (\omega - \theta) = \sin \omega \cos a$, $\theta$ be small, show that its approximate value is

$$2 \tan \omega \sin^2 \frac{\omega a}{2} \left(1 - \tan^2 \omega \sin^2 \frac{\omega a}{2}\right) \quad \text{[L.C.S.]}
$$

11. A small error $x$ is made in measuring the side $a$ of a triangle, a small error $y$ in measuring $b$, and a small error $\xi$ in measuring $c$. Prove that the consequent errors in $A$ and $B$ are each $\frac{1}{2} \mu$, provided the relation

$$2 \frac{b \mu - ay}{a^2 - b^2} \sin \xi = \mu \sin 1''$$

be satisfied. [L. C. S., 1892.]
CHAPTER VIII.

TANGENTS AND NORMALS.

82. Equation of TANGENT.

It was shown in Art. 18 that the equation of the
tangent at the point \((x, y)\) on the curve \(y = f(x)\) is

\[ Y - y = \frac{dy}{dx} (X - x) \]  

\((1)\),

\(X\) and \(Y\) being the current co-ordinates of any point on
the tangent.

Suppose the equation of the curve to be given in the
form \(f(x, y) = 0\).

It is shown in Art. 58 that

\[ \frac{dy}{dx} = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}. \]

Substituting this expression for \(\frac{dy}{dx}\) in \((1)\) we obtain

\[ Y - y = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} (X - x), \]

\[ \left( \right. \]

\[ \left. \right. \]
or
\[(X - x) \frac{\partial f'}{\partial x} + (Y - y) \frac{\partial f'}{\partial y} = 0.\] *(2)*

for the equation of the tangent.

If the partial differential coefficients \(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y},\) etc. be denoted by \(f_x, f_y,\) etc., equation (2) may then be written
\[(X - x)f_x + (Y - y)f'_y = 0.\]

**83. Simplification for Algebraic Curves.**

If \(f(x, y)\) be an algebraic function of \(x\) and \(y\) of degree \(n\), suppose it made homogeneous in \(x, y,\) and \(z\) by the introduction of a proper power of the linear unit \(z\) wherever necessary. Call the function thus altered \(f(x, y, z)\). Then \(f(x, y, z)'\) is a homogeneous algebraic function of the \(n^{th}\) degree; hence we have by Euler's Theorem (Art. 59)
\[xf_x + yf_y + zf_z = nf(x, y, z) = 0,
\]
by virtue of the equation to the curve.

Adding this to equation (2), the equation of the tangent takes the form
\[Xf_x + Yf_y + zf_z = 0.\] *(3)*

where the \(z\) is to be put = 1 after the differentiations have been performed.

We often for the sake of symmetry write \(Z\) instead of \(z\) in this equation and write the tangent in the form
\[Xf_x + Yf_y + Zf_z = 0.\]

**Ex.**

\[f(x, y) = x^4 + ax^2y^2 + b^2y^4 + c^4 = 0.\]

The equation, when made homogeneous in \(x, y, z\) by the introduction of a proper power of \(z\), is
\[f(x, y, z) = x^4 + ax^2yz^2 + b^2y^4 + c^4z^4 = 0,
\]
and
\[f_x = 4x^3 + a^2y^2z^2,
\]
\[f_y = a^2xz^2 + b^2z^4,
\]
\[f_z = 2a^2xyz + 3b^2yz^2 + 4c^4z^3.
\]
Substituting these in Equation 3, and putting \( z = x = 1 \), we have for the equation of the tangent to the curve at the point \((x, y)\)

\[
X (4x^3 + a^2y) + Y (a^2x + b^3) + 2a^2xy + 3b^2y + 4c^4 = 0.
\]

With very little practice the introduction of the \( z \) can be performed mentally. It is generally more advantageous to use equation (3) than equation (2), because (3) gives the result in its simplest form, whereas if (2) be used it is often necessary to reduce by substitutions from the equation of the curve.

84. NORMAL.

Def. The normal at any point of a curve is a straight line through that point and perpendicular to the tangent to the curve at that point.

Let the axes be assumed rectangular. The equation of the normal may then be at once written down. For if the equation of the curve be

\[ y = f(x), \]

the tangent at \((x, y)\) is

\[ Y - y = \frac{dy}{dx} (X - x), \]

and the normal is therefore

\[ (X - x) + (Y - y) \frac{dy}{dx} = 0. \]

If the equation of the curve be given in the form

\[ f(x, y) = 0, \]

the equation of the tangent is

\[ (X - x) f_x + (Y - y) f_y = 0, \]

and therefore that of the normal is

\[ \frac{X - x}{f_x} = \frac{Y - y}{f_y}. \]
Ex. 1. Consider the ellipse \( \frac{x^2}{a^2} \frac{y^2}{b^2} = 1 \).

This requires \( z^2 \) in the last term to make a homogeneous equation in \( x, y, \) and \( z \). We have then
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} - z^2 = 0.
\]

Hence the equation of the tangent is
\[
X \cdot \frac{2x}{a^2} + Y \cdot \frac{2y}{b^2} - z \cdot 2z = 0,
\]
where \( z \) is to be put \( = 1 \). Hence we get
\[
\frac{X \cdot x}{a^2} + \frac{Y \cdot y}{b^2} = 1 \text{ for the tangent,}
\]
and therefore
\[
\frac{X - x}{a^2} = \frac{Y - y}{b^2} \text{ for the normal.}
\]

Ex. 2. Take the general equation of a conic
\[
a x^2 + 2 h x y + b y^2 + 2 g x + 2 f y + c = 0.
\]
When made homogeneous this becomes
\[
a x^2 + 2 h x y + b y^2 + 2 g x: + 2 f y: + c z^2 = 0.
\]
The equation of the tangent is therefore
\[
X \left( a x + h y + g \right) + Y \left( h x + b y + f \right) + g x + f y + c = 0,
\]
and that of the normal is
\[
\frac{X - x}{a x + h y + g} = \frac{Y - y}{h x + b y + f}.
\]

Ex. 3. Consider the curve \( y = \log \sec \frac{x}{a} \).

Then
\[
\frac{dy}{dx} = \tan \frac{x}{a},
\]
and the equation of the tangent is
\[
Y - y = \tan \frac{x}{a} (X - x),
\]
and of the normal
\[
(Y - y) \tan \frac{x}{a} + (X - x) = 0.
\]
85. If \( f(x, y) = 0 \) and \( F(x, y) = 0 \) be two curves intersecting at the point \( x, y \), their respective tangents at that point are

\[
X f_x + Y f_y + Z f_z = 0,
\]

and

\[
XF_x + YF_y + ZF_z = 0.
\]

The angle at which these lines cut is

\[
\tan^{-1} \frac{f_x F_y - f_y F_x}{f_x F_x + f_y F_y}.
\]

Hence if the curves touch

\[
f_x/f_x = f_y/F_y;
\]

and if they cut orthogonally,

\[
f_x F_x + f_y F_y = 0.
\]

Ex. Find the angle of intersection of the curves

\[
x^3 - 3xy^2 = a,
\]

\[
3x^2y - y^3 = b.
\]

Calling the left-hand members \( f \) and \( F \) respectively, we have

\[
f_x = 3(x^2 - y^2) = F_y,
\]

\[
f_y = -6xy = -F_x.
\]

Hence clearly

\[
f_x F_x + f_y F_y = 0,
\]

and the curves cut orthogonally.

86. If the form of a curve be given by the equations

\[
x = \phi(t), \quad y = \psi(t)
\]

the tangent at the point determined by the third variable \( t \) is by equation 1, Art. 82,

\[
Y - \psi(t) = \frac{\psi'(t)}{\phi'(t)} \{X - \phi(t)\},
\]

or

\[
X\psi'(t) - Y\phi'(t) = \phi(t)\psi'(t) - \psi(t)\phi'(t).
\]

Similarly by Art. 84 the corresponding normal is

\[
X\phi'(t) + Y\psi'(t) = \phi(t)\phi'(t) + \psi(t)\psi'(t).
\]
EXAMPLES.

1. Find the equations of the tangents and normals at the point \((x_0, y_0)\) on each of the following curves:
   - \(x^2 + y^2 = a^2\)
   - \(y^2 = 4ax\)
   - \(xy = k^2\)
   - \(y = c \cosh \frac{r}{c}\)
   - \(x^2 y + xy^2 = a^3\)
   - \(e^y = \sin x\)
   - \(x^3 - 3axy + y^3 = 0\)
   - \((x^2 + y^2)^2 = a^2(x^2 - y^2)\)

2. Write down the equations of the tangents and normals to the curve \(y(x^2 + a^2) = ax^2\) at the points where \(y = \frac{a}{4}\).

3. Prove that \(\frac{x}{a} + \frac{y}{b} = 1\) touches the curve \(y = be^{-\frac{x}{a}}\) at the point where the curve crosses the axis of \(y\).

4. Find where the tangent is parallel to the axis of \(x\) and where it is perpendicular to that axis for the following curves:
   - \(a^2 x^2 + 2hxy + by^2 = 1\)
   - \(y = \frac{x^3 - a^3}{ax}\)
   - \(y^3 = x^2(2a - x)\)

5. Find the tangent and normal at the point determined by \(\theta\) on
   - The ellipse \(x = a \cos \theta\) \(y = b \sin \theta\)
   - The cycloid \(x = a(\theta + \sin \theta)\) \(y = a(1 - \cos \theta)\)
   - The epicycloid \(x = A \cos \theta - B \cos \frac{A}{B} \theta\) \(y = A \sin \theta - B \sin \frac{A}{B} \theta\)

6. If \(p = x \cos a + y \sin a\) touch the curve
   \[\frac{x^m}{a^m} + \frac{y^n}{b^m} = 1,\]
prove that
   \[p^{m-1} = (a \cos a)^{m-1} + (b \sin a)^{m-1}.\]
Hence write down the polar equation of the locus of the foot of the perpendicular from the origin on the tangent to this curve.

Examine the cases of an ellipse and of a rectangular hyperbola.

7. Find the condition that the conics
\[ a\xi^2 + b\eta^2 = 1, \quad a'\xi^2 + b'y^2 = 1 \]
shall cut orthogonally.

8. Prove that, if the axes be oblique and inclined at an angle \( \omega \), the equation of the normal to \( y = f(x) \) at \( (x, y) \) is
\[
(y - \xi) \left( \cos \omega + \frac{dy}{dx} \right) + (x - \eta) \left( 1 + \cos \omega \frac{dy}{dx} \right) = 0.
\]

9. Show that the parabolas \( x^2 = ay \) and \( y^2 = 2ax \) intersect upon the Folium of Descartes \( x^3 + y^3 = 3axy \); and find the angles between each pair at the points of intersection.

87. Tangents at the Origin.

It will be shown in a subsequent article (124) that in the case in which a curve, whose equation is given in the rational algebraic form, passes through the origin, the equation of the tangent or tangents at that point can be at once written down by inspection; the rule being to equate to zero the terms of lowest degree in the equation of the curve.

Ex. 1. In the curve \( x^2 + y^2 + ax + by = 0, \ ax + by = 0 \) is the equation of the tangent at the origin; and in the curve \( (x^2 + y^2)^2 = a^2 (x^2 - y^2), \ x^2 - y^2 = 0 \) is the equation of a pair of tangents at the origin.

Ex. 2. Write down the equations of the tangents at the origin in the following curves:

(a) \( (x^2 + y^2)^2 = a^2x^2 - b^2y^2. \)

(\( \beta \)) \( x^3 + y^3 = 5ax^2y^2. \)

(\( \gamma \)) \( (y - a)^2 \frac{x^2 + y^2}{y^2} = b^2. \)

From the equation \( Y - y = \frac{dy}{dx} (X - x) \)

it is clear that the intercepts which the tangent cuts off from the axes of \( x \) and \( y \) are respectively

\[
 x - \frac{y}{\frac{dy}{dx}} \quad \text{and} \quad y - x \frac{dy}{dx},
\]

for these are respectively the values of \( X \) when \( Y = 0 \) and of \( Y \) when \( X = 0 \).

Let \( PN, PT, PG \) be the ordinate, tangent, and normal to the curve, and let \( PT \) make an angle \( \psi \) with the axis of \( x \); then \( \tan \psi = \frac{dy}{dx} \). Let the tangent cut the axis of \( y \) in \( t \), and let \( OY, OY_1 \) be perpendiculars from \( O \), the origin, on the tangent and normal. Then the above values of the intercepts are also obvious from the figure.

\( \Xi \; D. \; C. \)
89. Subtangent, etc.

Def. The line $TN$ is called the subtangent and the line $NG$ is called the subnormal.

From the figure

Subtangent $= TN = y \cot \psi = \frac{y}{\frac{dy}{dx}}$.

Subnormal $= NG = y \tan \psi = y \frac{dy}{dx}$.

Normal $= PG = y \sec \psi = y \sqrt{1 + \tan^2 \psi}$

$= y \sqrt{1 + \left(\frac{dy}{dx}\right)}$

Tangent $= TP = y \csc \psi = y \frac{\sqrt{1 + \tan^2 \psi}}{\tan \psi}$

$= y \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{dy}{dx}}$.

$OY = O\ell \cos \psi = \frac{y - x \frac{dy}{dx}}{\sqrt{1 + \tan^2 \psi}} = \frac{y - x \frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$.

$OY_1 = OG \cos \psi = \frac{ON + NG}{\sqrt{1 + \tan^2 \psi}} = \frac{x + y \frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$.

These and other results may of course also be obtained analytically from the equation of the tangent.
Thus if the equation of the curve be given in the form
\[ f(x, y) = 0, \]
the tangent \[ Xf_x + Yf_y + Zf_z = 0 \]
makes intercepts \(-f_x/f_x^2\) and \(-f_y/f_y^2\) upon the co-ordinate axes, and the perpendicular from the origin upon the tangent is
\[ f_x/\sqrt{f_x^2 + f_y^2}, \]
and indeed, any lengths or angles desired may be written down by the ordinary methods and formulae of analytical geometry.

Ex. 1. For the "chainette"
\[ y = e^{\frac{c}{2}(e^x + e^{-x})}, \]
we have
\[ y_1 = e^{\frac{c}{2}(e^x - e^{-x})}. \]

Hence
\[ \text{Subtangent} = \frac{y}{y_1} = e^{\frac{c}{2} \left( \frac{e^x + e^{-x}}{e^x - e^{-x}} \right)}. \]

\[ \text{Subnormal} = yy_1 = e^{\frac{2x}{4}(e^x - e^{-x})}. \]

\[ \text{Normal} = y \sqrt{1 + y_1^2} = \frac{y^2}{c}, \text{ etc.} \]

Ex. 2. Find that curve of the class \( y = \frac{x^n}{a^{n-1}} \) whose subnormal is constant.

Here
\[ y_1 = \frac{x^{n-1}}{a^{n-1}}, \]
and
\[ \text{subnormal} = yy_1 = \frac{x^{2n-1}}{a^{2n-2}}. \]

Thus if \( 2n = 1 \) the \( x \) disappears and leaves
\[ \text{subnormal} = \frac{a}{2}, \]
and the curve is the ordinary parabola
\[ y^2 = ax. \]
90. **Values of** $\frac{ds}{dx}$, $\frac{dx}{ds}$, **etc.**

Let $P$, $Q$ be contiguous points on a curve. Let the co-ordinates of $P$ be $(x, y)$ and of $Q (x + \delta x, y + \delta y)$.

Then the perpendicular $PR = \delta x$, and $RQ = \delta y$. Let the arc $AP$ measured from some fixed point $A$ on the curve be called $s$ and the arc $AQ = s + \delta s$. Then arc $PQ = \delta s$. When $Q$ travels along the curve so as to come indefinitely near to $P$, the arc $PQ$ and the chord $PQ$ differ ultimately by a quantity of higher order of smallness than the arc $PQ$ itself. (Art. 81.)

Hence, rejecting infinitesimals of order higher than the second, we have

$$\delta s^2 = (\text{chord } PQ)^2 = (\delta x^2 + \delta y^2),$$

or

$$\frac{\delta s^2}{\delta x^2} = \frac{(dx)^2}{(ds)^2} + \frac{(dy)^2}{(ds)^2}.$$ 

Similarly

$$\frac{\delta s^2}{\delta x^2} = \frac{(dx)^2}{(ds)^2} + \frac{(dy)^2}{(ds)^2},$$

or

$$\frac{dy}{dx} = 1 + \frac{(dy)^2}{(dx)^2};$$

and in the same manner

$$\frac{(ds)^2}{(dy)^2} = 1 + \frac{(dx)^2}{(dy)^2}.$$
TANGENTS AND Normals.

If \( \psi \) be the angle which the tangent makes with the axis of \( x \) we have as in Art. 18,

\[
\tan \psi = \lim \frac{RQ}{PR} = \lim \frac{\delta y}{\delta x} = \frac{dy}{dx},
\]

and also

\[
\cos \psi = \lim \frac{PR}{\text{chord } PQ} = \lim \frac{PR}{\text{arc } PQ} = \lim \frac{\delta x}{\delta s} = \frac{dx}{ds},
\]

and

\[
\sin \psi = \lim \frac{RQ}{\text{chord } PQ} = \lim \frac{RQ}{\text{arc } PQ} = \lim \frac{\delta y}{\delta s} = \frac{dy}{ds}.
\]

**Examples.**

1. Find the length of the perpendicular from the origin on the tangent at the point \( x, y \) of the curve \( x^4 + y^4 = c^4 \).

2. Show that in the curve \( y = bc^n \) the subtangent is of constant length.

3. Show that in the curve \( by^2 = (x+a)^3 \) the square of the subtangent varies as the subnormal.

4. For the parabola \( y^2 = 4ax \), prove

\[
\frac{ds}{dx} = \sqrt{1 + \frac{a}{x}}.
\]

5. Prove that for the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), if \( x = a \sin \phi \)

\[
\frac{ds}{d\phi} = a \sqrt{1 - e^2 \sin^2 \phi}.
\]

6. For the cycloid \( x = a \, \text{vers} \, \theta \)

\[
y = a \left( \theta + \sin \theta \right) \}
\]

prove

\[
\frac{ds}{dx} = \sqrt{\frac{2a}{x}}.
\]

7. In the curve \( y = a \log \sec \frac{x}{\alpha} \),

prove \( \frac{ds}{dx} = \sec \frac{x}{\alpha} \), \( \frac{ds}{dy} = \cosec \frac{x}{\alpha} \), and \( x = u \psi \).
8. Show that the portion of the tangent to the curve
\[ x^\frac{2}{3} + y^\frac{2}{3} = a^\frac{2}{3}, \]
which is intercepted between the axes, is of constant length.

Find the area of the portion included between the axes and the tangent.

9. Find for what value of \( n \) the length of the subnormal of the curve \( xy^n = a^{n+1} \) is constant. Also for what value of \( n \) the area of the triangle included between the axes and any tangent is constant.

10. Prove that for the catenary \( y = c \cosh \frac{t}{c} \), the length of the perpendicular from the foot of the ordinate on the tangent is of constant length.

11. In the tractory
\[ x = \sqrt{c^2 - y^2} + \frac{c}{2} \log \frac{c - \sqrt{c^2 - y^2}}{c + \sqrt{c^2 - y^2}}, \]
prove that the portion of the tangent intercepted between the point of contact and the axis of \( x \) is of constant length.

.91. Polar Co-ordinates.

If the equation of the curve be referred to polar co-ordinates, suppose \( O \) to be the pole and \( P, Q \) two contiguous points on the curve. Let the co-ordinates of \( P \) and \( Q \) be \((r, \theta)\) and \((r + \delta r, \theta + \delta \theta)\) respectively. Let \( PN \) be the perpendicular on \( OQ \), then \( NQ \) differs from
\( \delta r \) and \( NP \) from \( r \delta \theta \) by a quantity of higher order of smallness than \( \delta \theta \). (Art. 79.)

Let the arc measured from some fixed point \( A \) to \( P \) be called \( s \) and from \( A \) to \( Q, s + \delta s \). Then arc \( PQ = \delta s \). Hence, rejecting infinitesimals of order higher than the second, we have

\[
\delta s^2 = (\text{chord } PQ)^2 = (NQ^2 + PN^2) = (\delta r^2 + r^2 \delta \theta^2).
\]

and therefore

\[
\left( \frac{dr}{ds} \right)^2 + r^2 \left( \frac{d\theta}{ds} \right)^2 = 1, \quad \text{or} \quad \left( \frac{ds}{dr} \right)^2 = 1 + r^2 \left( \frac{d\theta}{dr} \right)^2,
\]

or

\[
\frac{(ds)^2}{d\theta} = r^2 + \frac{dr}{d\theta}
\]

according as we divide by \( \delta s^2, \delta r^2 \), or \( \delta \theta^2 \) before proceeding to the limit.

92. Inclination of the Radius Vector to the Tangent.

Next, let \( \phi \) be the angle which the tangent at any point \( P \) makes with the radius vector, then

\[
\tan \phi = r \frac{d\theta}{dr}, \quad \cos \phi = \frac{dr}{ds}, \quad \sin \phi = \frac{r d\theta}{ds}.
\]

For, with the figure of the preceding article, since, when \( Q \) has moved along the curve so near to \( P \) that \( Q \) and \( P \) may be considered as ultimately coincident, \( QP \) becomes the tangent at \( P \) and the angles \( OQT \) and \( OPT \) are each of them ultimately equal to \( \phi \), and

\[
\tan \phi = Lt \tan NQP = Lt \frac{NP}{QN} = Lt \frac{r \delta \theta}{\delta r} = r \frac{d\theta}{dr};
\]

\[
\cos \phi = Lt \cos NQP = Lt \frac{NQ}{\text{chord } QP} = Lt \frac{NQ}{\text{arc } QP} = Lt \frac{\delta r}{\delta s} = \frac{dr}{ds};
\]
\[
\sin \phi = Lt \sin NQP = Lt \frac{NP}{\text{chord } QP} \\
= Lt \frac{NP}{\text{arc } QP} = Lt \frac{r \delta \theta}{s} = \frac{vd\theta}{ds}.
\]

Ex. Find the angle \(\phi\) in the case of the curve
\[r^n = a^n \sec (n\theta + a),\]
and prove that this curve is intersected by the curve
\[r^m = b^m \sec (n\theta + \beta)\]
at an angle which is independent of \(a\) and \(b\). \[\text{[I. C. S., 1886.]}\]

Taking the logarithmic differential,
\[\frac{1}{r} \frac{dr}{d\theta} = \tan (n\theta + a),\]
whence
\[\frac{\pi}{2} - \phi = n\theta + a.\]

In a similar manner for the second curve
\[\frac{\pi}{2} - \phi' = n\theta + \beta,\]
\(\phi'\) being the angle which the radius vector makes with the tangent to the second curve. Hence the angle between the tangents at the point of intersection is \(a - \beta\).

\[93. \textbf{Polar Subtangent, Subnormal.}\]

Let \(OY\) be the perpendicular from the origin on the tangent at \(P\).
Let \(T0t\) be drawn through \(O\) perpendicular to \(OP\)

and cutting the tangent in \(T\) and the normal in \(t\). Then
OT is called the "Polar Subtangent" and \(Ot\) is called the "Polar Subnormal."

It is clear that
\[
OT = OP \tan \phi = r^2 \frac{d\theta}{dr}
\]
and that
\[
Ot = OP \cot \phi = \frac{dr}{d\theta}
\]

94. It is often found convenient when using polar co-ordinates to write \(\frac{1}{u}\) for \(r\), and therefore \(-\frac{1}{u^2} \frac{du}{d\theta}\) for \(\frac{dr}{d\theta}\). With this notation,

\[
\text{Polar Subtangent} = r^2 \frac{d\theta}{dr} = -\frac{d\theta}{du}
\]

Ex. In the conic \(lu = 1 + e \cos \theta\)
we have
\[
l = e \sin \theta \frac{d\theta}{du}
\]

Thus the length of the polar subtangent is \(l/e \sin \theta\).

Also, from the figure, the angular co-ordinate of its extremity is
\[
\theta = \frac{\pi}{2}
\]

Hence the co-ordinates of \(T(r_1, \theta_1)\) satisfy the equation
\[
r_1 = l/e \sin \left(\frac{\pi}{2} + \theta_1\right)
\]

The locus of the extremity is therefore
\[lu = e \cos \theta;
\]
that is, the directrix corresponding to that focus which is taken as origin.

95. Perpendicular from Pole on Tangent.

Let \(OY = p\).

Then \[p = r \sin \phi,\]
and therefore

\[ \frac{1}{p^2} = \frac{1}{r^2} \cosec^2 \phi = \frac{1}{r^2} \left( 1 + \cot^2 \phi \right) = \frac{1}{r^2} \left( 1 + \frac{1}{r^2} \left( \frac{dr}{d\theta} \right)^2 \right); \]

therefore

\[ \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left( \frac{dr}{d\theta} \right)^2 \]

\[ \therefore = u^2 + \left( \frac{du}{d\theta} \right)^2 \]  

(1) \hspace{1cm} (2).

Ex. In the spiral \( r = a \theta^\nu \)

we have \( \theta \frac{du}{d\theta} = 1 - \theta^{-2}, \)

whence

\[ a \frac{du}{d\theta} = 2 \theta^{-3}; \]

and therefore, squaring and adding,

\[ \frac{a^2}{p^2} = 1 - 2\theta^{-2} + \theta^{-4} + 4\theta^{-6}. \]

Thus, corresponding to \( \theta = \pm 1, \) we have

\[ a^2 = 4 \text{ and } p = \frac{a}{2}. \]

96. The Pedal Equation.

The relation between \( p \) and \( r \) often forms a very convenient equation to the curve. It is called the Pedal equation.

(1) If the curve be given in Cartesians,

\[ F(x, y) = 0 \]

the tangent is

\[ XF_x + YF_y + ZF_z = 0 \]

and

\[ p^2 = \frac{F_z^2}{F_x^2 + F_y^2} \]  

(2).

If \( x, y \) be eliminated between equations (1), (2) and

\[ x^2 + y^2 = r^2 \]

(3),

the required equation will result.
TANGENTS AND NORMALS

Ex. If \( x^2 + y^2 = 2ax \),
\[ X(x - a) + Yy = ax \]
is the equation of the tangent, and
\[ p^2 = \frac{a^2 x^2}{(x - a)^2 + y^2} = \frac{1}{4} \frac{r^4}{a^2} \]
or
\[ r^2 = 2ap. \]

This result will also be evident geometrically.

(2) If the curve be given in Polars we may first obtain \( p \) in terms of \( r \) and \( \theta \) by Art. 95, and then eliminate \( \theta \) between this result and the equation to the curve.

Ex. Required the pedal equation of \( r^m = a^m \sin m\theta \).

By logarithmic differentiation,
\[ \frac{m}{r} \frac{dr}{d\theta} = m \cot m\theta, \]

\[ \therefore \cot \phi = \cot m\theta \text{ or } \phi = m\theta, \]
whence
\[ p = r \sin \phi = r \sin m\theta = r^m \frac{a^m}{r}, \]
or
\[ pa^m = r^{m+1}. \]

**EXAMPLES.**

1. In the equiangular spiral \( r = ae^{\theta \cot a} \), prove
\[ \frac{dr}{ds} = \cos a \text{ and } p = r \sin a. \]

2. For the involute of a circle, viz.,
\[ \theta = \sqrt{r^2 - a^2} - \cos^{-1} \frac{a}{r}, \]
prove
\[ \cos \phi = \frac{a}{r}. \]

3. In the parabola \( \frac{2a}{r} = 1 - \cos \theta \), prove the following results:

\[ (a) \quad \phi = \pi - \frac{\theta}{2}. \]
(β) \[ p = \frac{a}{\sin \frac{\theta}{2}} \]

(γ) \[ p^2 = ar. \]

(δ) Polar subtangent = \(2a\cos \theta \).

4. For the cardioid \( r = a(1 - \cos \theta) \), prove

(a) \[ \phi = \frac{\theta}{2} \]

(β) \[ p = 2a \sin^3 \frac{\theta}{2} \]

(γ) \[ p^2 = \frac{r^2}{2a} \]

(δ) Polar subtangent = \(2a \frac{\sin^3 \frac{\theta}{2}}{\cos \frac{\theta}{2}}\).

97. Maximum number of tangents from a point to a curve of the \(n^{th}\) degree.

Let the equation of the curve be \( f(x, y) = 0 \). The equation of the tangent at the point \((x, y)\) is

\[ Xf_x + Yf_y + Zf_z = 0, \]

where \(z\) is to be put equal to unity after the differentiation is performed. If this pass through the point \(h, k\) we have

\[ h f_x + k f_y + f_z = 0. \]

This is an equation of the \((n - 1)^{th}\) degree in \(x\) and \(y\) and represents a curve of the \((n - 1)^{th}\) degree passing through the points of contact of the tangents drawn from the point \((h, k)\) to the curve \( f(x, y) = 0 \). These two curves have \(n(n - 1)\) points of intersection, and therefore there are \(n(n - 1)\) points of contact corresponding to \(n(n - 1)\) tangents, real or imaginary, which can be drawn from a given point to a curve of the \(n^{th}\) degree.
Thus for a conic, a cubic, a quartic, the maximum number of tangents which can be drawn from a given point is 2, 6, 12 respectively.

98. Number of Normals which can be drawn to a Curve to pass through a given point.

Let \( h, k \) be the point through which the normals are to pass.

The equation of the normal to the curve \( f(x, y) = 0 \) at the point \( (x, y) \) is

\[
\frac{X - x}{f_x} = \frac{Y - y}{f_y}.
\]

If this pass through \( h, k \),

\[
(h - x)f_y = (k - y)f_x.
\]

This equation is of the \( n^\text{th} \) degree in \( x \) and \( y \) and represents a curve which goes through the feet of all normals which can be drawn from the point \( h, k \) to the curve. Combining this with \( f(x, y) = 0 \), which is also of the \( n^\text{th} \) degree, it appears that there are \( n^2 \) points of intersection, and that therefore there can be \( n^2 \) normals, real or imaginary, drawn to a given curve to pass through a given point.

For example, if the curve be an ellipse, \( n = 2 \), and the number of normals is 4. Let \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) be the equation of the curve, then

\[
(h - x) \frac{y}{b^2} = (k - y) \frac{x}{a^2}.
\]

is the curve which, with the ellipse, determines the fact of the normals drawn from the point \( (h, k) \). This is a rectangular hyperbola which passes through the origin and through the point \( (h, k) \).

The student should consider how it is that an infinite number of normals can be drawn from the centre of a circle to the circumference.

99. The curves

\[
(h - x)f_x + (k - y)f_y = 0 \ldots \ldots (1),
\]

and

\[
(h - x)f_y - (k - y)f_x = 0 \ldots \ldots (2),
\]
on which lie the points of contact of tangents and the
feet of the normals respectively, which can be drawn to
the curve \( f(x, y) = 0 \) so as to pass through the point
\((h, k)\), are the same for the curve \( f(x, y) = a \). And, as
equations (1) and (2) do not depend on \( a \), they represent
the loci of the points of contact and of the feet of the
normals respectively for all values of \( a \), that is, for all
members of the family of curves obtained by varying \( a \)
in \( f(x, y) = a \) in any manner.

EXAMPLES.

1. Through the point \( h, k \) tangents are drawn to the curve
\[ Ax^3 + By^3 = 1; \]
show that the points of contact lie on a conic.

2. If from any point \( P \) normals be drawn to the curve whose
equation is \( y^n = mx^n \), show that the feet of the normals lie on a
conic of which the straight line joining \( P \) to the origin is a dia-
meter. Find the position of the axes of this conic.

3. The points of contact of tangents from the point \( h, k \) to
the curve \( x^3 + y^3 = 3axy \) lie on a conic which passes through the
origin.

4. Through a given point \( h, k \) tangents are drawn to curves
where the ordinate varies as the cube of the abscissa. Show that
the locus of the points of contact is the rectangular hyperbola
\[ 2xy + kx - 3hy = 0, \]
and the locus of the remaining point in which each tangent cuts
the curve is the rectangular hyperbola
\[ xy - 4kx + 3hy = 0. \]

EXAMPLES.

1. Find the points on the curve
\[ y = (x - 1)(x - 2)(x - 3) \]
at which the tangent is parallel to the axis of \( x \).

Show also that the tangents at the first and third inter-
sections with the \( x \)-axis are parallel, and at the middle inter-
section the tangent makes an angle 135° with that axis.
EXAMPLES.

2. In any Cartesian curve the rectangle contained by the subtangent and the subnormal is equal to the square on the corresponding ordinate.

3. Show that the only Cartesian locus in which the ratio of the subtangent to the subnormal is constant is a straight line.

4. If the ratio of the subnormal to the subtangent vary as the square of the abscissa the curve is a parabola.

5. Show that in any curve

\[ \frac{\text{Subnormal}}{\text{Subtangent}} = \left( \frac{\text{Normal}}{\text{Tangent}} \right)^2. \]

6. Find that normal to

\[ \sqrt{xy} = a + x, \]

which makes equal intercepts upon the co-ordinate axes.

7. Prove that the sum of the intercepts of the tangent to

\[ \sqrt{x} + \sqrt{y} = \sqrt{u} \]

upon the co-ordinate axes is constant.

8. Show that in the curve

\[ y = a \log (x^2 - a^2), \]

the sum of the tangent and the subtangent varies as the product of the co-ordinates of the point.

9. Show that in the curve

\[ x^m + y^n = \alpha^m - \alpha^n y^{2n}, \]

the \( m \)th power of the subtangent varies as the \( n \)th power of the subnormal.

10. In the curve \( y^n = a^{n-1}x \) the subnormal \( \propto y^2 \) and the subtangent \( \propto x \).

11. Show that in the curve \( y = be^{-x} \) the subtangent varies as the square of the abscissa.

12. If in a curve the normal varies as the cube of the ordinate, find the subtangent and the subnormal.

13. Show that in the curve for which

\[ s = c \log \frac{e}{y} \]

the tangent is of constant length.
14. Show that in the curve for which
\[ y^2 = c^2 + s^2, \quad \text{(The Catenary)} \]
the perpendicular from the foot of the ordinate upon the tangent
is of constant length.

15. Show that the polar subtangent in the curve \( r = a\theta \) (The
Spiral of Archimedes) varies as the square of the radius vector,
and the polar subnormal is constant.

16. Show that the polar subtangent is constant in the curve
\[ r\theta = a. \quad \text{(The Reciprocal Spiral.)} \]

17. Show that in the curve
\[ r = ae^{\theta \cot \alpha} \quad \text{(The Equiangular Spiral.)} \]

(1) the tangent makes a constant angle with the radius
vector;

(2) the Polar Subtangent \( = r \tan a; \)
the Polar Subnormal \( = r \cot a; \)

(3) the loci of the extremities of the polar subtangent,
the polar subnormal, the perpendicular upon the tangent from
the pole are curves of the same species as the original.

18. Show that each of the several classes of curves (Cotes's
Spirals)
\[ r = ae^{n\theta}, \quad r\theta = a, \quad r \sin n\theta = a, \quad r \sinh n\theta = a, \]
\[ r \cosh n\theta = a, \]
have pedal equations of the form
\[ \frac{1}{p^2} = \frac{A}{r^2} + B, \]
where \( A \) and \( B \) are certain constants.

19. Find the angle of intersection of the Cardioides
\[ r = a (1 + \cos \theta), \]
\[ r = b (1 - \cos \theta). \]

20. Find the angle of intersection of
\[ \begin{aligned}
  x^2 - y^2 &= a^2, \\
  x^2 + y^2 &= a^2 \sqrt{2},
\end{aligned} \]
EXAMPLES.

21. Show that the condition of tangency of

\[ x \cos \alpha + y \sin \alpha = p, \]

with

\[ x^m y^n = a^{m+n}, \]

is

\[ p^{m+n} \cdot m^m \cdot n^n = (m+n)^{m+n} \cdot e^{m+n} \cdot \alpha \sin \alpha. \]

Hence write down the equation of the locus of the foot of the perpendicular from the origin upon a tangent.

22. Show that in the curve (the cycloid)

\[ x = a (\theta + \sin \theta), \]
\[ y = a (1 - \cos \theta), \]

\[ \frac{dx}{d\theta} = 2a \cos \theta \]
\[ \frac{dy}{dx} = \sqrt{2a/y}. \]

23. Show that in the curve (an epicycloid)

\[ x = (a+b) \cos \theta - b \cos \frac{a+b}{b} \theta, \]
\[ y = (a+b) \sin \theta - b \sin \frac{a+b}{b} \theta. \]

we have

\[ \mu = (a+2b) \sin \frac{a+b}{2b} \theta; \quad \Psi = \frac{a+2b}{2b} \theta; \quad \mu = (a+2b) \sin \frac{a+2b}{a+2b} \theta; \]

and that the pedal equation is

\[ r^2 = a^2 + 4 \frac{(a+b)}{(a+2b)^2} p^2. \]

24. Show that the normal to \( y^2 = 4ax \) touches the curve

\[ 27ay^2 = 4 (x-2a)^3. \]

25. Show that the locus of the extremity of the polar sub-tangent of the curve

\[ u = f(\theta), \]
\[ u + f' \left( \frac{n}{a} + \theta \right) = 0 \]

E. D. C.
26. Show that the locus of the extremity of the polar sub-
normal of the curve

\[ r = f(\theta), \]

is

\[ r = f' \left( \theta - \frac{\pi}{2} \right). \]

27. In the curve

\[ r \left( m + n \tan \frac{\theta}{2} \right) = 1 + \tan \frac{\theta}{2}, \]

show that the locus of the extremity of the polar subtangent is

\[ r = \frac{m - n}{2}, \quad r = 1 + \cos \theta. \]
CHAPTER IX.

ASYMPTOTES.

100. Def. If a straight line cut a curve in two points at an infinite distance from the origin and yet is not itself wholly at infinity, it is called an asymptote to the curve.

101. To obtain the Asymptotes.

If \( \phi (x, y) = 0 \) ..................(1)

be the equation of any rational algebraic curve of the \( n^{th} \) degree, and

\[ y = mx + c \] ..................(2)

that of any straight line, the equation

\[ \phi (x, mx + c) = 0 \] ..................(3)

obtained by substituting the expression \( mx + c \) for \( y \) gives the abscissae of the points of intersection.

This equation is in general of the \( n^{th} \) degree, showing that a curve of the \( n^{th} \) degree is in general cut in \( n \) points real or imaginary by any straight line.

The two constants of the straight line, viz. \( m \) and \( c \), are at our choice. We are to choose them so as to make two of the roots of equation (3) infinite. We then have a line cutting the given curve so that two of the points of intersection are at an infinite distance from the origin.
Imagine equation (3) expanded out and expressed in
descending powers of \( x \) as
\[
Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots + K = 0 \quad \ldots \ldots (4),
\]
\( A, B, C, \ldots \) being certain functions of \( m \) and \( c. \)

The equation whose roots are the reciprocals of the
roots of this equation is
\[
A + Bz + Cz^2 + \ldots + Kz^n = 0
\]

(by putting \( x = \frac{1}{z} \));

and it is evident that if \( A \) and \( B \) be both zero two roots
of this equation for \( z \) will become evanescent, and
therefore two roots of the equation for \( x \) become infinite. If then we choose \( m \) and \( c \) to satisfy the equations
\[
A = 0, \quad B = 0,
\]
and substitute their values in the equation
\[
y = mx + c,
\]
we shall obtain the equation of an asymptote.

102. It will be found in examples (and it admits
of general proof) that the equation \( A = 0 \) contains \( m \)
only and in a degree not higher than \( n \). Also that
\( B = 0 \) contains \( c \) in the first degree. Hence a curve of
the \( n \)th degree does not possess more than \( n \) asymptotes.

Ex. Find the asymptotes of the curve
\[
y^3 - x^2y + 2y^2 + 4y + x = 0.
\]
Putting \( y = mx + c, \)
\[
(m^2 + c)x^3 + (3m^2c - c + 2m) + x = 0,
\]
or
\[
3m^2c - c + 2m = 0.
\]

We now are to choose \( m \) and \( c \) so that
\[
m^3 = 0
\]
and
\[
3m^2c - c + 2m = 0
\]
The first equation is a cubic for \( m \) and gives \( m = 0, 1 \) or \(-1\).
ASYMPTOTES.

The second equation is of the first degree in \( c \) and gives

\[
c = \frac{2m^2}{1 - 3m^2}.
\]

If \( m = 0 \) we have \( c = 0 \); if \( m = 1 \) we have \( c = -1 \); if \( m = -1 \) we have \( c = 1 \).

Hence we obtain three asymptotes, viz.

\[
y = 0,
\]

\[
y = x - 1,
\]

\[
y = -x - 1.
\]

EXAMPLES.

Find the asymptotes of

1. \( y^2 + 6xy^2 + 11x^2y - 6y^3 + x + y = 0 \).
2. \( y^2 - 4x^2y - 4y^2 + 4x^3 + 4xy - 4x^2 = 5 \).
3. \( y^2 + 3x^2y + xy^2 - 3x^3 + 2y^2 + 2xy + 4x + 5y + 6 = 0 \).
4. \( (y + x + 1)(y + 2x + 2)(y + 3x + 3)(y - x^2 + x^2 + y^2) = 0 \).
5. \( (2x + 3y)(3x + 4y)(4x + 5y) + 26x^2 + 70xy + 47y^2 + 2x + 3y = 1 \).

103. The case of parallel Asymptotes.

After having formed equation (4) of Art. 101 by substitution of \( m \alpha + c \) for \( y \) and rearrangement, it sometimes happens that one or more of the values of \( m \), deduced from the equation \( A = 0 \), will make \( B \) vanish identically, and therefore any value of \( c \) will give a line cutting the curve in two points at infinity. In this case as the letter \( c \) is still at our choice, it may be chosen so as to make the third coefficient \( C \) vanish. It will be seen from examples that each such value of \( m \) now gives rise to two values of \( c \). This is the case of parallel asymptotes. The two lines thus obtained each cut the curve at three points at infinity.
Ex. Find the asymptotes of the cubic curve,
\[ y^3 - 5xy^2 + 8x^2y - 4x^3 - 3y^2 + 9xy - 6x^2 + 2y - 2x = 1. \]

Putting \( mx + c \) for \( y \) and rearranging,
\[
(m^3 - 6m^2 + 8m - 4)x^3 + (3m^2c - 10mc + 8c - 3m^2 + 9m - 6)x^2 \\
+ (3mc^2 - 5c^2 - 6mc + 9c + 2m - 2)x + c^3 - 3c^2 + 2c - 1 = 0.
\]

Choosing \( m^3 - 5m^2 + 8m - 4 = 0 \)
and \( 3m^2c - 10mc + 8c - 3m^2 + 9m - 6 = 0 \),
the first gives \((m - 1)(m - 2)^2 = 0\),
whence \( m = 1, 2 \) or 2.

If \( m = 1 \) the second equation gives \( c = 0 \) and the corresponding asymptote is \( y = x \).

If \( m = 2 \) we have \( 12c - 20c + 8c - 12 + 18 = 6 \) which vanishes identically for all finite values of \( c \). Thus any line parallel to \( y = 2x \) will cut the curve in two points at infinity. We may however choose \( c \) so that the next coefficient
\[ 3mc^2 - 5c^2 - 6mc + 9c + 2m - 2 \]
vanishes for the value \( m = 2 \), giving
\[ c^3 - 3c + 2 = 0, \text{ i.e. } c = 1 \text{ or } 2. \]

Thus each of the system of lines parallel to \( y = 2x \) cuts the curve in two points at infinity. But of all this infinite system of parallel straight lines the two whose equations are
\[ y = 2x + 1, \]
and
\[ y = 2x + 2, \]
are the only ones which cut the curve in three points at infinity and therefore the name asymptote is confined to them.

The asymptotes are therefore
\[
\begin{align*}
  y &= x \\
  y &= 2x + 1 \\
  y &= 2x + 2
\end{align*}
\]

EXAMPLES.

Find the asymptotes of
1. \( y^3 - xy^2 - x^3y + x^3 + x^2 - y^2 = 1. \)
2. \( y^4 - 2x^2y + 2x^3y - x^4 - 3x^2 + 3x^2y + 3x^2y^3 - 3y^3 - 2x^2 + 2y^2 = 1. \)
3. \( (y^2 - x^2)^2 - 2(y^2 + x^2) = 1. \)
104. Those asymptotes which are parallel to the $y$-axis will not be discovered by the above processes for their equations are of the form $x = a$, and are not included in the form $y = mx + c$ for a finite value of $m$. We, therefore, specially consider the case of those asymptotes which may be parallel to one or other of the co-ordinate axes.

105. Asymptotes Parallel to the Axes.

Let the equation of the curve be
\[ a_n x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots + a_{n-1} x y^{n-1} + a_n y^n + b_1 x^{n-1} + b_2 x^{n-2} y + \ldots + b_{n-1} x y^{n-2} + b_n y^{n-1} + \ldots + \ldots = 0 \] \hspace{1cm} (1).

If arranged in descending powers of $x$ this is
\[ a_n x^n + (a_1 y + b_1) x^{n-1} + \ldots = 0 \hspace{1cm} (2). \]

Hence, if $a_n$ vanish, and $y$ be so chosen that
\[ a_1 y + b_1 = 0, \]
the coefficients of the two highest powers of $x$ in equation (2) vanish, and therefore two of its roots are infinite. Hence the straight line $a_1 y + b_1 = 0$ is an asymptote.

In the same way, if $a_n = 0$, $a_{n-1} x + b_n = 0$ is an asymptote.

Again, if $a_n = 0$, $a_1 = 0$, $b_1 = 0$, and if $y$ be so chosen that
\[ a_2 y^2 + b_2 y + c_2 = 0, \]
three roots of equation (2) become infinite, and the lines represented by
\[ a_2 y^2 + b_2 y + c_2 = 0 \]
represent a pair of asymptotes, real or imaginary, parallel to the axis of $x$. 
Hence the rule to find those asymptotes which are parallel to the axes is, "equate to zero the coefficients of the highest powers of \( x \) and \( y \)."

**Ex. 1.** Find the asymptotes of the curve

\[ x^3y^2 - x^2y - xy^2 + x + y + 1 = 0. \]

Here the coefficient of \( x^3 \) is \( y^2 - y \) and the coefficient of \( y^2 \) is \( x^3 - x \). Hence \( x = 0, \ y = 1 \), \( y = 0 \), and \( y = 1 \) are asymptotes. Also, since the curve is one of the fourth degree, we have thus obtained all the asymptotes.

**Ex. 2.** Find the asymptotes of the cubic curve

\[ x^3 + 2x^2y + xy^2 - x^2 - xy + 2 = 0. \]

Equating to zero the coefficient of \( y^2 \) we obtain \( x = 0 \), the only asymptote parallel to either axis.

Putting \( mx + c \) for \( y \),

\[ x^3 + 2x^2(mx + c) + x(mx + c)^2 - x^2 - (mx + c) + 2 = 0, \]

or rearranging,

\[ x^3(1 + 2m + m^2) + x^2(2c + 2mc - 1 - m) + x(c^2 - c) + 2 = 0, \]

1 + 2m + m^2 = 0 gives two roots \( m = \pm 1 \). 2c + 2mc - 1 - m = 0 is an identity if \( m = -1 \) and this fails to find \( c \). Proceeding to the next coefficient \( c^2 - c \) = 0 gives \( c = 0 \) or 1.

Hence the three asymptotes are \( x = 0 \), and the pair of parallel lines

\[ y + x = 0, \]
\[ y + x = 1. \]

**EXAMPLES.**

1. The asymptotes of \( y^2 (x^2 - a^2) = x \) are

\[ \begin{align*}
  y &= 0 \\
  x &= \pm a
\end{align*} \]

2. The co-ordinate axes are the asymptotes of

\[ xy^3 + x^3y = a^4. \]

3. The asymptotes of the curve \( x^2y^2 - a^2(x^2 + y^2) \) are the sides of a square.

'106. The methods given above will obtain all linear asymptotes.' It is often more expeditious however to
obtain the oblique asymptotes as an approximation of the curve to a linear form at infinity as described in the next article.


Let \( P_r, F_r \) be used to denote rational algebraical expressions which contain terms of the \( r^{th} \) and lower, but of no higher degrees.

Suppose the equation of a curve of the \( n^{th} \) degree to be thrown into the form

\[
(ax + by + c) P_{n-1} + F_{n-1} = 0 \quad \cdots (1).
\]

Then any straight line parallel to \( ax + by = 0 \) obviously cuts the curve in one point at infinity; and to find the particular member of this family of parallel straight lines which cuts the curve in a second point at infinity, let us examine what is the ultimate linear form to which the curve gradually approximates as we travel to infinity in the above direction, thus obtaining the ultimate direction of the curve and forming the equation of the tangent at infinity. To do this we make the \( x \) and \( y \) of the curve become large in the ratio given by

\[
x : y = -b : a,
\]

and we obtain the equation

\[
ax + by + c + \lim \left( \frac{F_{n-1}}{P_{n-1}} \right) = 0.
\]

If this limit be finite we have arrived at the equation of a straight line which at infinity represents the limiting form of the curve, and which satisfies the definition of an asymptote.

To obtain the value of the limit it is advantageous to put \( x = -\frac{b}{t} \) and \( y = \frac{a}{t} \), and then after simplification make \( t = 0 \).
Ex. Find the asymptote of
\[ x^3 + 3x^2y + 3xy^2 + 2y^3 - x^2 + y^2 + x. \]
We may write this curve as
\[ (x + 2y)(x^2 + xy + y^2) = x^2 + y^2 + x, \]
whence the equation of the asymptote is given by
\[ x + 2y = \lim_{t \to -\infty} \frac{x^2 + y^2 + x}{x^2 + xy + y^2}, \]
and putting \( x = -\frac{2}{t}, \ y = \frac{1}{t} \) we have
\[ x + 2y = \lim_{t \to 0} \frac{-2 + \frac{1}{t}}{-\frac{8}{t^2} + \frac{1}{t^2} + \frac{1}{t^2}} = \lim_{t \to 0} \frac{5 - 2t}{3} = \frac{5}{3}. \]
\[ i.e., \quad x + 2y = \frac{5}{3}. \]

Example. Show that \( x + y = \frac{a}{2} \) is the only real asymptote of the curve
\( (x + y)(x^2 + y^2) = a(x^2 + ay). \)

108. Next, suppose the equation of a curve put into the form
\[ (ax + by + c)P_{n-1} + F_{n-2} = 0, \]
then the line \( ax + by + c = 0 \) cuts the curve in two points at infinity, for no terms of the \( n^{th} \) or \( (n-1)^{th} \) degrees remain in the equation determining the points of intersection. Hence in general the line
\[ ax + by + c = 0 \]
is an asymptote. We say, in general, because if \( P_{n-1} \) be of the form \( (ax + by + c)P_{n-2} \), itself containing a factor \( ax + by + c \), there will be a pair of asymptotes parallel to \( ax + by + c = 0 \), each cutting the curve in three points at infinity. The equation of the curve then becomes
\[ (ax + by + c)^2P_{n-2} + F_{n-2} = 0, \]
and the equations of the parallel asymptotes are

\[ ax + by + c = \pm \sqrt{-\frac{\text{Lt} \cdot \frac{F_{n-2}}{P_{n-2}}}{L}} \]

where \( x \) and \( y \) in the limit on the right-hand side become infinite in the ratio \( \frac{x}{y} = -\frac{b}{a} \).

Or, if the curve be written in the form

\[
(ax + by)^2 P_{n-2} + (ax + by) F_{n-2} + f_n = 0,
\]

in proceeding to infinity in the direction \( ax + by = 0 \), we have

\[
(ax + by)^2 + (ax + by) \cdot \text{Lt} \frac{F_{n-2}}{P_{y-2}} + \text{Lt} \frac{f_{n-2}}{P_{n-2}} = 0,
\]

when the limits are to be obtained by putting \( x = -\frac{b}{t}, \)

\( y = \frac{a}{t} \), and then diminishing \( t \) indefinitely. We thus obtain a pair of parallel asymptotes,

\[ ax + by = \alpha \] and \[ ax + by = \beta, \]

where \( \alpha \) and \( \beta \) are the roots of

\[ \rho^2 + \rho \text{Lt} \frac{F_{n-2}}{P_{n-2}} + \text{Lt} \frac{f_{n-2}}{P_{n-2}} = 0. \]

And other particular forms which the equation of the curve may assume can be treated similarly.

**Ex. 1.** To find the pair of parallel asymptotes of the curve

\[
(2x - 3y + 1)^2 (x + y) - 8x + 2y - 9 = 0.
\]

Here

\[ 2x - 3y + 1 = \pm \sqrt{L \cdot \frac{8x - 2y + 9}{x + y}}, \]

where \( x \) and \( y \) become infinite in the direction of the line \( 2x = 3y \).

Put \( x = \frac{3}{t}, \) \( y = \frac{2}{t} \), the right side becomes \( +2 \). Hence the asymptotes required are \( 2x - 3y = 1 \) and \( 2x - 3y + 3 = 0. \)
2. Find the asymptotes of

\[(x - y)^2 (x^2 + y^2) - 10 (x - y) x^2 + 12y^2 + 2x + y = 0.\]

Here \[(x - y)^2 - 10 (x - y) L \to r^2 + y^2 + 12L \to r^2 + y^2 = 0,\]

or \[(x - y)^2 - 5 (x - y) + 6 = 0,\]

giving the parallel asymptotes \(x - y = 2\) and \(x - y = 3.\)

109. Asymptotes by Inspection.

It is now clear that if the equation \(F' = 0\) break up into linear factors so as to represent a system of \(n\) straight lines, no two of which are parallel, they will be the asymptotes of any curve of the form

\[F' + F'' = 0.\]

Ex. 1. \((x - y)(x + y)(x + 2y - 1) = 3x + 4y + 5\)

is a cubic curve whose asymptotes are obviously

\[x = 0,\]
\[y = 0,\]
\[x + y = 0,\]
\[x + 2y - 1 = 0.\]

Ex. 2. \((x - y)^2 (x + 2y - 1) = 3x + 4y + 5.\)

Here \(x + 2y - 1 = 0\) is one asymptote. The other two asymptotes are parallel to \(y = x.\) Their equations are

\[x = y - \sqrt{\frac{3 + 4 + 5t}{1 + 2 - t}} = \frac{3}{\sqrt{7}}.\]

110. Case in which all the Asymptotes pass through the Origin.

If then, when the equation of a curve is arranged in homogeneous sets of terms, as

\[u_n + u_{n-2} + u_{n-3} + \ldots = 0,\]

it be found that there are no terms of degree \(n - 1,\) and if also \(u_n\) contain no repeated factor, the \(n\) straight lines passing through the origin, and whose equation is \(u_n = 0,\) are the \(n\) asymptotes.
EXAMPLES.

Find the asymptotes of the following curves:

1. \( y^2 = x^2 (2a - x) \).
2. \( y^2 = x (a^2 - x^2) \).
3. \( x^3 + y^3 = a \).
4. \( y (a^2 + x^2) = a^2 x \).
5. \( u.v_y = x^3 - a^3 \).
6. \( x^2 (2a - x) = x^3 \).
7. \( x^3 + y^3 = 3a.x.y \).
8. \( x^2 y + y^2 x = a^3 \).
9. \( x^2 y^2 = (x + y)^2 (a^2 - y^2) \).
10. \( x^2 y^2 = a^2 y^2 - b^2 x^2 \).
11. \( xy (x - y) = a (a^2 - y^2) - b \).
12. \( (a^2 - a^2) y^2 = x^3 (a^2 + x^2) \).
13. \( x^2 y^2 = 2a^2 (2a - x) \).
14. \( y^2 (a - x) = x (b - x)^2 \).
15. \( x^2 y = x^3 + x + y \).
16. \( x^2 y^2 + a y^2 = x^3 + m.v^2 + n.v + \mu \).
17. \( x^2 + 2a^2 y - x^2 y^2 - 2y^2 + 4y^2 + 2xy + y = 0 \).
18. \( x^3 - 2a^2 y + y^2 + x^2 - xy + y = 2 - 0 \).
19. \( y (x - y)^2 - y (x - y) + 2 \).
20. \( x^3 + 2x^2 y - 4xy^2 - 8y^3 - 4x + 8y = 1 \).
21. \( (x + y)^2 (x + 2y + 2) = x + 9y - 2 \).
22. \( 3x^3 + 17x^2 y + 21x^2 y^2 - 9y^3 - 2x^3 - 12xv^2 - 18xy^2 + 3x^2 v + 4x^2 y = 0 \).

**111. Intersections of a Curve with its Asymptotes.**

If a curve of the \( n \)th degree have \( n \) asymptotes, no two of which are parallel, we have seen in Art. 109 that the equations of the asymptotes and of the curve may be respectively written

\[
F_n = 0,
\]

and

\[
F_n + F_n' = 0.
\]

The \( n \) asymptotes therefore intersect the curve again at points lying upon the curve \( F_{n-2} = 0 \). Now each asymptote cuts its curve in two points at infinity, and therefore in \( n - 2 \) other points. Hence these \( n (n - 2) \) points lie on a certain curve of degree \( n - 2 \). For example,
1. The asymptotes of a cubic will cut the curve again in three points lying in a straight line;
2. The asymptotes of a quartic curve will cut the curve again in eight points lying on a conic section;
and so on with curves of higher degree.

EXAMPLES.

1. Find the equation of a cubic which has the same asymptotes as the curve \( x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0 \), and which touches the axis of \( y \) at the origin, and goes through the point (3, 2).

2. Show that the asymptotes of the cubic
\[ x^3y - xy^2 + xy + y^2 + r - y = 0 \]
cut the curve again in three points which lie on the line \( r + y = 0 \).

3. Find the equation of the conic on which lie the eight points of intersection of the quartic curve
\[ xy(x^2 - y^2) + a^2y^2 + b^2x^2 - a^2b^2 \]
with its asymptotes.

4. Show that the four asymptotes of the curve
\[ (x^2 - y^2)(y^2 - 4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - x^2 + 3y - 1 = 0 \]
cut the curve again in eight points which lie on a circle.

112. **Polar co-ordinates.**

When the equation of a curve is given in the form
\[ rf_1(\theta) + f_0(\theta) = 0 \],
it is clear that the directions given by
\[ f_1(\theta) = 0 \],
are those in which \( r \) becomes infinite.

Let this equation be solved, and let the roots be \( \alpha, \beta, \gamma, \) etc.
Let $\overline{XO} = \alpha$. Then the radius $OP$, the curve, and the asymptote meet at infinity towards $P$. Let $OY (= \rho)$

be the perpendicular upon the asymptote. Since $OY$ is at right angles to $OP$ it is the polar subtangent; and 

\[ \rho = -\frac{d\theta}{du}. \]

Let $\overline{XO} = \alpha'$, and let $Q$ be any point whose co-ordinates are $r, \theta$ upon the asymptote. Then the equation of the asymptote is

\[ \rho = r \cos (\theta - \alpha') \ldots \ldots \ldots \ldots \ldots \ldots (3). \]

It is clear from the figure that \( \alpha' = \alpha - \frac{\pi}{2} \).

To find the value of \(-\frac{d\theta}{du}\) when \( u = 0 \), write \( \frac{1}{u} \) for \( r \) in equation (1), and we have

\[ f_1(\theta) + uf_0(\theta) = 0. \]

Whence differentiating

\[ f_1'(\theta) + uf_0'(\theta) + \frac{du}{d\theta} f_0'(\theta) = 0. \]
Putting $\theta = \alpha$, and therefore $u = 0$, we have (if $f_0'(\alpha)$ be finite)

$$
\left( - \frac{d\theta}{du} \right)_{u=0} = \frac{f_0(\alpha)}{f_1'(\alpha)} \quad \text{(4)}
$$

Substitute this value of $\left( - \frac{d\theta}{du} \right)_{u=0}$ for $p$ in equation (3) and we obtain

$$
\frac{f_0(\alpha)}{f_1'(\alpha)} = r \cos \left( \theta - \alpha + \frac{\pi}{2} \right) = r \sin (\alpha - \theta).
$$

Hence the equations of the asymptotes are

$$
r \sin (\alpha - \theta) = \frac{f_0(\alpha)}{f_1'(\alpha)},
$$

$$
r \sin (\beta - \theta) = \frac{f_0(\beta)}{f_1'(\beta)},
$$

etc.

113. Rule for Drawing the Asymptote.

After having found the value of $\left( - \frac{d\theta}{du} \right)_{u=1}$ imagine we stand at the origin looking in the direction of that value of $\theta$ which makes $u = 0$. Draw a line at right angles to that direction through the origin and of length equal to the calculated value of $\left( - \frac{d\theta}{du} \right)_{u=0}$ to the right or to the left, according as that value is positive or negative. Through the end of this line draw a perpendicular to it of indefinite length. This straight line will be the asymptote.

Ex. Find the asymptotes of the curve

$$
r \cos \theta - a \sin \theta = 0.
$$

Here $f_1(\theta) = \frac{r}{a}$.  
