with respect to these axes are: \( \Omega + \mu \cos \theta, -\mu \sin \theta, \) and \( \dot{\theta} \); therefore, the components of the moment of momentum are \( C(\Omega + \mu \cos \theta), -A\mu \sin \theta, \) and \( A\dot{\theta} \). Thus, the kinetic energy is given by

\[
T = \frac{1}{2}[C(\Omega + \mu \cos \theta)^2 + A\mu^2 \sin^2 \theta + A\dot{\theta}^2] \tag{5.8}
\]

We introduce the coordinate \( \beta \), which we ignored up to this point, \( \text{viz.} \), the angular displacement of the top around the \( \gamma \)-axis. We have \( \Omega = \dot{\beta} \)

Substituting \( \mu = \varphi \) and \( \Omega = \beta \), we have

\[
T = \frac{1}{2}[C(\dot{\beta} + \varphi \cos \theta)^2 + A\varphi^2 \sin^2 \theta + A\dot{\theta}^2] \tag{5.9}
\]

and the potential energy is given by

\[
U = mgs \cos \theta \tag{5.10}
\]

Hence, Lagrange's equations of motion for the generalized coordinates \( \beta, \varphi, \) and \( \theta \) are

\[
\frac{d}{dt} \left[ C(\dot{\beta} + \varphi \cos \theta) \right] = 0 \tag{5.11}
\]

\[
\frac{\dot{\varphi}}{dt} \left[ C(\dot{\beta} + \varphi \cos \theta) \cos \theta + A\varphi^2 \sin \theta \right] = 0 \tag{5.12}
\]

\[
\frac{\ddot{\theta}}{dt}(A\dot{\theta}) + C(\dot{\beta} + \varphi \cos \theta) \dot{\varphi} \sin \theta - A\varphi^2 \sin \theta \cos \theta = mgs \sin \theta \tag{5.13}
\]

From Eq. (5.11) it follows that \( C(\dot{\beta} + \varphi \cos \theta) = \text{const.} = H_z \). Then we have

\[
-H_z \sin \theta \cdot \dot{\theta} + A\varphi \sin^2 \theta + 2A\varphi \dot{\theta} \sin \theta \cos \theta = 0 \tag{5.14}
\]

\[
A\dot{\beta} + H_z \varphi \sin \theta - A\varphi^2 \sin \theta \cos \theta = mgs \sin \theta \tag{5.15}
\]

We put \( \dot{\varphi} = r \) and take into account that \( \theta \) and \( r \) are small. Then, multiplying (5.14) by \( a \), we obtain

\[
-H_z \dot{\varphi} + A\varphi r + 2A\varphi \dot{\theta} = 0 \tag{5.16}
\]

and from (5.15)

\[
A\dot{r} + H_z \varphi - Ar \varphi^2 = mgsr \tag{5.17}
\]

or

\[
\frac{d}{dt}(r^2 \varphi) = \frac{H_z}{A} \dot{r} \tag{5.18}
\]

\[
\dot{r} - r^2 \varphi = \frac{mgs}{A} \tau - \frac{H_z}{A} r \dot{\varphi}
\]

It is seen that Eqs. (5.18) represent the equations of motion of a particle of unit mass whose polar coordinates in a plane are \( r \) and \( \varphi \) under the action of a repulsive radial force \(\frac{mgs}{A} \tau\) and a force \(\frac{H_z}{A} \sqrt{r^2 + r^2 \varphi^2}\) whose direction is normal to the velocity of the mass point. Hence, Eqs. (5.18) are equivalent to (5.7).
We shall investigate the stability of the motion defined by Eqs. (5.7). Substituting \( x_s = X e^{\lambda t} \) and \( y_s = Y e^{\lambda t} \), we obtain

\[
X \left( \lambda^2 - \frac{mgs}{A} \right) = -Y \frac{H_z}{A} \lambda \\
Y \left( \lambda^2 - \frac{mgs}{A} \right) = X \frac{H_z}{A} \lambda
\]

or by multiplication,

\[
\left( \lambda^2 - \frac{mgs}{A} \right)^2 = - \left( \frac{H_z}{A} \right)^2 \lambda^2
\]

The roots of Eq. (5.20) are

\[
\lambda = \pm \frac{H_z}{2A} i \pm \sqrt{\frac{mgs}{A} - \left( \frac{H_z}{2A} \right)^2}
\]

It is seen that if \((H_z/2A)^2 > mgs/A\), all four roots of Eq. (5.20) are pure imaginary quantities, and we obtain oscillations with constant amplitude. If \((H_z/2A)^2 < mgs/A\), two roots have positive real parts and two have negative. In this case the motion is unstable. The condition for stability is, therefore,

\[
H_z^2 > 4Amgs
\]

Hence, the rotation of a symmetrical top around a vertical axis is stable if its moment of momentum is sufficiently large. The minimum amount required is given by Eq. (5.22).

If \( s \) is less than zero, the center of gravity is below the fixed point of the top. Then Eqs. (5.7) give the small oscillations of a so-called gyroscopic pendulum. The reader will be able to discuss this case, which is also quite interesting, without difficulty.

6. Stability Conditions for Oscillating Systems.—A motion of the type \( q = \text{const.} e^{\lambda t} \) is stable if the exponent \( \lambda \) has no real positive part. In the two examples (sections 2 and 5) in which we discussed the stability of systems, the characteristic equations were bi-quadratic. In such cases the character of the roots can easily be decided. In this section we discuss the case in which the equation for \( \lambda \) is a cubic or an arbitrary quartic. A few remarks will be devoted to higher degree equations, which we encounter when dealing with three or more degrees of freedom.

Stability Conditions for the Cubic Equation.—We assume that the equation

\[
\lambda^3 + a\lambda^2 + b\lambda + c = 0
\]
is the characteristic equation of a system; we shall find the condition for the stability of the system. For stability it is necessary that none of the roots has a positive real part. Now, if \( \lambda_1, \lambda_2, \lambda_3 \) are the roots, the left side of Eq. (6.1) is equal to

\[
(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)
\]

Then it is seen that \(-a\) is equal to the sum of the real parts of the roots. Hence, \(-a\) must be negative, i.e., \(a > 0\). The same holds for the coefficient \(c\), because \(-c\) is equal to the product of the three roots. If the three roots are all real and negative, their product is certainly negative, and \(c > 0\). If two roots are complex conjugates, e.g., \(\lambda_1 = \alpha + i\beta\) and \(\lambda_2 = \alpha - i\beta\), \(-c\) is the product of the real root \(\lambda_3\) and of the product \(\lambda_1\lambda_2 = \alpha^2 + \beta^2\). Hence, if \(\lambda_3\) is negative, again \(c > 0\). Finally, consider the coefficient \(b = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1\). If all roots are real and negative, \(b > 0\); if, for instance, \(\lambda_1\) and \(\lambda_2\) are complex conjugates, we write \(b = \lambda_1\lambda_2 + \lambda_3(\lambda_1 + \lambda_2)\). Now \(\lambda_1\lambda_2\) is always positive; if the real parts of \(\lambda_1\) and \(\lambda_2\) are negative, \(\lambda_1 + \lambda_2\) is negative and multiplied by the negative \(\lambda_3\) again makes \(b\) positive. It follows that for stability all coefficients of the equation must be positive. However, this condition is not sufficient.

If \(a > 0\), \(b > 0\), \(c > 0\), one of the roots, say \(\lambda_1\), is certainly real and negative; however, the two other roots, say \(\lambda_2\) and \(\lambda_3\), could be conjugate complex with a positive real part. For example, if \(c\) is very large compared to \(a\) and \(b\), \(\lambda\) will have a positive real part, since, approximately, \(\lambda^3 + c = 0\), or \(\lambda = c^{1/3}\sqrt[3]{-1}\), i.e.,

\[
\lambda_1 \cong -c^{\frac{1}{3}}, \quad \lambda_2 \cong \left(\frac{1}{2} + \frac{i}{2}\sqrt{3}\right)c^{\frac{1}{3}}, \quad \lambda_3 \cong \left(\frac{1}{2} - \frac{i}{2}\sqrt{3}\right)c^{\frac{1}{3}}.
\]

If we now vary the coefficient \(c\), we pass through a value of \(c\) for which the real parts of \(\lambda_2\) and \(\lambda_3\) become negative. For this limiting value of \(c\), the roots \(\lambda_2\) and \(\lambda_3\) are pure imaginary, for example, \(\lambda_2 = \beta i\), \(\lambda_3 = -\beta i\). Physically speaking the system is at the stability limit, carrying out pure harmonic vibration of the frequency \(\beta\). Substituting \(\lambda = \pm i\beta\) in Eq. (6.1), we have

\[
-\beta^3 + h\beta = 0
\]

\[
-a\beta^2 + c = 0
\]

or eliminating \(\beta\),

\[
\frac{c}{a} = b
\]
If we consider \( a \) and \( b \) as fixed values and \( c \) variable, the roots are continuous functions of \( c \). Then it follows that the complex roots will have a positive real part for

\[
c > ab
\]

and a negative real part if

\[
c < ab
\]

The condition (6.5), together with \( a > 0 \), \( b > 0 \), \( c > 0 \), is necessary and sufficient for stability.

**Stability Conditions for the Quartic Equation.**—Let us write the characteristic quartic equation in the form:

\[
f(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0
\]

Then a reasoning analogous to that used in the case of the cubic shows that in case of stability \( a > 0 \) and \( d > 0 \) because the sum of the roots must be negative and their product must be positive. The coefficient \( b = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 \) can be written in the form:

\[
b = (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4) + \lambda_1\lambda_2 + \lambda_3\lambda_4
\]

Now, if all roots are real and negative, \( b > 0 \). If \( \lambda_1 \) and \( \lambda_2 \) are a pair of conjugate complex roots, \( \lambda_1 + \lambda_2 \) is real and negative, and \( \lambda_1\lambda_2 \) is real and positive; hence, again \( b > 0 \). If both \( \lambda_1 \), \( \lambda_2 \) and \( \lambda_3 \), \( \lambda_4 \) are pairs of conjugate complex roots, both \( \lambda_1 + \lambda_2 \) and \( \lambda_3 + \lambda_4 \) are real and negative, and \( \lambda_1\lambda_2 \) and \( \lambda_3\lambda_4 \) are real and positive; consequently, again \( b > 0 \) holds. It is easy to show in a similar way that under the assumption of negative real parts \( c > 0 \). If the real parts of all roots are equal to zero, \( a = c = 0 \). This is the case of the biquadratic equation discussed in the previous section.

Hence, for stability it is necessary that none of the coefficients of (6.6) should be negative. However, this condition is not sufficient. There is an additional condition which can be obtained by the method of the so-called test function.*

Let us consider the equation

\[
g(\lambda) = \lambda^4 + a(\alpha)\lambda^3 + b(\alpha)\lambda^2 + c(\alpha)\lambda + d(\alpha) = 0
\]

* The reasoning applied above to the cubic equation can be applied also to the quartic and yields immediately the condition (6.10). However, we prefer to introduce the test function in view of later applications.
whose coefficients are functions of a parameter $\alpha$ and satisfy the following conditions:

a. The values $a(0)$, $b(0)$, $c(0)$, and $d(0)$ for $\alpha = 0$ are equal to the values of the coefficients in Eq. (6.6).

b. The roots of (6.7) are equal to $\lambda_1 + \alpha$, $\lambda_2 + \alpha$, $\lambda_3 + \alpha$, $\lambda_4 + \alpha$, where $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$ are the roots of Eq. (6.6). [How $a(\alpha)$, $b(\alpha)$, . . . can be calculated is shown later in Eq. (7.2)].

It is seen that if, for example, $\lambda_1 = -\alpha_1 + i\beta_1$, Eq. (6.7) has for $\alpha = \alpha_1$ two pure imaginary roots, viz., $\pm \beta_1 i$. For such a value of $\alpha$, the functions $a(\alpha)$, $b(\alpha)$, $c(\alpha)$, and $d(\alpha)$ must satisfy simple conditions. Specifically, if (6.7) has a pure imaginary root $\beta i$, we have

$$\beta^4 - b\beta^2 + d = 0 \tag{6.8}$$

and

$$a\beta^2 - c = 0$$

since the real and imaginary part of (6.6) must vanish separately.

Eliminating $\beta^2$, we have

$$\frac{c^2}{a^2} - \frac{b c}{a} + d = 0$$

or

$$c^2 - abc + a^2 d = 0 \tag{6.9}$$

We call the function

$$F(\alpha) = c^2 - abc + a^2 d$$

where $a$, $b$, $c$, and $d$ are functions of $\alpha$, as defined above, the test function of the quartic (6.6).

It is evident that as often as $-\alpha$ is equal to the real part of a root, of $f(\lambda) = 0$, we have $F(\alpha) = 0$, i.e., this equation has a real root. If the real parts of all the roots of $f(\lambda)$ are negative, all real roots of $F(\alpha)$ must be positive, i.e., must lie between $\alpha = 0$ and $\alpha = +\infty$, and no root of $F(\alpha)$ can lie between $\alpha = -\infty$ and $\alpha = 0$. Now if $\alpha$ is a very large negative number, say $-N$, the four roots of the polynomial $g(\lambda)$ will be approximately equal to $-N$, and $g(\lambda)$ will have the approximate form $g(\lambda) \approx (\lambda + N)^4$. Expanding this expression, we have $a(\alpha) \approx 4N$, $b(\alpha) \approx 6N^2$, $c(\alpha) \approx 4N^3$, and $d(\alpha) = N^4$. Hence, $F(-N) = c^2 - abc + a^2 d \approx -64N^6$, and therefore $F(-\infty)$ is certainly negative. Consequently, if $F(\alpha)$ has no negative real roots, $F(0)$ must be negative too.
We obtain the result that if the roots of \( f(\lambda) = 0 \) have no positive real part,
\[
F(0) = c^2 - abc + a^2d < 0^*
\]
where \( a, b, c, \) and \( d \), are the values of \( a(\alpha), b(\alpha), c(\alpha), \) and \( d(\alpha) \) for \( \alpha = 0, \) i.e., the coefficients of \( f(\lambda) \).

Condition (6.10) is necessary for stability and, together with the rules which refer to the signs of the coefficients, is sufficient.

That the conditions \( a > 0, b > 0, c > 0, d > 0, \) and \( F(0) < \alpha \)
are sufficient for stability of the system whose characteristic equation is \( f(\lambda) = 0 \), can be shown in the following way: If all coefficients of \( f(\lambda) \) are positive, \( f(\lambda) = 0 \) can have no real positive roots; however, it would be possible to have complex roots with positive real parts. Now, since \( F(\infty) \) and \( F(0) \)
have identical signs, as was shown above, if \( F(\alpha) = 0 \) has real roots between \( \alpha = -\infty \) and \( \alpha = 0 \), it must have an even number of them. Hence, there must be two pairs of conjugate complex roots with positive real parts, if any, e.g., \( \lambda_{1,2} = \alpha_1 \pm i\beta_1 \) and \( \lambda_{3,4} = \alpha_2 \pm i\beta_2 \), where \( \alpha_1 \) and \( \alpha_2 \) are positive. But in this case \( a = -2(2\alpha_1 + 2\alpha_2) \) would be negative which contradicts the condition \( a > 0 \). Hence, if \( a > 0, b > 0, c > 0, d > 0, \) and \( F(0) < \alpha, \) none of the roots can have positive real parts.

The test-function method can also be applied to equations of higher degree; for instance, in the case of a sextic, the expression \( F(0) \) is the so-called resolvent of two algebraic equations resulting from the substitution of \( \lambda = i\beta \) and the separation of the real and imaginary terms.

7. Calculation of Complex Roots of Algebraic Equations.— There are exact formulas for the calculation of the roots of cubic equations. However, in general, it will be simpler to calculate real roots by one of the methods given in Chapter V, or if there is a pair of conjugate complex roots to calculate first the real root, \( \lambda_1 \), divide the equation by \( \lambda - \lambda_1 \), and solve the remaining equation of second degree.

In the case of quartics the following methods are recommended:

a. The Test-function Method.—The test function introduced in the last section can also be used for numerical computation of the complex roots by plotting \( F(\alpha) \) as function of \( \alpha \) and determining the real roots of \( F(\alpha) = 0 \). The roots of \( F(\alpha) = 0 \) determine

* Equation (6.10) for \( d = 0 \) becomes identical with Eq. (6.5).
the real parts of the roots of \( f(\lambda) = 0 \). Then the complex parts are determined by one of the equations (6.8), e.g., by the relation \( \beta^2 = c/a \).

In order to plot \( F(\alpha) \), the coefficients \( a(\alpha), b(\alpha), c(\alpha), \) and \( d(\alpha) \) must be computed as functions of \( \alpha \). Let us denote the roots of \( f(\lambda) = 0 \), by \( \lambda_1, \lambda_2, \lambda_3, \) and \( \lambda_4 \); then the polynomial whose roots are equal to \( \lambda_1 + \alpha, \lambda_2 + \alpha, \lambda_3 + \alpha, \) and \( \lambda_4 + \alpha \) is given by

\[
g(\lambda) = f(\lambda - \alpha) - f(-\alpha) + f'(-\alpha)\lambda + \frac{f'''(-\alpha)}{1 \cdot 2} \lambda^2 \\
+ \frac{f''''(-\alpha)}{1 \cdot 2 \cdot 3} \lambda^3 + \frac{f'''''}{1 \cdot 2 \cdot 3 \cdot 4} \lambda^4 \tag{7.1}
\]

For example, \( g(\lambda_1 + \alpha) = f(\lambda_1 + \alpha - \alpha) = f(\lambda_1) = 0 \). We obtain

\[
g(\lambda) = \lambda^4 + (a - 4\alpha)\lambda^3 + (b - 3u\alpha + 6a\alpha^2)\lambda^2 \\
+ (c - 2b\alpha + 3\alpha^2 - 4a\alpha^3)\lambda + (d - ca + ba^2 - a\alpha^3 + \alpha^4) \tag{7.2}
\]

The coefficients of \( \lambda^3, \lambda^2, \lambda, 1 \) are the functions \( a(\alpha), b(\alpha), c(\alpha), d(\alpha) \) and, substituting them in \( F(\alpha) = c^2 - a(bc - ad) \), we obtain the test function sought for.

b. **Approximate Factorization.**—Let us assume that the quartic equation

\[
f(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0 \tag{7.3}
\]

has two pairs of conjugate complex roots \( \lambda_1, \lambda_2 \) and \( \lambda_3, \lambda_4 \) and in addition, that \( |\lambda_1| = |\lambda_2| > |\lambda_3| = |\lambda_4| \). Then it is easily shown by a reasoning quite similar to that applied in Chapter V to equations with real roots that \( \lambda_1 \) and \( \lambda_2 \) are well approximated by the roots of the equation \( \lambda^2 + a\lambda + b = 0 \) and the small roots \( \lambda_3 \) and \( \lambda_4 \) by the roots of the equation \( b\lambda^2 + c\lambda + d = 0 \). Hence, the expression

\[
f(\lambda) \simeq (\lambda^2 + a\lambda + b)\left(\lambda^2 + \frac{c}{b}\lambda + \frac{d}{b}\right) = 0 \tag{7.4}
\]

represents an approximate factorization of Eq. (7.3).

If the ratio \( |\lambda_1|/|\lambda_2| \) is not sufficiently large, the method of squaring the roots can be applied in a manner similar to that used in the case of real roots. The method is not very efficient in the case of complex roots because, if we apply the squaring \( n \) times, we obtain an equation for \( \lambda^{2n} \); and if \( \lambda \) is complex, it is rather hard to determine which of the \( 2n \) roots of \( \lambda^{2n} \) is the correct
root of the original equation (7.3). In the case of real roots, only the sign of \( \lambda \) is dubious; in the case of complex roots we obtain \( 2n \) different complex numbers as possible values for \( \lambda \).

The following general theorem, which is quoted here without proof, is useful for equations of higher degree and for equations with some real and some complex roots:

If the absolute values of \( r \) roots (real or complex) of the equation

\[
f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0 \tag{7.5}
\]

are very large in comparison with the rest of the roots, the \( r \) large roots are approximately given by the roots of the equation:

\[
\lambda^r + a_1\lambda^{r-1} + \cdots + a_r = 0
\]

and the rest of the roots are approximated by the roots of the equation

\[
a_r\lambda^{n-r} + a_{r+1}\lambda^{n-r-1} + \cdots + a_n = 0
\]

c. Extension of Newton's Method to Complex Roots.—Newton's original method can be applied to the calculation of complex roots if we start from an approximate complex value. Starting from a real value, the method can never lead to complex roots as is seen by Eq. (8.1) in Chapter V. However, the method can be modified by approximating the function \( f(\lambda) \) by a quadratic function instead of a linear one. Using Taylor's expansion, we write

\[
f(\lambda) = f(\lambda_1) + f'(\lambda_1)(\lambda - \lambda_1) + \frac{1}{2}f''(\lambda_1)(\lambda - \lambda_1)^2 \tag{7.6}
\]

Choosing now an arbitrary value of \( \lambda_1 \) as first approximation, we obtain, solving Eq. (7.6) for \( \lambda \),

\[
\lambda = \lambda_1 - \frac{f'(\lambda_1)}{f''(\lambda_1)} \pm \sqrt{\frac{f'(\lambda_1)^2 - 2f(\lambda_1)f''(\lambda_1)}} {f''(\lambda_1)} \tag{7.7}
\]

If the radical in the last term is negative we obtain a complex value for the next approximation of the root. By the process of iteration we are able to obtain successive approximations for a pair of conjugate complex roots.

This method has the disadvantage that very complicated numerical work has to be done with complex quantities. It is an advantage to have schemes for the numerical calculation which
involve only real quantities. If $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ are one pair of complex roots of the equation $f(\lambda) = 0$, the polynomial $f(\lambda)$ is divisible by the quadratic factor

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2$$

Hence, if we assume approximate values for the coefficients $2\alpha$ and $\alpha^2 + \beta^2$ and improve these coefficients by the process of iteration, we also improve the approximations for the corresponding roots. Methods for successive correction of the coefficients of a quadratic factor have been worked out by L. Bairstow, J. I. Craig, and others.

8. Longitudinal Stability of an Airplane.—Assume that every point of an airplane moves in a vertical plane parallel to its plane of symmetry; we call such a motion the **longitudinal motion** of the airplane. The longitudinal motion of the airplane is described by the translation of its center of gravity and the rotation of the airplane body around an axis normal to the plane of motion. The equations of motion were given in Chapter IV [Eqs. (9.1) to (9.3)]

$$m \frac{dv}{dt} = -D + mg \sin \theta$$
$$mv \frac{d\theta}{dt} = -L + mg \cos \theta$$
$$I \frac{d^2\varphi}{dt^2} = -M$$

In these equations the following notations are used:

- $v$ the magnitude of the velocity of the center of gravity.
- $\theta$ the inclination of the flight path, positive downward.
- $\varphi$ the angle between the longitudinal axis of the airplane and the horizontal, positive upward.
- $m$ the mass of the airplane.
- $I$ the moment of inertia of the airplane around a transversal axis through the center of gravity.
- $D$ the drag, i.e., the aerodynamic force opposite to the flight direction.
- $L$ the lift, i.e., the aerodynamic force normal to the flight direction.
- $M$ the moment of the aerodynamic forces, diving moment being measured positive.
To investigate the small oscillations of the airplane in the neighborhood of uniform level flight with the velocity \( v_0 \) we assume that the velocity \( v = v_0 + u \), where \( u \) is small compared to \( v_0 \) and \( \theta, \varphi \) are small angles. Furthermore, we assume that the drag \( D \) can be neglected or that it is balanced at every instant by a propeller thrust of equal magnitude. Then Eqs. (8.1) upon neglecting terms of higher order give us the following relations:

\[
\begin{align*}
\frac{1}{g} \frac{d}{dt} u &= \theta \\
\frac{v_0}{g} \frac{d}{dt} \theta &= -\frac{L}{mg} + 1 \\
\frac{d^2 \varphi}{dt^2} &= -\frac{M}{m}
\end{align*}
\]  
(8.2)

where \( i^2 = I/m \), i.e., \( i \) is equal to the radius of gyration of the airplane around its transversal axis.

In level flight \( L = mg \) and \( M = 0 \). Hence, if \( \Delta L \) and \( \Delta M \) are the increments of the lift and the moment, we have

\[
L = mg + \Delta L, \quad M = \Delta M
\]

Let us calculate the increments \( \Delta L \) and \( \Delta M \) as functions of \( u, \theta, \) and \( \varphi \). It is assumed in general that the lift is proportional to the square of the velocity of flight and is a linear function of the angle of attack \( \alpha \). We measure the angle of attack from the value corresponding to zero lift and assume that \( \alpha = \alpha_0 \) and \( \varphi = 0 \) correspond to level flight. Then we write

\[
L = mg \left( \frac{v}{v_0} \right)^2 \frac{\alpha}{\alpha_0}
\]
(8.3)

or substituting \( v = v_0 + u \) and \( \alpha = \alpha_0 + \Delta \alpha \) and neglecting terms of higher order

\[
L = mg \left( 1 + 2 \frac{u}{v_0} + \frac{\Delta \alpha}{\alpha_0} \right)
\]
(8.4)

Taking into account that \( \Delta \alpha = \theta + \varphi \),

\[
L = mg \left( 1 + 2 \frac{u}{v_0} + \frac{\theta + \varphi}{\alpha_0} \right)
\]
(8.5)
The aerodynamic moment $M$ consists of the wing moment and the moment of the aerodynamic force acting on the tail surface. The tail force consists of two parts: one due to the change of the angle of attack, the other due to the rotation of the airplane. We combine the first part of the tail force and the wing moment into one term, in that we write for the total stabilizing moment produced by a change $\Delta\alpha$ of the angle of attack
\[
\Delta M_1 = mgck_m\Delta\alpha
\] (8.6)
where $c$ is the mean chord length of the wing and $k_m$ is a numerical factor. We say that $k_m$ determines the static stability of the airplane, i.e., the magnitude of the restoring or stabilizing moment.

The contribution of the rotation to the tail force corresponds to an apparent increase of the angle of attack. If the airplane rotates with the angular velocity $d\varphi/dt$ and the distance between the center of pressure of the tail and the center of gravity of the airplane is $l$, the vertical velocity of the tail is equal to $l\frac{d\varphi}{dt}$. This is equivalent to a change in angle of attack of the amount $\frac{1}{v_0}\frac{l}{dt}d\varphi$. Thus the rotation produces a force on the tail which is proportional to $\frac{1}{v_0}\frac{l}{dt}d\varphi$, and we can write the corresponding moment about the center of gravity of the airplane in the form:
\[
\Delta M_2 = k_i mgl^2\frac{\ddot{\varphi}}{v_0}
\] (8.7)
where $k_i$ is a numerical constant that depends especially on the ratio between tail and wing surface.

Substituting (8.5), (8.6), and (8.7) into Eq. (8.2), we obtain
\[
\begin{align*}
\frac{1}{g}\dot{u} & = \theta \\
v_0\dot{\theta} & = -2\frac{u}{v_0} - \frac{1}{\alpha_0}(\theta + \varphi) \\
\frac{1}{g}\ddot{\varphi} & = -k_m\frac{c}{\alpha_2}(\varphi + \theta) - k_i l^2\frac{1}{v_0}\dot{\varphi}
\end{align*}
\] (8.8)
We eliminate $u$ and obtain the following two equations for $\theta$ and $\varphi$:
\[ \frac{v_0^2}{g} \dot{\theta} = -2\theta - \frac{v_0}{g\alpha_0} (\theta + \phi) \] (8.9)

\[ \frac{v_0^2}{g} \ddot{\phi} = -k_m \frac{cv_0^2}{g t^2} (\phi + \theta) - k_t \frac{l^2}{t^2} v_0 \phi \]

In order to obtain a dimensionless form of these equations we introduce the dimensionless time parameter \( \tau = tg/v_0 \). This amounts to using \( v_0/g \) as an appropriate time unit. Then we have

\[ \frac{d^2\theta}{d\tau^2} = -2\theta - \frac{1}{\alpha_0} \frac{d\theta}{d\tau} - \frac{1}{\alpha_0} \frac{d\phi}{d\tau} \] (8.10)

\[ \frac{d^2\phi}{d\tau^2} = -\sigma (\phi + \theta) - \delta \frac{d\phi}{d\tau} \]

where \( \sigma \) and \( \delta \) are used as abbreviations for the two dimensionless quantities occurring in the second of Eqs. (8.9). It is seen that the solution of the stability problem depends on the following three dimensionless parameters:

\( \alpha_0 \) is a characteristic aerodynamic parameter of the airplane; it depends on the lift coefficient employed in level flight.

\( \sigma = k_m \frac{cv_0^2}{gt^2} \) is a parameter which is characteristic for the static stability of the airplane. In fact, \( \sigma > 0 \) means that a rotation of the airplane by an angle \( \phi \) produces a restoring moment. If \( \sigma < 0 \), the moment tends further to increase the angle \( \phi \). Hence, we say that if \( \sigma > 0 \), the airplane is statically stable; if \( \sigma < 0 \), it is statically unstable; if \( \sigma = 0 \), the airplane has neutral static stability.

\( \delta = k_t \frac{l^2}{t^2} \) is a parameter which determines the damping effect of the tail.

Substituting \( \theta = Ae^{\lambda \tau}, \phi = Be^{\lambda \tau} \), we obtain the following frequency equation:

\[ \begin{vmatrix} \lambda^2 + \frac{\lambda}{\alpha_0} + 2 & \frac{1}{\alpha_0} \lambda \\ \sigma & \lambda^2 + \delta \lambda + \sigma \end{vmatrix} = 0 \] (8.11)

or, expanding the determinant,

\[ \lambda^4 + \left( \frac{1}{\alpha_0} + \delta \right) \lambda^3 + \left( 2 + \sigma + \delta \alpha_0 \right) \lambda^2 + 2\delta \lambda + 2\sigma = 0 \] (8.12)
Before we investigate the stability criteria, let us assume that $\sigma$ is very large, so that all terms which do not contain $\sigma$ are small compared with the terms containing $\sigma$. Then, neglecting all the rest of the terms, we obtain

$$\lambda^2 + 2 = 0$$  \hspace{1cm} (8.13)

or $\lambda = \pm \sqrt{-2}$. Taking into account that $\tau = tg/v_0$, this value of $\lambda$ corresponds to harmonic oscillations with the period

$$T' = 2\pi \frac{v_0}{g\sqrt{\sigma}}$$

In addition to the roots $\lambda = \pm \sqrt{-2}$ Eq. (8.12) must have two other roots. It is seen that these roots must be of the order of magnitude $\sqrt{\sigma}$. In that case $\lambda^4$ and $\sigma \lambda^2$ are of the same order and are large compared with the rest of the terms. Thus, we have approximately

$$\lambda^2 + \sigma = 0$$  \hspace{1cm} (8.14)

or $\lambda = \pm \sqrt{-\sigma}$. The corresponding period is equal to

$$T = 2\pi \frac{v_0}{g\sqrt{\sigma}} = 2\pi \sqrt{\frac{2}{gck_m}}$$

Let us investigate the corresponding modes of oscillation: If $\sigma$ is very large compared with $\delta$, the two equations (8.10) are reduced to

$$\frac{d^2 \theta}{d\tau^2} + 2\theta = -\frac{1}{\alpha_0} \frac{d}{d\tau} (\theta + \varphi)$$

$$\frac{d^2 \varphi}{d\tau^2} + \sigma \varphi = -\sigma \theta$$  \hspace{1cm} (8.15)

If $\lambda = \pm \sqrt{-2}$, the second equation is satisfied with sufficient approximation by $(\varphi + \theta) = 0$, and the first equation gives (with $\tau = gt/v_0$)

$$\theta = C \sin \frac{\sqrt{2}gt}{v_0} + D \cos \frac{\sqrt{2}gt}{v_0}$$  \hspace{1cm} (8.16)

This result corresponds exactly to the result obtained in Chapter IV for the so-called phugoid motion. In fact, there the assumption was made that the airplane is so stable that every deviation
in the angle of attack is instantaneously corrected. This means that $\sigma$ is very large and $\Delta x = \varphi + \theta = 0$.

For $\lambda = \pm \sqrt{-\sigma}$, the first equation is satisfied with sufficient approximation by $\theta = 0$, and we are left with the equation

$$\frac{d^2\varphi}{dt^2} + \sigma \varphi = 0 \quad (8.17)$$

which means that the airplane oscillates about its center of gravity as a pendulum oscillates about its axis of suspension.

Hence, in the case of excessive static stability the motion of the airplane consists of a phugoid-like motion of the center of gravity and of an oscillation of the airplane body around the center of gravity.* The wave length of the phugoid motion is equal to $2\pi \frac{v_0^2}{g \sqrt{2}}$, where $v_0$ is the speed of level flight, and the period of the rotational oscillation is equal to $2\pi \sqrt{\frac{i^2}{gck_m}}$.

Let us now drop the assumption of large $\sigma$ and investigate the conditions for the stability of the oscillations defined by the frequency equation (8.12). We found in section 6 that in the case of stability, the coefficients

$$a = \frac{1}{\alpha_0} + \delta, \quad b = 2 + \sigma + \frac{\delta}{\alpha_0}, \quad c = 2\delta, \quad d = 2\sigma$$

of Eq. (8.12) must be positive. In addition, the expression

$$F = c^2 - a(bc - ad) \quad (8.18)$$

has to be negative.

It is seen that the coefficients $a$, $b$, $c$, and $d$ are positive if $\sigma > 0$, since $\delta$ and $\alpha_0$ are positive by their definition. Hence, an airplane can be stable only if $\sigma > 0$, i.e., if it has static stability.

Substituting the values of $a$, $b$, $c$, and $d$ into (8.18), we obtain as an additional stability criterion

$$4\delta^2 - \left(\frac{1}{\alpha_0} + \delta\right) \left(4\delta + \frac{2\delta^2}{\alpha_0} - \frac{2\sigma}{\alpha_0}\right) < 0$$

or

$$\sigma < \delta^2 + \frac{2\delta\alpha_0}{1 + \delta\alpha_0} \quad (8.19)$$

* As a matter of fact, both modes of oscillation are slightly damped if the stability criterion (8.19) is satisfied; the amplitude of the first mode increases slowly if the stability criterion is violated.
The line \( \sigma = \delta^2 + \frac{2\delta \alpha_0}{1 + \delta \alpha_0} \) is plotted in Fig. 8.1 corresponding to \( \alpha_0 = \frac{1}{10} \). The abscissa in this diagram is \( \delta \) (damping parameter), the ordinate is \( \sigma \) (parameter of static stability). It is seen that the airplane is stable only if it has a certain damping. The magnitude of \( \delta \) which is required for stability increases with increasing static stability. If the drag \( D \) is taken into account, the limiting curve which separates the stable and unstable regions in the \( \delta \sigma \) plane intersects the positive \( \sigma \)-axis. Hence, in reality, if the parameter of the static stability \( \sigma \) is beyond a certain limit, the airplane becomes dynamically stable also for \( \delta = 0 \).

**Problems**

1. A shaft rotates in an abundantly lubricated sleeve bearing. Assume that a displacement \( \rho \) of the center of the shaft from the center of the bearing produces a reaction of the magnitude \( R \) directed at an angle of \( 180° - \theta \) with the direction of the displacement (Fig. P.1). Also assume that the shaft is unloaded so that in the equilibrium position the center of the shaft coincides with the center of the bearing. Show that if damping is neglected, this equilibrium position is unstable unless \( \theta = 0 \).

2. A U-tube pressure gauge consists of two vertical tubes of diameter \( d_1 \) and a horizontal capillary tube of diameter \( d_2 = d_1/10 \) and length \( l \). The height of the liquid in the columns is equal to \( h_0 \) when the same pressure acts on both columns. The specific weight of the fluid is \( \gamma \), the coefficient of viscosity \( \mu \). Find the amplitude of the oscillations of the liquid columns if a variable pressure acts on one of the columns, when the pressure difference is given by \( p \left( 1 + \varepsilon \sin \frac{2\pi l}{T} \right) \) and \( \varepsilon \ll 1 \). The fluid resistance in the vertical tubes can be neglected.

*Hint:* The pressure drop in a capillary tube of diameter \( d \) and length \( l \) is, according to Poiseuille’s law, \( \Delta p = \frac{128 \mu l}{\pi d^4} Q \) where \( Q \) is the quantity of fluid flowing through per unit time.
3. Condenser Microphone.—The electromechanical setup known as the condenser microphone is shown schematically in Fig. P.3. The circuit consists of a battery of constant voltage $E$, a coil of inductance $L$, a resistance $R$, and a condenser of variable capacity $C$. The condenser has one fixed plate to which a movable plate of mass $m$ is attached elastically. This movable electrode is the receiving membrane of the microphone. It vibrates under the variable pressure $F$ of the acoustic waves. The system has two degrees of freedom. The charge $Q$ of the condenser and the displacement $q$ of the membrane are the two generalized coordinates. Set up the differential equations for these quantities using Lagrange’s equations.

Solution: This is a mixed problem in which one of the variables is a mechanical and the other an electrical quantity. However, we know that no distinction needs to be made between them in setting up Lagrange’s equations. When the system is at rest, the condenser is charged by an amount $Q_0$, therefore the electrodes are attracted and compress the springs by an amount $q_0$. When the system oscillates, the charge and the displacement are $Q + Q_0$ and $q + q_0$. The capacity of the condenser is inversely proportional to the distance between the electrodes; we put $C = \frac{A}{a - q}$ where $A$ and $a$ are constants. The total potential energy (electrostatic and elastic) is

$$U = \frac{1}{2}\left(\frac{a - q}{A}\right)(Q_0 + Q)^2 + \frac{1}{2}k(q_0 + q)^2 - EQ$$

Taking into account the equilibrium conditions $\frac{\partial U}{\partial Q} = \frac{\partial U}{\partial q} = 0$, for $Q = q = 0$, and dropping terms higher than quadratic in $Q$ and $q$, we write

$$U = \frac{1}{2}\left(\frac{a}{A}Q^2 - \frac{Q_0}{A}Qq + kq^2\right) + \text{const.}$$

The term $\frac{2Q_0}{A}Qq$ represents the electromechanical coupling. The reader will notice that the coupling coefficient $2Q_0/A$ is proportional to $E$. The sum of the magnetic and kinetic energies is $T = \frac{1}{2}LQ^2 + \frac{1}{2}m\dot{q}^2$; the dissipation function is $D = \frac{1}{2}RQ^2$. Lagrange’s equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = -\frac{\partial U}{\partial q} + F; \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right) = -\frac{\partial D}{\partial Q} - \frac{\partial U}{\partial Q}$$

4. A short electric train is made up of three units: a locomotive and two passenger cars. The weight of each unit is 40,000 lb., the spring constants of the springs connecting them are equal to 15,000 lb./in. Determine the smallest value of the damping factor for two identical shock absorbers placed between the three units and acting by viscous friction such that the relative motion of the cars is not oscillatory.
Solution. The characteristic equation of the system is

\[(m\lambda^2 + \beta \lambda + k)(m\lambda^2 + 3\beta \lambda + 3k) = 0\]

where \(m\) is the mass of one car, \(k\) is the spring factor, and \(\beta\) the damping factor. The general solution is the superposition of two damped oscillations. The critical damping for the oscillation corresponding to the first factor of the characteristic equation is equal to

\[\beta = 2\sqrt{km} = 2\sqrt{\frac{40,000 \times 15,000}{386}}\text{ lb.-sec./in.}\]

The condition for the nonoscillatory character of the motion corresponding to the second factor of the characteristic equation is \(3\beta > 2\sqrt{3km}\). This condition is satisfied by the above value of \(\beta\).

6. The vertical axis of symmetry of the gyroscope shown in Fig. P.5 is free to rotate about the point A and is restrained by two springs at the point B. The axes of the springs are perpendicular to each other; their spring constants are \(k_1\) and \(k_2\). Calculate the magnitude of the moment of momentum of the gyroscope required for stable rotation.

7. An airplane engine is suspended in an elastic mounting as shown in Chapter V, Prob. 8, in such a manner that the three translatory and rotational degrees of freedom are not coupled. Show that the gyroscopic moment of the propeller introduces a coupling between the rotations, and calculate the precession of the axis of the engine for the case of a four-bladed propeller.

7. The stability of a monorail car is obtained by a vertical gyroscope. The axis of the gyroscope is hinged to the car at point \(P\) in such a way that it can move only in the plane of symmetry of the car. We have the following data:

- \(W_c\) total weight of the car and the gyroscope.
- \(W_g\) weight of the gyroscope.
- \(I\) moment of inertia of the car with its mounted gyroscope about the rail.
- \(C\) moment of inertia of the gyroscope about its spin axis.
- \(A\) moment of inertia of the gyroscope about an axis through \(P\) and perpendicular to the plane of symmetry of the car.
- \(h\) height of the center of gravity of the gyroscope above \(P\).
- \(s\) height of the center of gravity of the car and its gyroscope above the rail.

Set up the equations of motion for small oscillations of the gyroscope around the vertical and determine the spin velocity \(\Omega\) of the gyroscope required for stability.

Hint: Calculate the rate of change of the moment of momentum of the gyroscope about a horizontal axis passing through \(P\). Because the oscillations are small, this axis may be considered as fixed. We have
\[ A\dot{\theta} = C\Omega\dot{\phi} + W_\phi k\theta \]

where \( \theta \) is the inclination of the gyroscope in the plane of symmetry of the car and \( \phi \) the inclination of the car. Similarly, taking moments about the rail, \( I\ddot{\phi} = -C\Omega\dot{\theta} + W_\phi \beta \phi \).

8. The equation of motion for the sleeve of a centrifugal steam-engine governor is given by

\[ m\ddot{x} + \beta \dot{x} + kx = C_1(\omega - \omega_0) \]

where \( m \) is an inertia factor which accounts for the mass of the flyballs, the link arms, and the sleeve, \( \beta \) is a damping factor, \( k \) the constant of the governor spring, \( \omega \) the instantaneous velocity, \( \omega_0 \) the mean angular velocity of the engine, and \( C_1 \) a characteristic constant of the governor, which determines the force acting on the sleeve if the angular velocity is different from its mean value. The equation of the engine is of the form:

\[ I \frac{d\omega}{dt} = -C_2 x \]

where \( I \) is the equivalent moment of inertia of the moving parts of the engine and \( C_2 \) is a characteristic constant of the engine, which determines the torque as a function of the displacement \( x \) of the sleeve of the governor. Find the condition of stability of the coupled system consisting of the engine and the governor.

9. Solve the algebraic equation

\[ x^4 - 9x^3 + 30x^2 - 51x + 26 = 0 \]

10. Solve the algebraic equation

\[ x^4 - 7x^3 + 22x^2 - 38x^2 + 37x - 16 = 0 \]

References

1. Reference 2, Chapter III; references 2 to 4, Chapter IV; and references 2 and 4, Chapter V.

Special problems:

2. References 5 and 6, Chapter III (for section 5).

CHAPTER VII

THE DIFFERENTIAL EQUATIONS OF THE THEORY OF STRUCTURES

_Salviati._—In this discussion I shall take for granted the well-known mechanical principle, which has been shown to govern the behavior of a bar, which we call a lever, namely, that the force bears to the resistance the inverse ratio of the distances which separate the fulcrum from the force and resistance, respectively.

_Simplicio._—This was demonstrated first of all by Aristotle in his "Mechanics."

_Salviati._—Yes, I am willing to concede his priority in point of time; but as regards rigor of demonstration the first place must be given to Archimedes.

—_GALILEO GALILEI_, "Dialogues concerning Two New Sciences —The Second Day" (1638).

**Introduction.**—This chapter is concerned with the equilibrium and with the harmonic oscillations of cords and beams. Practically all problems of this chapter are boundary problems, i.e., they ask for particular solutions which satisfy certain conditions at the boundaries. We treat in detail differential equations which have constant coefficients or can be reduced to the standard form of Bessel’s differential equation. However, methods are shown for solving problems involving arbitrary differential equations also.

In the differential equations treated in Chapters IV to VI, the *time* appeared as an independent variable. In the differential equations encountered in this chapter the independent variable is a *space coordinate*. The unknown function is in most cases the deflection of a one-dimensional structure, such as a string or a beam. The deflection under given loads is governed by *non-homogeneous* differential equations; the particular physical setup, e.g., the type of support or the type of connection with other parts of the structure, furnishes the *boundary conditions*.

The next group of problems deals with the determination of the natural frequencies of one-dimensional structures. In the oscillation problems of Chapter V we were concerned with the
oscillations of a finite number of masses, mass points, or rigid bodies, whereas we shall now consider the masses continuously distributed over the whole elastic structure. To be sure, we restrict ourselves to harmonic oscillations of structures, i.e., we do not treat transient vibrations which are governed by partial differential equations where the time and at least one space coordinate enter as independent variables.

Returning to equilibrium problems, we treat the classical problem of elastic instability or buckling of columns—a problem first treated by Euler. There are two methods of approach to this problem. In one method the existence of equilibrium positions in the vicinity of the undeflected shape of the column is investigated, i.e., the existence of solutions of the differential equation in addition to the trivial solution which represents the undeflected shape. The second method consists of the investigation of the stability of the undeflected shape by comparison of the total potential energy of the system in undeflected and various deflected positions. This method leads to the calculus of variations. This chapter contains a short discussion of the stability problem from this point of view, and we give in Chapter VIII a practical, so-called direct method for solving simple variation problems.

The free oscillation and buckling problems lead to homogeneous differential equations containing unknown parameters and require the determination of certain characteristic values of these parameters for which the given boundary conditions can be satisfied. These characteristic values represent the natural frequencies, the critical speeds, or the critical loads of the structure.

Several examples of forced oscillations and combined axial and lateral loadings of structures are included. These problems lead to nonhomogeneous equations, however, with a parameter in each equation representing either the frequency of the impressed force or the axial load. The character of the solution depends greatly on the value of this parameter. If certain characteristic values are approached, the structure develops resonance or may collapse owing to excessive deflections.

1. Deflection of a String under Vertical Load.—We consider a perfectly flexible string submitted to vertical loads (Fig. 1.1a). Since such a string has no resistance to bending, the only internal force is the tension $F$ acting in the direction of the tangent of the
deflection curve. Let us denote the ordinates of the deflection curve by \( w(x) \), the angle between the horizontal \( x \)-direction and the tangent to the deflection curve by \( \theta \), the load per unit length of the horizontal projection of the deflection curve by \( p(x) \).

![Diagram](image)

Fig. 1.1.—Equilibrium of a string under a vertical distributed load.

Then the equilibrium equations for an element between \( x \) and \( x + dx \) are (Fig. 1.1b)

\[
\frac{d}{dx} (F \cos \theta) = 0
\]

\[
\frac{d}{dx} (F \sin \theta) \, dx + p \, dx = 0
\]  

(1.1)

According to the first equation, the horizontal component of the tension, \( H = F \cos \theta \) is constant, and substituting \( H \) in the second equation, we have

\[
H \frac{d}{dx} (\tan \theta) = -p(x)
\]  

(1.2)

or with \( \tan \theta = dw/dx \)

\[
H \frac{d^2w}{dx^2} = -p(x)
\]  

(1.3)

Let us assume that \( w = 0 \) and \( dw/dx = \tan \theta_0 \) for \( x = 0 \). Then by repeated integration
\[ w = -\frac{1}{H} \int_0^x d\xi \int_0^\xi p(\xi) \, d\xi + x \tan \theta_0 \quad (1.4) \]

Integrating by parts, we have

\[ \int_0^x d\xi \int_0^\xi p(\xi) \, d\xi = - \int_0^x xp(\xi) \, d\xi - \int_0^x \xi p(\xi) \, d\xi \]

and substituting this expression in Eq. (1.4),

\[ w = -\frac{1}{H} \int_0^x (x - \xi)p(\xi) \, d\xi + x \tan \theta_0 \quad (1.5) \]

If we write Eq. (1.5) in the form:

\[ -Hw + Hx \tan \theta_0 + \int_0^x (\xi - x)p(\xi) \, d\xi = 0 \quad (1.6) \]

it is easily seen that Eq. (1.6) expresses the condition that the resulting moment of the forces acting on the string to the left of an arbitrary point whose coordinates are \( x \) and \( w \) (cf. Fig. 1.1), is equal to zero. In fact, \(-Hw\) represents the moment of the horizontal component \( H \) of the tension; the term \( Hx \tan \theta_0 \) represents the moment of the vertical reaction \( V \), which is equal to \( H \tan \theta_0 \); and finally, the integral \( \int_0^x (\xi - x)p(\xi) \, d\xi \) represents the moment of the load \( p \).

Let us assume that a concentrated force \( P_i \) acts at the point \( x = \xi_i \). This means that the distributed load \( p(x) \) per unit length has very large values in a small interval between \( x = \xi_i - \epsilon \) and \( x = \xi_i + \epsilon \). Let us write \( \xi = \xi_i + \xi' \); then

\[ \int_{\xi_i - \epsilon}^{\xi_i + \epsilon} (\xi - x)p(\xi) \, d\xi = \int_{\xi_i - \epsilon}^{\xi_i + \epsilon} (\xi_i - x)p(\xi) \, d\xi + \int_{\xi_i - \epsilon}^{\xi_i + \epsilon} \xi'p(\xi') \, d\xi' \quad (1.7) \]

If we proceed to the limit \( \epsilon \to 0 \), the first integral on the right side of Eq. (1.7) becomes \((\xi_i - x)\int_{\xi_i - \epsilon}^{\xi_i + \epsilon} p(\xi) \, d\xi = (x - \xi_i)P_i\), and the second term vanishes, as \( \xi' \) is of the order of \( \epsilon \). The expression \((\xi_i - x)P_i\), is equal to the moment of \( P_i \) with respect to the point \( x \). Hence, if we include concentrated forces, Eq. (1.6) becomes

\[ -Hw + Hx \tan \theta_0 + \sum_i (\xi_i - x)P_i + \int_0^x (\xi - x)p(\xi) \, d\xi = 0 \quad (1.8) \]
It follows from Eq. (1.8) that the deflection \( w \) of a string loaded by vertical forces—distributed or concentrated—multiplied by the horizontal tension \( H \) gives the value of the moment of the vertical forces whose lines of action are to the left of \( x \), including the vertical reaction \( V \), the concentrated forces \( P_i \), and the distributed load \( p(\xi) \).

The *string polygon method* commonly used for the construction of bending-moment diagrams of beams is based on Eq. (1.8). The string polygon is the actual equilibrium configuration of a string under the action of the loads acting on the beam and of two end tensions \( F_A \) and \( F_B \), whose horizontal component \( H \) is arbitrary. The integral in Eq. (1.8) can be evaluated graphically by replacing the distributed loads by concentrated forces whose magnitudes \( P_i \) are given by the areas of the corresponding sections of the load diagram \( p(x) \) and whose points of application are the centroids of those sections.

Figure 1.2 shows the construction of the string polygon for a beam simply supported at the two ends and acted upon by the vertical loads \( P_1 \), \( P_2 \), and \( P_3 \). The ordinates \( w \) of the string polygon correspond to the equation (cf. Fig. 1.1.)

\[
w = x \tan \theta_0 + \frac{1}{H} \sum (\xi_i - x)P_i
\]  

(1.9)

The summation includes the forces acting to left of \( \xi = x \). To construct the polygon we first draw a *force diagram* with an arbitrarily chosen value of \( H \). The sides of the string polygon
are parallel to the lines connecting the pole $O$ with the end points of the forces. Since the vertical reaction at $x = 0$ is unknown, we start with an arbitrary value of $\theta_0$. Then, of course, $w(l)$ is not necessarily zero, and to satisfy this condition we connect the end points $A'$ and $B'$ of the string polygon by a straight line. Then the ordinates between the sides of the polygon and the line $A'B'$ are equal to the moment divided by $H$. The vertical reactions at the supports are obtained from the force diagram by drawing a straight line from the pole parallel to $A'B'$.

*Small Deflections of a String Held under Tension.*—Let us now assume that the deflection $w$ of a horizontal string is very small. In this case $\cos \theta \simeq 1$, and in Eq. (1.3) $H$ can be replaced by the resultant tension $F$. Hence Eq. (1.3) becomes

$$F \frac{d^2w}{dx^2} = -p(x)$$  \hspace{1cm} (1.10)

and the solution with $w = 0$ for $x = 0$ is

$$w = -\frac{1}{F} \int_0^x (x - \xi) p(\xi) \, d\xi + Cx$$  \hspace{1cm} (1.11)

where $C$ is a constant of integration, which is determined by the second boundary condition. In the case of a constant load $p_0$,

$$w = -\frac{p_0 x^2}{2F} + Cx$$

and if $w = 0$ for $x = l$, $C = p_0 l/2F$ and

$$w = \frac{p_0 x}{2F} (l - x)$$

The maximum deflection occurs at $x = l/2$ and is equal to $w_{\text{max}} = p_0 l^2/8F$.

In the case of an arbitrary load $p(x)$, putting $w = 0$ for $x = l$ in Eq. (1.11), we have

$$-\frac{1}{F} \int_0^l (l - \xi) p(\xi) \, d\xi + Cl = 0$$

and substituting the value of $C$ in (1.11),

$$w = \frac{1}{F} \int_0^x \xi \left(1 - \frac{x}{l}\right) p(\xi) \, d\xi + \frac{1}{F} \int_x^l x \left(1 - \frac{\xi}{l}\right) p(\xi) \, d\xi$$  \hspace{1cm} (1.12)
This equation can be interpreted in the following way: Let us assume as the only load a concentrated load $P$ acting at the point $\xi$ (Fig. 1.3). Then $d^2w/dx^2 = 0$, except for $x = \xi$, and the solution $w(x)$ consists of two linear functions. The deflection $f$ at the point $x = \xi$ is given by the condition that

$$F(\tan \theta_2 + \tan \theta_1) = P$$

or

$$F \left( \frac{f}{\xi} + \frac{f}{l - \xi} \right) = P \quad (1.13)$$

$$f = \frac{P}{F} \xi (l - \xi) \quad (1.14)$$

Hence, the solution for $x < \xi$ is given by

$$w = \int_{\xi}^{x} = \frac{P}{F} x \left( 1 - \frac{\xi}{l} \right)$$

and for $x > \xi$

$$w = \int_{l - \xi}^{l - x} = \frac{P}{F} \xi \left( 1 - \frac{x}{l} \right) \quad (1.15)$$

We call the function obtained by substituting $P = 1$ in (1.15) the influence function $g(x, \xi)$ (Fig. 1.4b). It gives the deflection at $x$ for a unit load applied to the point $\xi$. It is a function of two variables $x$ and $\xi$, and its derivative with respect to $x$ is discontinuous for $x = \xi$.

Let us now subdivide the load $p(\xi)$ into sections $p(\xi) \Delta \xi$ and replace $p(\xi) \Delta \xi$ by concentrated forces (Fig. 1.4a). Then the contribution of the loads between $\xi = 0$ and $\xi = x$ to the deflection amounts to

$$\sum_{\xi=0}^{x} p(\xi) \Delta \xi \xi \left( 1 - \frac{x}{l} \right)$$

and that of the loads
between $\xi = x$ and $\xi = l$ to

$$
\frac{1}{F} \sum_{\xi = z}^{\xi = l} p(\xi) \Delta \xi x \left(1 - \frac{\xi}{l}\right)
$$

Proceeding to the limit $\Delta \xi \to 0$, we obtain the expression (1.12), which we write in the form (Fig. 1.4c):

$$
w = \frac{1}{F} \int_0^l p(\xi) g(x, \xi) \, d\xi \quad (1.16)
$$

In Fig. 1.4c the application of Eq. (1.16) is shown for the case in which the load is applied between two points, $\xi = a$ and $\xi = b$. This method of solution is called the method of the influence function.

2. String with Elastic Support.—We assume that the vertical deflection of the string is restrained by a large number of springs (Fig. 2.1) such that their action can be replaced by a distributed restoring force per unit length equal to $kw$, where $k$ is a proportionality factor and $w$ is the deflection. In this case we must add the amount $-kw$ to the vertical load and obtain the differential equation

$$
F \frac{d^2w}{dx^2} = -p(x) + kw
$$

or

$$
F \frac{d^2w}{dx^2} - kw = -p(x) \quad (2.1)
$$

The solution of (2.1) consists of the general solution of the homogeneous equation, $w_1$, and an arbitrary particular solution $w_2$ of the nonhomogeneous equation. The general solution of the homogeneous equation is
\[ w_1 = Ae^{\sqrt{\frac{x}{p}}} - Be^{-\sqrt{\frac{x}{p}}} \] (2.2)

If \( p(x) \) has a simple form, the particular solution can often be easily guessed or calculated. For example, if \( p(x) \) is equal to a constant \( p_0 \), then \( w_2 = p_0/k \) is a particular solution.

As an example, we give the solution corresponding to the deflection of an infinite string under the action of a concentrated

\[ w = -\sqrt{\frac{k}{f}} \] (2.3)

or \( w = fe^{-\sqrt{\frac{k}{f}}(x-\xi)} \) for \( x > \xi \) and \( w = fe^{\sqrt{\frac{k}{f}}(x-\xi)} \) for \( x < \xi \). The deflection \( f \) is determined by the condition that

\[ \lim_{\xi \to 0} F \left[ \left( \frac{dw}{dx} \right)_{\xi+\epsilon} - \left( \frac{dw}{dx} \right)_{\xi-\epsilon} \right] = -P \] (2.4)

This condition yields the relation \( 2f\sqrt{kk} = P \). Hence, the deflection of the string is given by the following expressions:

\[ w_1 = \frac{P}{2\sqrt{kk}} \sqrt{\frac{x}{f}} \text{ for } x > \xi \]

\[ w_2 = \frac{P}{2\sqrt{kk}} c \sqrt{\frac{x}{f}} \text{ for } x < \xi \] (2.5)

3. Bending of Beams. General Theory.—A beam is a prismatical or approximately prismatical body with a resistance to bending and twisting. A beam is called a straight beam when
the centers of gravity of all cross sections lie on a straight line which is called the \textit{axis} of the beam.

We give in this section a short review of the results of the elastic theory of straight beams. We assume that the \(x\)-axis of a rectangular coordinate system coincides with the beam axis and the cross sections are parallel to the \(yz\) plane. We consider an arbitrary cross section and denote the components of the resultant and of the resultant moment of the external forces acting on the portion of the beam to the left of the cross section considered by \(X, Y, Z, M_x, M_y, M_z\). The center of gravity of the cross section is chosen as base point for the moments. We call \(X\) the \textit{axial thrust}; \(Y\) and \(Z\), the \textit{shear forces}; \(M_z\), the \textit{twisting moment}; \(M_y\) and \(M_x\), the \textit{bending moments} acting around the \(y\)- and \(z\)-axes, respectively.

Let us first assume that the cross section is symmetrical with respect to the \(xz\) plane and the lines of action of all forces are parallel to the \(z\)-axis. Hence, \(X = Y = 0\) and \(M_y = M_z = 0\). In this case we shall call \(M_y = M\) simply the \textit{bending moment}; the resultant force \(Z = S\), the \textit{shear force}. We measure the normal deflection \(w\) positive downward, \(M\) positive clockwise, and \(S\) positive upward. We denote the load per unit length of the beam by \(p(x)\), the concentrated loads by \(P_1, P_2, \ldots, P_n\); \(p\) and the \(P\)'s are taken positive downward.

The equilibrium condition for the vertical forces acting on an element \(dx\) of the beam requires that

\[
\frac{dS}{dx} dx + p \, dx = 0 \tag{3.1}
\]

The equilibrium condition for the moments acting on the same element is given by

\[
\frac{dM}{dx} \, dx \quad S \, dx = 0 \tag{3.2}
\]

Hence, \(S = \frac{dM}{dx}\), \(p = -\frac{dS}{dx}\), and, therefore,

\[
p = -\frac{d^2M}{dx^2} \tag{3.3}
\]

The analysis of the deformation produced by the bending leads to the result that the \textit{curvature} of the beam axis is equal to

\[
\frac{1}{R} = \frac{M}{EI}
\]
where $R$ is the radius of curvature (positive if the deflected axis is convex seen from below), $I$ is the moment of inertia of the cross section with respect to the $y$-axis, and $E$ denotes Young's modulus.

The curvature of the line $w = w(x)$ is given by

$$\frac{1}{R} = -\frac{d^2w}{dx^2} \left[ 1 + \left( \frac{dw}{dx} \right)^2 \right]^{-\frac{3}{2}}$$

or for small deflections approximately by $\frac{1}{R} = -\frac{d^2w}{dx^2}$. Hence,

$$\frac{d^2w}{dx^2} = -\frac{M}{EI}$$  \hspace{1cm} (3.4)

Substituting $M$ from (3.4) into Eqs. (3.2) and (3.3), we obtain

$$-\frac{d}{dx} \left( EI \frac{d^2w}{dx^2} \right) = S$$  \hspace{1cm} (3.5)

$$\frac{d^2}{dx^2} \left( EI \frac{d^2w}{dx^2} \right) = p(x)$$  \hspace{1cm} (3.6)

Equation (3.6) is called the differential equation for the deflection of the beam. The quantity $EI$ is called the flexural rigidity.

We consider the following boundary conditions:

a. **Hinged Support.**—If the hinged support is at one end of the beam, at that point $w = 0$, and the moment is equal to zero, and, therefore, $d^2w/dx^2 = 0$. If the beam extends beyond a hinged support, at the support $w = 0$, and the moment is continuous.

b. **Clamped Support.**—$w = 0$ and $dw/dx = 0$.

c. **Free End.**—$M = S = 0$ and, therefore, $EI \frac{d^2w}{dx^2} = 0$, and

$$\frac{d}{dx} \left( EI \frac{d^2w}{dx^2} \right) = 0.$$

A hinged support involves a reaction; a clamped support, a reaction and a reaction moment. The beam is statically determine if the total number of reactions and reaction moments is equal to the number of equilibrium equations. In this case, the moment $M$ can be determined from the equilibrium conditions, and instead of Eq. (3.6) we can use Eq. (3.4) for the computation of the deflection $w$.

The elastic energy per unit length of the beam is equal to

$$\frac{1}{2} M \frac{1}{R} = \frac{1}{2} EI \left( \frac{1}{R} \right)^2 = \frac{1}{2} EI \left( \frac{d^2w}{dx^2} \right)^2$$  \hspace{1cm} (3.7)
In general, the resistance of a section of the beam to bending is determined by its two moments of inertia $I_v = \int_A x^2 \, dA$ and $I_z = \int_A y^2 \, dA$, and by its product of inertia $I_{yz} = \int_A yz \, dA$, where $dA$ is an element of the cross-sectional area. If $M_v$ and $M_z$ are the bending moments with respect to the $y$- and $z$-axes, respectively, if $v$ and $w$ are the deflections in the negative $y$- and $z$-directions, and if $\frac{1}{R_v} = -\frac{d^2v}{dx^2}$ and $\frac{1}{R_z} = -\frac{d^2w}{dx^2}$ are the curvatures of the deflection curve in the $xy$ and $xz$ planes, we have

$$M_v = -EI_v \frac{d^2w}{dx^2} - EI_{yz} \frac{d^2v}{dx^2},$$

$$M_z = -EI_z \frac{d^2v}{dx^2} + EI_{yz} \frac{d^2w}{dx^2}$$

(3.8)

If $I_{yz} = 0$, the $y$- and $z$-axes are called the principal inertia axes of the cross section. In this case

$$M_v = -EI_v \frac{d^2w}{dx^2}, \quad M_z = -EI_z \frac{d^2v}{dx^2}$$

(3.9)

and the quantities $EI_v$ and $EI_z$ are called the principal flexural rigidities of the beam. It is seen that the planes of the resultant moment and the resultant deflection coincide only if

(a) $EI_v = EI_z,$

or

(b) either $M_v$ or $M_z$ is equal to zero.

The influence of an axial thrust $X$ on the bending of beams will be taken into account in the sections dealing with the theory of the suspension bridge, the theory of buckling, and the theory of a combined axial and lateral load.

The twisting moment is supposed to be proportional to the angle of twist per unit length of the beam. If the angle of rotation of an arbitrary cross section is $\theta(x)$,

$$M_z = -C \frac{d\theta}{dx}$$

(3.10)

where $C$ is called the torsional rigidity of the beam. It is the
product of the shear modulus $G$ and a quantity which has the dimension of a moment of inertia of the cross section.

4. Deflection of Beams. Beams on Elastic Foundation.—We assume that the $y$- and $z$-axes are principal axes of the beam and the load is acting in the $xz$ plane. Then, according to Eq. (3.6), the deflection is given by the equation

$$\frac{d^2}{dx^2} \left( EI \frac{d^2 w}{dx^2} \right) = p(x) \quad (4.1)$$

If $p(x)$ is a given function of $x$, $w$ can be calculated by repeated quadratures. Several graphical and numerical methods have been developed for solving Eq. (4.1). We remember that

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} \quad \text{and} \quad d^2 M/dx^2 = -p.$$ 

If we consider the function $M/EI$ as a load applied to the beam, we may consider the deflection $w$ as the bending moment corresponding to this load. Hence, the construction of the string polygon explained in section 1 can be applied also for the determination of the deflection.

Let us assume now that a uniform beam $B$ is attached to a fixed base $A$ by means of a large number of springs $S$. We shall take into account their action—as was done in section 2—by a restoring force $-kw$ per unit length acting in a direction opposite to $w$ (Fig. 4.1). The factor $k$ is called the modulus of the foundation. If a distributed external load $p(x)$ acts on the beam, the total load will be $p(x) - kw$, and we obtain from Eq. (4.1)

$$EI \frac{d^4 w}{dx^4} = p(x) - kw \quad (4.2)$$
We first calculate the general solution of the homogeneous equation

\[ \frac{d^4w}{dx^4} + \beta^4w = 0 \]  

(4.3)

where \( \beta^4 = k/EI \). The characteristic equation of (4.3) is \( \lambda^4 + \beta^4 = 0 \) and we have

\[ \lambda = \beta \sqrt{-1}. \]

Let us write \( -1 = e^{(2k+1)i\pi} \) where \( k \) is an integer. Then it is seen that the four values of \( \sqrt{-1} \) are \( e^{\frac{i\pi}{4}}, e^{\frac{3i\pi}{4}}, e^{\frac{5i\pi}{4}}, \) and \( e^{\frac{7i\pi}{4}} \), i.e., the four values of the fourth root of \(-1\) are represented in the complex plane by points located at \( A, B, C, \) and \( D \) on the unit circle (Fig. 4.2). The general solution of Eq. (4.3) is, therefore,

\[ w = C_1 e^{\frac{\beta}{\sqrt{2}}(1+i)x} + C_2 e^{\frac{\beta}{\sqrt{2}}(-1+i)x} + C_3 e^{\frac{\beta}{\sqrt{2}}(-1-i)x} + C_4 e^{\frac{\beta}{\sqrt{2}}(1-i)x} \]

(4.4)
or in real form

\[ w = e^{\frac{\beta}{\sqrt{2}}x} \left( A \cos \frac{\beta}{\sqrt{2}}x + B \sin \frac{\beta}{\sqrt{2}}x \right) \]
\[ + e^{-\frac{\beta}{\sqrt{2}}x} \left( A' \cos \frac{\beta}{\sqrt{2}}x + B' \sin \frac{\beta}{\sqrt{2}}x \right) \]

(4.5)

To find \( w \) for a given \( p(x) \), a particular solution of Eq. (4.2) must be added to Eq. (4.5). The four arbitrary constants are determined by four boundary conditions.

Let us in particular investigate the case of a concentrated load \( P \) applied at a point \( \xi \), i.e., \( p(x) = 0 \) for \( x \neq \xi \). We assume the length of the beam unlimited in both directions. Then, owing to symmetry, \( dw/dx = 0 \) at \( x = \xi \); also \( w \) must be finite (or zero) at \( x = \pm \infty \). The solution satisfying these conditions is given by

\[ w = Ce^{\frac{\beta}{\sqrt{2}}(x-\xi)} \left[ \cos \frac{\beta}{\sqrt{2}}(x-\xi) \mp \sin \frac{\beta}{\sqrt{2}}(x-\xi) \right] \]

(4.6)
where the upper sign holds for \( x < \xi \) and the lower sign for \( x > \xi \). The constant \( C \) is determined by the condition that the difference between the shearing force \( S \) at \( x = \xi - \epsilon \) and \( x = \xi + \epsilon \) is equal to the load if \( \epsilon \to 0 \). The shearing force is given by \( S = -EI \frac{d^3w}{dx^3} \). We obtain by differentiation

\[
\left( \frac{d^3w}{dx^3} \right)_{x=\xi-\epsilon} = -\sqrt{2\beta^3}C, \quad \left( \frac{d^3w}{dx^3} \right)_{x=\xi+\epsilon} = +\sqrt{2\beta^3}C
\]

Hence, \( P = 2\sqrt{2\beta^3}CEI \), from which

\[
C = -\frac{P}{\sqrt{8\beta^3EI}} \quad \text{(4.7)}
\]

It is seen that the deflection curve has the shape of damped waves with a discontinuous third derivative at the point \( x = \xi \)

Fig. 4.3.—Deflection of an elastically supported beam under a concentrated load.

(Fig. 4.3). The distance of the first zero points \( O' \) and \( O'' \) of the deflection from the point of action of the concentrated load \( P \) is equal to \( (3\pi/2\beta)\sqrt{2} \). The values of the function

\[
\phi(z) = e^{-z}(\cos z + \sin z) \quad \text{(4.8)}
\]

and its second derivative, which is needed for computation of the bending moment, are tabulated in Timoshenko’s “Strength of Materials,” vol. II, pages 405-406.

There are many classes of engineering problems leading to the equation treated in this section; among the important problems that can be reduced to that of the bending of a beam on an elastic foundation is the axially symmetrical deformation of a circular pipe.
Circular Pipe with Reinforcing Ring.—As an example of this second application, we consider a circular pipe of thickness \( t \) and mean radius \( r \); the latter is defined as the mean value of the internal and the external radii. The pressures acting on the pipe and, therefore, the deformations are assumed symmetrical about the \( x \)-axis (Fig. 4.4). We consider a strip of the pipe between two planes passing through the \( x \)-axis, with the small angle \( \Delta \alpha \) between them. Such a strip will behave like a beam of width \( r \Delta \alpha \) and height \( t \), provided we consider the strip subjected not only to the pressure acting on the pipe, but also to the forces acting on the two sides of the strip, owing to the elastic shrinkage or expansion of the annular sections of the pipe. The resultant of these forces is directed radially and represents an elastic restoring force for the beam considered. Let us measure the radial deflection \( w \) of the strip positive in the outward direction. The expansion of an annular section from the radius \( r \) to \( r + w \) produces a tensile stress in the annulus equal to \( E^w_r \), where \( E \) is a modulus of elasticity of the material.* Then the forces acting on the side faces per unit length of the strip are equal to \( E^w_r \), and the radial resultant acting toward the axis is equal to \( E^w r \Delta \alpha \) per unit length of the strip. It is equivalent to a restoring

* Actually \( E \) is equal to Young's modulus divided by \( 1 - \mu^2 \), where \( \mu \) is Poisson's ratio.
force whose spring constant is equal to

\[ k = \frac{Et}{r} \Delta \alpha \]

The external load per unit length is equal to \( pr \Delta \alpha \) where \( p \) is the internal pressure per unit area of the cylindrical surface.

(The exact value of the load would be \( \left( r - \frac{t}{2} \right) p \Delta \alpha \); however, in the case of thin-walled pipes \( t/2 \) can be neglected in comparison with \( r \).) Finally, the moment of inertia of the cross section of the strip is equal to \( r \Delta \alpha \frac{t^3}{12} \).

With these values Eq. (4.2) becomes

\[ \frac{d^4w}{dx^4} + \frac{12}{r^2t^2w} = \frac{12}{Et^3p} \]  

(4.9)

We introduce \( a = \sqrt{rt} \) as a characteristic length parameter of the problem. Its physical meaning will be illustrated by the application given below.

With this notation Eq. (4.9) becomes

\[ \frac{d^4w}{dx^4} + \frac{12w}{a^4} = \frac{12}{Et^3p} \]  

(4.10)

Let us apply Eq. (4.10) to a pipe of infinite length subjected to a uniform internal pressure \( p \) and reinforced at the section \( x = 0 \) by a stiffener ring whose stiffness is so excessive that the change of the diameter of the reinforced section can be neglected (Fig. 4.5). Then we have the boundary conditions \( x = 0 \) and \( dw/dx = 0 \) at \( x = 0 \); at infinity \( w \) is finite and \( dw/dx = 0 \).

The general solution of (4.10) contains a particular solution of the nonhomogeneous equation. For this particular solution we use

\[ w = \frac{ap}{Et^3} = \frac{pr^2}{Et} \]  

(4.11)

The solution of the homogeneous equation which together with (4.11) satisfies the boundary conditions at infinity and the condition \( dw/dx = 0 \) at \( x = 0 \) can be obtained from Eq. (4.6)
by substituting \( \beta = \sqrt{12}/a \) and \( \xi = 0 \). Hence, we have

\[
w = \frac{pr^2}{Et} + C e^{\pm \frac{x}{a} \sqrt{3}} \left[ \cos \left( \frac{x}{a} \sqrt{3} \right) \mp \sin \left( \frac{x}{a} \sqrt{3} \right) \right]
\] (4.12)

where the upper signs hold for \( x < 0 \) and the lower signs for \( x > 0 \). The remaining constant \( C \) is determined by the condition that \( w = 0 \) at \( x = 0 \). We finally obtain

\[
w = \frac{pr^2}{Et} \left\{ 1 - e^{\pm \frac{x}{a} \sqrt{3}} \left[ \cos \left( \frac{x}{a} \sqrt{3} \right) \mp \sin \left( \frac{x}{a} \sqrt{3} \right) \right] \right\}
\] (4.13)

The deflection \( w \) is plotted as a function of \( x \) in Fig. 4.5. The radial deflection at infinity is equal to \( w_\infty = \frac{pr^2}{Et} \), i.e., to the

![Diagram of deflection of a reinforced circular pipe due to internal pressure.](image)

Fig. 4.5.—Deflection of a reinforced circular pipe due to internal pressure.

deflection that would occur at any section of the pipe without the stiffener ring.

If we use the stiffener ring as a stress-relieving device it is important to know the length of the pipe that is materially affected by the ring. To estimate this length we calculate the load carried by the ring. This can be done by computing the shearing forces for \( x = + \epsilon \), as we have done on page 273 or by calculating for each section the partial pressure \( p \), which is in equilibrium with the annular stresses in the pipe. If \( p \) is the total pressure, the rest of the load, i.e.,

\[
P = \int_{-\epsilon}^{\epsilon} (p - p_1) \, dx = 2\int_{0}^{\epsilon} (p - p_1) \, dx
\] (4.14)

must be carried by a unit length of the ring. The annular stress in the pipe is equal to \( E_t \); hence, \( p_1 = \frac{E_t}{r^2} \). Substituting this value into Eq. (4.14) and taking \( w \) from Eq. (4.13), we obtain
\[ P = 2p \int_{0}^{\infty} e^{-\frac{x\sqrt{3}}{a}} \left( \cos \frac{x\sqrt{3}}{a} + \sin \frac{x\sqrt{3}}{a} \right) dx \]  

(4.15)

Carrying out the integration, we have

\[ P = \frac{2pa}{\sqrt{3}} = 2p \frac{\sqrt{\pi}}{\sqrt{3}} \]  

(4.16)

Hence, the pressure acting over a length \( 2l = 2\sqrt{\pi l / \sqrt{3}} \) is carried by the stiffener. Therefore, to reduce the stress in a long pipe materially by a series of stiffener rings, the spacing must be of the order of magnitude \( 2l \). It is seen that this length is proportional to the square root of the radius of the pipe and to the square root of the wall thickness.

5. The Theory of the Suspension Bridge.—The so-called deflection theory of the suspension bridge considers the bridge structure as a combination of a string (the suspension cable) and a beam (the bridge truss) (Fig. 5.1). The peculiar feature of this combination is that, whereas the deflection of the beam may be considered small, the deflection of the string, \( i.e., \) the deviation of its shape from a straight line, has to be considered as of finite magnitude. However, if we assume—as is usually done in the analysis of suspension bridges—that the cable is adjusted in such a way that it carries its own weight, the weight of the hangers, and the dead weight of the truss without producing a bending moment in the beam, then all additional deformations of the cable and the truss due to the live load are of small magnitude and can be calculated by means of linear equations.

We shall use the following notations:

\( a \). The dead load of the system per unit length is equal to \( q \); the live load applied to the truss is equal to \( p \).
b. The horizontal tension in the cable, when loaded by the dead weight only, is denoted by \( H \); the additional tension produced by application of the live load by \( h \). The shape of the cable corresponding to the initial loading condition is given by \( y = y(x) \), \( y \) being considered positive downward from the horizontal line connecting the end points of the cable. The additional deflection of the cable is \( w(x) \). It will be assumed that the elastic deformation of the hangers can be neglected, and the vertical deflection of the truss at the point \( x \) is equal to the vertical deflection of the cable at the same point \( x \). It is understood that this implies, in addition to the assumption of rigid connection between cable and truss, disregard of the horizontal deflections of the cable.

c. The moment of inertia of the cross section of the truss is assumed to be constant and equal to \( I \); Young's modulus is denoted by \( E \).

We consider a cable extended between \( x = 0 \) and \( x = l \) and a beam hinged at \( x = 0 \) and \( x = l \).

The differential equation for the shape of the cable in the initial loading condition is, according to Eq. (1.3),

\[
H \frac{d^2y}{dx^2} = -q
\]  
(5.1)

If a live load \( p \) is added, a certain portion \( p_1 \) of the live load is carried by the cable, while \( p - p_1 \) is carried by the bending stiffness of the truss. The horizontal cable tension is increased to \( H + h \), and the deflection \( w \) is added to the ordinate \( y \). Hence, the equation for this condition is

\[
(H + h) \frac{d^2(y + w)}{dx^2} = -q - p_1
\]  
(5.2)

On the other hand, the differential equation of the beam is, according to (3.6),

\[
EI \frac{d^4w}{dx^4} = p - p_1
\]  
(5.3)

Three unknown quantities occur in Eqs. (5.2) and (5.3); viz., the two functions \( w(x) \) and \( p_1(x) \) and the parameter \( h \). Hence, an additional relation is necessary to determine the parameter \( h \). This is given by the following condition: obviously, if \( w(x) \) is known, the change in the length is determined by the additional
tension $h$ and the elasticity of the cable. Calculating the change in length in both ways, and comparing the two expressions, we obtain the relation which determines $h$.

Let us assume first that $h$ is known. Then by substituting $p_1$ from Eq. (5.2) into Eq. (5.3), we obtain

$$EI\frac{d^4w}{dx^4} - (H + h)\left(\frac{d^2y}{dx^2} + \frac{d^2w}{dx^4}\right) = p + q$$  \hspace{1cm} (5.4)

Taking into account Eq. (5.1), this reduces to

$$EI\frac{d^4w}{dx^4} - (H + h)\frac{d^2w}{dx^2} = p + h\frac{d^2y}{dx^2}$$

or substituting again $\frac{q}{H}$ for $-\frac{d^2y}{dx^2}$, we have

$$EI\frac{d^4w}{dx^4} - (H + h)\frac{d^2w}{dx^2} = p - q\frac{h}{H}$$  \hspace{1cm} (5.5)

This is the fundamental equation of the theory of the suspension bridge. It is seen by comparison with Eq. (5.3) that the portion of the live load that is transmitted to the cable and not carried directly by the truss is equal to

$$p_1 = q\frac{h}{H} - (H + h)\frac{d^2w}{dx^2}$$  \hspace{1cm} (5.6)

The meaning of the first term on the right side of Eq. (5.6) is easily expressed: The truss is relieved by a certain portion of the dead load $q$; the reduction in per cent involved is equal to the increase in per cent of the cable tension.

The second term also allows a physical interpretation. Let us assume that an axial force $X = H + h$ acts along the axis of the truss beam. If we assume that the beam is deflected into the shape $w = w(x)$ and its radius of curvature is equal to $1/R \approx -d^2w/dx^2$, the action of an axial tension $X$ is equivalent to that of a normal load of the amount $X/R$, and it represents a restoring force if $X$ is positive. Hence, the effect of the hangers is the same as if an axial tension of the magnitude of the cable tension were applied along the axis of the beam.

The general solution of Eq. (5.5) is, with $\mu = \sqrt{\frac{H + h}{EI}}$,

$$w = A + Bx + Ce^{\mu x} + De^{-\mu x} + w_p(x)$$  \hspace{1cm} (5.7)

where $w_p(x)$ is a particular solution of Eq. (5.5).
Let us assume \( p - q \frac{h}{H} \) constant. Then it is more convenient to choose the center of the beam as the origin of the coordinate system and assume \( w = 0 \) and \( \frac{d^2w}{dx^2} = 0 \) for \( x = \pm \frac{l}{2} \). If the symmetry of the problem is taken into account, only an even function of \( w \) can occur in the solution. Hence, we write

\[
w = C_1 + C_2 \cosh \mu x - \frac{1}{H + h} \left( p - q \frac{h}{H} \right) \frac{x^2}{2}\tag{5.8}
\]

It is readily shown by differentiation that the last term in Eq. (5.8) is a particular solution of Eq. (5.5). The boundary conditions call for

\[
C_1 + C_2 \cosh \mu l/2 - \frac{p - q \frac{h}{H}}{H + h} l^2 = 0
\]

\[
- \mu^2 C_2 \cosh \mu l/2 - \frac{p - q \frac{h}{H}}{H + h} = 0
\]

Hence,

\[
\mu^2 C_1 = \frac{p - q \frac{h}{H}}{H + h} \left( \frac{\mu^2 l^2}{8} + 1 \right)
\]

\[
- \mu^2 C_2 = \frac{p - q \frac{h}{H}}{H + h} \frac{1}{\cosh \mu l/2}
\]

Let us now calculate the bending moment at the center. We find from Eq. (5.8)

\[
M_{\text{max}} = -EI \frac{d^2w}{dx^2} = EI \left( \mu^2 C_2 + \frac{p - q \frac{h}{H}}{H + h} \right)
\]

or if \( C_2 \) is substituted from Eq. (5.9),

\[
M_{\text{max}} = EI \frac{p - q \frac{h}{H}}{H + h} \left( 1 - \frac{1}{\cosh \mu l/2} \right)\tag{5.10}
\]

Substituting \( \frac{H + h}{EI} = \mu^2 \), we have

\[
M_{\text{max}} = pl^2 \left( 1 - \frac{q}{p \frac{h}{H}} \right) \left( 1 - \frac{1}{\cosh \mu l/2} \right) \tag{5.11}
\]
It will be shown later that \( h/H \) is a function of the dimensionless quantity \( Hl^3/EI \) and the ratio \( p/q \). Hence, the moment at the center is expressed in the form:

\[
M_{\text{max}} = pl^2f\left(\frac{p}{q}, \frac{Hl^3}{EI}\right)
\]  

(5.12)

The ratio of the live load to the dead load and the dimensionless quantity \( Hl^3/EI \) are the two governing parameters of the problem.

The particular solution of Eq. (5.5) for arbitrary loads \( p \) can be found either by superposition of concentrated loads or by expanding \( p \) in Fourier series. The latter method will be discussed in detail in Chapter VIII. The method of superposition is explained at the end of this chapter in connection with Prob. 6.

The Calculation of the Additional Cable Tension.—We consider two cases. In the first, we assume that the cable is inextensible, and, in the second, we assume a certain given modulus of elasticity for the extension of the cable. The initial total length of the cable is equal to \( L = \int_0^l \sqrt{1 + (dy/dx)^2} \, dx \) where \( y(x) \) is the deflection in the initial loading condition. (Note that the origin \( x = 0 \) is again moved to the left tower.) If we replace \( y(x) \) by \( y(x) + w(x) \), expand the radical in a Taylor series, and neglect higher terms in \( w \), the variation of \( L \) will be

\[
\Delta L = \int_0^l (dy/dx) \left( \frac{dw}{dx} / \sqrt{1 + (dy/dx)^2} \right) \, dx
\]  

(5.13)

We integrate Eq. (5.13) by parts and write \( y' = dy/dx \),

\[
y'' = \frac{d^2y}{dx^2}
\]

Then we have

\[
\Delta L = \left[ w \frac{y'}{\sqrt{1 + y'^2}} \right]_0^l - \int_0^l \frac{wyy''}{(1 + y'^2)^{3/2}} \, dx
\]  

(5.14)

The first term on the right side of Eq. (5.14) vanishes, because \( w \) vanishes at the end points. If we now neglect \( y'^2 \) in comparison

* We remember that \( \sqrt{1 + (x + \epsilon)^2} - \sqrt{1 + x^2} \) can be written \( 1 + (x + \epsilon)^2 - (1 + x^2) \) or approximately \( \frac{x \epsilon}{\sqrt{1 + x^2}} \) for small \( \epsilon \).
with unity and substitute \( y'' = -\frac{q}{H} \) from Eq. (5.1), we obtain
\[
\Delta L = \int_0^l \frac{wq}{H} \, dx
\]  
(5.15)

or
\[
H\Delta L = \int_0^l wq \, dx
\]  
(5.16)

Equation (5.16) can be interpreted as the expression of the theorem of the conservation of energy, applied to the cable. In the initial loading condition the normal load on the cable is equal to \( q \), the horizontal, or approximately the total, tension is equal to \( H \). If now the deflection is increased by the small quantity \( w \), the expression on the right side gives the work done by the load \( q \); the expression on the left side is the work done by the tension \( H \) if the cable is stretched by the amount \( \Delta L \). The two amounts of work must be equal, contributions of the deflection involving higher than the first power being neglected.

If we assume the cable inextensible, we obtain the following relation:
\[
\int_0^l wq \, dx = 0
\]  
(5.17)

This means that we have to substitute in Eq. (5.17) for \( w \) the solution of Eq. (5.5), which itself contains the parameter \( h \), and in this way, we obtain an equation for the determination of \( h \). This equation is a transcendental equation, because \( h \) is involved also in \( \mu \), and, therefore, in \( \cosh \mu l / 2 \). In order to obtain a first approximation, we put \( \mu = \sqrt{H/EI} \), neglecting \( h \) in comparison with \( H \). Then, we calculate \( h \) from (5.17), correct \( \mu \), and repeat the calculation. It is seen that, as was said above, \( h/H \) will be determined in this way as function of \( Hh^2/ EI \) and \( p/q \).

If the cable is extensible, we assume that \( \Delta L = hL/EA \), where \( A \) is the effective cross-sectional area of the cable and \( E \) the modulus of elasticity of the material. Then, substituting \( \Delta L \) in Eq. (5.16), we obtain
\[
\frac{HhL}{EA} = \int_0^l wq \, dx
\]  
(5.18)

This relation gives again an equation for the calculation of \( h \). We remember that \( h \) occurs also in this case in the expression for
In practical applications it is necessary to take into account also the extension of the cable due to the change of temperature.

6. Problems of Harmonic Vibrations Reduced to Statical Problems.—Sections 7 to 11 will deal with harmonic vibrations of one-dimensional continuous systems. As a matter of fact, the equation of motion of such a system is a partial differential equation with the time $t$ and the space coordinate $x$ as independent variables. Consider, for example, the small vibrations of a string. We denote the deflection by $\xi$ and the mass of the string per unit length by $\rho$. According to d'Alembert's principle (Chapter III, section 10), any dynamical problem may be treated as a statical one by adding the appropriate inertia forces to the given external forces. The inertia force to be added per unit length of the string is equal to $-\rho \frac{\partial^2 \xi}{\partial t^2}$, and we obtain the equation of motion for the free vibrations by substituting this quantity for the load $p$ in Eq. (1.10):

$$F \frac{\partial^2 \xi}{\partial x^2} = \rho \frac{\partial^2 \xi}{\partial t^2}$$

(6.1)

This is a partial differential equation for the deflection $\xi(x,t)$. However, if we restrict ourselves to harmonic motion by putting

$$\xi(x,t) = w(x) \sin \omega t$$

the partial differential equation (6.1) is reduced to the ordinary differential equation

$$F \frac{d^2 w}{dx^2} + \rho \omega^2 w = 0$$

(6.2)

The new variable $w(x)$ can be interpreted as the maximum deflection or the amplitude of the harmonic motion and $\rho \omega^2 w$ as the inertia force per unit length at the instant of maximum deflection.

To find the equation for the free vibration of a beam, we consider again the inertia force $-\rho \frac{\partial^2 \xi}{\partial t^2}$ as the load, and using Eq. (3.6), we find

$$\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 \xi}{\partial x^2} \right) + \rho \frac{\partial^2 \xi}{\partial t^2} = 0$$

(6.3)

This partial differential equation is reduced to an ordinary differ-
ential equation if we limit ourselves to harmonic motion by putting $\xi(x,t) = w(x) \sin \omega t$. We find

$$\frac{d^2}{dx^2} \left( EI \frac{d^2w}{dx^2} \right) - \rho \omega^2 w = 0 \quad (6.4)$$

In the following sections we use the Eqs. (6.2) and (6.4) to determine the frequencies and the modes of vibration of strings and beams under given boundary conditions.

7. **Harmonic Vibration of a String under Tension.**—Taking up first the problem of the vibrating string, we find the general solution of Eq. (6.2), assuming that $\rho$ is constant,

$$w = C_1 \cos \sqrt{\frac{\rho}{l^2}} \omega x + C_2 \sin \sqrt{\frac{\rho}{l^2}} \omega x \quad (7.1)$$

If the string extends from $x = 0$ to $x = l$ and is fixed at the two ends, we have the boundary conditions $w = 0$ for $x = 0$ and $x = l$. The condition $w = 0$ for $x = 0$ requires that $C_1 = 0$. We are then left with the condition

$$w = C_2 \sin \sqrt{\frac{\rho}{l^2}} \omega l = 0 \quad (7.2)$$

This can be satisfied only if either $C_2 = 0$ or $\sqrt{\frac{\rho}{l^2}} \omega l = k\pi$ where $k$ is an integer. Hence the problem, in addition to the trivial solution $w = 0$ (position of rest), has an infinite number of solutions, each corresponding to a definite value of the parameter $\omega$. The values of $\omega$ which satisfy the condition (7.2) are

$$\omega_k = \frac{k\pi}{l} \sqrt{\frac{E}{\rho}} \quad (7.3)$$

The corresponding solutions of Eq. (6.2) are

$$w_k = C_k \sin \frac{k\pi x}{l}, \quad k = 1, 2, 3, \ldots, \infty \quad (7.4)$$

where the amplitude $C_k$ remains undetermined. The string can vibrate in an infinite number of sinusoidal shapes called *modes of vibration* (Fig. 7.1a); each mode corresponds to a certain frequency.
The lowest frequency \((n = 1)\) is called the **fundamental tone** of the string, the corresponding mode of vibration is called the fundamental mode of vibration. The modes corresponding to higher frequencies are known as the **harmonics**, or **overtones**. Figure 7.1b shows the **spectrum** of the string, *i.e.*, the distribution of the natural frequencies of the string over the frequency scale.

Let us go back to Eq. (2.1) for the string with elastic support and put \(p = 0\):

\[
F \frac{d^2w}{dx^2} - kw = 0
\]  

(7.5)

Comparing this equation with Eq. (6.2), it is seen that the sign of the coefficient of \(w\) has a decisive influence on the character of the respective solutions. In the case of a string with fixed ends and elastic support, \(w = 0\) is the only possible shape of the string when no load is applied, whereas the equation for a vibrating string has additional solutions for certain definite values of the parameter \(\omega\). The mathematical reason for the different behavior of the two equations is that the general solution of the Eq.
(6.2) is of the oscillatory type and goes through zero an infinite number of times between \(-\infty\) and \(+\infty\), while the general solution of Eq. (7.5) cannot have more than one zero point between the same limits; hence, if the solution of Eq. (7.5) has to satisfy the boundary condition \(w = 0\) at more than one point, \(w\) must be zero everywhere.

The modes of vibration of the string are analogous to the modes of principal oscillations of a system which has a finite number of degrees of freedom (Chapter V). Every mode of vibration is a pure harmonic oscillation. If we replace the string by \(n\) masses, the amplitudes of these masses would be determined by the coefficients of the normal modes. It is seen that the function \(w(x)\) which determines the shape of a mode of vibration replaces a set of such coefficients.

8. Vibration of Beams. The Critical Speed of a Rotating Shaft.—We now take up the problem of harmonic vibration of a beam. We restrict ourselves to beams of uniform cross section. Then Eq. (6.4) becomes

\[
EI \frac{d^4w}{dx^4} - \rho \omega^2 w = 0 \quad (8.1)
\]

Using the notation \(\beta^4 = \rho \omega^2 / EI\), we have

\[
\frac{d^4w}{dx^4} - \beta^4 w = 0 \quad (8.2)
\]

The characteristic equation of Eq. (8.2) is \(\lambda^4 - \beta^4 = 0\), and, therefore,

\[
\lambda = \beta \sqrt{1}
\]

The fourth roots of unity are located at the points \(A, B, C,\) and \(D\) of the unit circle as shown in Fig. (8.1), and the general solution of Eq. (8.2) is

\[
w = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{\beta x} + C_4 e^{-\beta x} \quad (8.3)
\]
or, in real form,

\[
w = A \cosh \beta x + B \sinh \beta x + C \cos \beta x + D \sin \beta x \quad (8.4)
\]

We consider various boundary conditions.

a. Let us assume that a vibrating beam is hinged (simply supported) at both ends \(x = 0\) and \(x = l\). Then we have
\( w = 0 \) and \( d^2w/dx^2 = 0 \) for \( x = 0 \) and \( x = l \). From the conditions at \( x = 0 \) it follows that \( A = C = 0 \), and we are left with the conditions

\[
\begin{align*}
B \sinh \beta l + D \sin \beta l &= 0 \\
B \sinh \beta l - D \sin \beta l &= 0
\end{align*}
\] (8.5)

Equations (8.5) are satisfied if \( B = D = 0 \) or if \( \sin \beta l \sinh \beta l = 0 \). The last condition leads to \( \beta l = k\pi \) where \( k \) is an integer. It follows from (8.5) that in this case \( B = 0 \) and \( D \) is undetermined. There are an infinite number of possible vibrations of sinusoidal shape \( w_k = D_k \sin \frac{k\pi x}{l} \). Their angular frequencies are given by the formula

\[
\omega_k = \frac{k^2\pi^2}{l^2} \sqrt{\frac{EI}{\rho}}
\] (8.6)

It is seen that the modes of vibration of a hinged beam (Fig. 8.2a) are identical with those of a vibrating string. However,
comparing expressions (7.3) and (8.6), we find an essential difference in the frequencies: The frequencies of the beam increase as the squares of successive integers (Fig. 8.2b), whereas the frequencies of the string are proportional to the integers themselves.

![Diagram](image)

**Fig. 8.3.**—The modes of oscillation and the frequency spectrum of a uniform cantilever beam.

b. Let us consider a cantilever beam of the length \( l \) clamped at \( x = 0 \) (Fig. 8.3). We must satisfy the boundary conditions \( w = dw/dx = 0 \) for \( x = 0 \) and \( d^2w/dx^2 = d^3w/dx^3 = 0 \) at the free end \( x = l \). From the boundary conditions at \( x = 0 \) it follows that \( A + C = B + D = 0 \), and, therefore,

\[
w = A(\cosh \beta x - \cos \beta x) + B(\sinh \beta x - \sin \beta x)
\]

The conditions for \( x = l \) lead to the following relations:

\[
\left( \frac{d^2w}{dx^2} \right)_{x=l} = A\beta^2(\cosh \beta l + \cos \beta l) + B\beta^2(\sinh \beta l + \sin \beta l) = 0 \quad (8.7)
\]

\[
\left( \frac{d^3w}{dx^3} \right)_{x=l} = A\beta^3(\sinh \beta l - \sin \beta l) + B\beta^3(\cosh \beta l + \cos \beta l) = 0 \quad (8.8)
\]
Eliminating $A$ and $B$ between (8.7) and (8.8), we find the following condition for the characteristic values of $\beta l$:

$$1 + \cosh \beta l \cdot \cos \beta l = 0 \quad (8.9)$$

or

$$\cos \beta l = -\frac{1}{\cosh \beta l} \quad (8.10)$$

There are an infinite number of values of $\beta l$ satisfying this equation. These values correspond to the abscissae of the points of intersection of the curve $\eta_1 = -\frac{1}{\cosh \beta l}$ and the curve $\eta_2 = \cos \beta l$ (Fig. 8.4).

The three first roots of the equation are $\beta l = 0.600\pi$, $1.49\pi$, $2.50\pi$. Since $1/\cosh \beta l$ becomes infinitely small for large values of $\beta l$, the higher roots are given with satisfactory accuracy by the equation

$$\cos \beta l = 0 \quad (8.11)$$

or

$$\beta l = (k - \frac{1}{2})\pi \quad (8.12)$$

The second root $\beta_2 l = 1.49\pi$ already differs only very slightly from the value $1.50\pi$, which corresponds to Eq. (8.12). The angular frequencies are given by $\omega_k = \beta_k^2 \sqrt{EI/\rho}$ and, therefore, the higher frequencies are

$$\omega_k = \frac{(k - \frac{1}{2})^2 \pi^2}{l^2} \sqrt{\frac{EI}{\rho}} \quad (8.13)$$
Equation (8.13) shows that the frequencies of the higher harmonics increase as the squares of the successive integers. But this rule is not true for the lowest frequencies.

The modes of vibration are obtained from Eq. (8.9) and either (8.7) or (8.8). We find

\[
w_k = C_k \left[ \frac{\cosh \beta_k x - \cos \beta_k x}{\cosh \beta_k l + \cos \beta_k l} - \frac{\sinh \beta_k x - \sin \beta_k x}{\sinh \beta_k l + \sin \beta_k l} \right]
\]

(8.14)

where \( C_k \) is an arbitrary constant. Some of these modes and the spectrum are plotted in Fig. 8.3. It can be seen that for the higher harmonics the deflection curves approach the sinusoidal shape.

Let us now assume that a shaft of uniform cross section is rotating around its axis with the angular velocity \( \omega \). If, owing to lateral deflection of the shaft, the center of gravity of a cross section does not lie on the axis of rotation, a centrifugal force equal to \( \rho \omega^2 w \) per unit length will act on the shaft as lateral load, where \( w(x) \) is the deflection (Fig. 8.5). We obtain as the condition of equilibrium

\[
EI \frac{d^4 w}{dx^4} = \rho \omega^2 w
\]

(8.15)

where \( EI \) is the flexural rigidity of the shaft. It is seen that the equation of equilibrium for a rotating shaft is identical with the equation for the modes of vibration of a beam. In general, the only equilibrium configuration will be \( w = 0 \), but there are certain values of the angular velocity \( \omega \) for which equilibrium shapes different from \( w = 0 \) are possible. These angular velocities are known as the critical speeds of the shaft.

9. Vibration of a Beam Carrying a Concentrated Mass.—A uniform beam of length \( l \) resting on two supports at \( x = 0 \) and \( x = l \) carries a mass \( m \) at its mid-point (Fig. 9.1). We shall
first consider the modes of vibration that are symmetrical on both sides of the concentrated mass. Then it is sufficient to compute \( w \) for one-half of the beam, for example, for \( 0 < x < l/2 \). We use Eq. (8.4) and satisfy the boundary condition at the origin by putting \( A = C = 0 \). Thus,

\[
w = B \sinh \beta x + D \sin \beta x
\]  

(9.1)

The boundary conditions at \( x = l/2 \) are:

a. \( dw/dx = 0 \), owing to the assumption of symmetrical modes of vibration.

![Fig. 9.1.—The modes of oscillation of a simply supported beam with a concentrated mass at the center.](image)

b. The shearing force at \( x = l/2 \) must be equal to half the load, which is equal to the inertia force due to the concentrated mass \( m \). The shearing force is equal to [cf. Eq. (3.5)]

\[
S = -EI \frac{d^3w}{dx^3}
\]

Hence,

\[
EI \frac{d^3w}{dx^3} = -\frac{1}{2}m\omega^2 w
\]  

(9.2)

These two conditions lead to the following expressions:

\[
R\beta \cosh \frac{\beta l}{2} + D\beta \cos \frac{\beta l}{2} = 0
\]  

(9.3)

\[
-EI \left( B\beta^3 \cosh \frac{\beta l}{2} - D\beta^3 \cos \frac{\beta l}{2} \right) = \frac{m\omega^2}{2} \left( R \sinh \frac{\beta l}{2} + D \sin \frac{\beta l}{2} \right)
\]  

(9.4)
Elimination of $B$ and $D$ between these two equations yields the following condition for $\beta$:

$$-4 = \frac{m \omega^2}{EI \beta^3} \left( \tanh \frac{\beta l}{2} - \tan \frac{\beta l}{2} \right)$$

(9.5)

Remembering that $\beta = \sqrt{\frac{\rho \omega^2}{EI}}$ and denoting the total mass of the beam $\rho l$ by $m_b$, Eq. (9.5) becomes

$$\frac{2m_b}{m} = \frac{\beta l}{2} \left( \tan \frac{\beta l}{2} - \tanh \frac{\beta l}{2} \right)$$

(9.6)

Equation (9.6) is a transcendental equation for $\beta l/2$ if the mass ratio $m_b/m$ is given. Since, according to (9.6),

$$\tan \frac{\beta l}{2} - \tanh \frac{\beta l}{2} = \frac{4}{\beta l/m}$$

(9.7)

the left side must be small for large values of $\beta l/2$, and because $\tanh \beta l/2 \to 1$ for $\beta l/2 \to \infty$, we have for large roots approximately $\tan \beta l/2 = 1$, or

$$\frac{\beta l}{2} = k\pi + \frac{\pi}{4}$$

(9.8)

The frequencies of the higher symmetrical harmonics are, therefore,

$$\omega_k = 4 \left( k + \frac{1}{4} \right)^2 \frac{\pi^2}{l^2} \sqrt{\frac{EI}{\rho}}$$

(9.9)

The shapes of vibration are found from either (9.3) or (9.4):

$$w_k = C_k \left( \frac{\sinh \beta_k x}{\cosh \frac{\beta_k l}{2}} - \frac{\sin \beta_k x}{\cos \frac{\beta_k l}{2}} \right)$$

(9.10)

where $C_k$ is an arbitrary constant. For higher modes the mass $m$ remains practically stationary (cf. Fig. 9.1); we have

$$(w_k)_{x = \frac{l}{2}} = C_k \left( \tanh \frac{\beta_k l}{2} - \tan \frac{\beta_k l}{2} \right)$$

The term in parentheses converges to zero for $k \to \infty$, according to Eq. (9.7), and, therefore, the deflection at $x = l/2$ tends to zero. As far as the higher symmetrical modes are concerned
the mass \( m \) has the same effect as if the beam were clamped at its mid-point.

If the mass \( m \) is very small and \( \beta l/2 \) is not very large, the right side of Eq. (9.7) becomes large. Then we have approximately
\[
\tan \left( \frac{\beta l}{2} \right) \approx \infty, \quad \text{or} \quad \beta k l = (2k - 1)\pi.
\]
These are the roots already found for the symmetric modes of a freely supported beam [Eq. (8.6)]; in this case the influence of the mass at the mid-point is negligible.

Let us now assume that the mass of the beam \( m_b \) is small compared to the mass \( m \). In this case the right side of Eq. (9.6) must be small. It is seen that a small value of \( \beta l/2 \) can satisfy this condition. This will give us the lowest, so-called fundamental frequency of the system. To find \( \beta l/2 \) we expand the right side of Eq. (9.6) in a power series in \( \xi = \beta l/2 \). Substituting the power series
\[
\tan \xi = \xi + \frac{1}{3} \xi^3 + \frac{2}{15} \xi^5 + \frac{17}{315} \xi^7 + \frac{62}{2,835} \xi^9 + \frac{1,382}{155,925} \xi^{11} + \cdots
\]
\[
\tanh \xi = \xi - \frac{1}{3} \xi^3 + \frac{2}{15} \xi^5 - \frac{17}{315} \xi^7 + \frac{62}{2,835} \xi^9 - \frac{1,382}{155,925} \xi^{11} + \cdots
\]
in Eq. (9.6), we obtain
\[
\frac{m_b}{m} = \xi^4 + \frac{17}{105} \xi^8 + \frac{1,382}{51,975} \xi^{12} + \cdots.
\]
We want to solve this equation with respect to \( \xi^4 \), i.e., we want to invert the series and express \( \xi^4 \) as a function of \( m_b/m \). We put \( 3m_b/m = \eta \) and write
\[
\xi^4 = \eta + a\eta^2 + b\eta^3 + \cdots
\]
Introducing this expression into Eq. (9.11), we find
\[
\eta = \eta + \left( a + \frac{17}{105} \right) \eta^2 + \left( b + \frac{34}{105} a + \frac{1,382}{51,975} \right) \eta^3 + \cdots
\]
By equating the coefficients of the same powers of \( \eta \) we obtain the following relations for the coefficients \( a, b, etc. \):
\[
a = -\frac{17}{105}
\]
\[
b = -\frac{34}{105} a - \frac{1,382}{51,975} = \frac{376}{14,553}
\]
Hence,
\[
\xi^4 = \frac{\beta l^4}{16} = \frac{m_b}{m} \left[ 1 - \frac{17}{35} \left( \frac{m_b}{m} \right) + \frac{376}{1,617} \left( \frac{m_b}{m} \right)^3 + \cdots \right]
\]
Let us denote the fundamental angular frequency of the system by \( \omega_1 \). Remembering that \( \omega = \beta z \sqrt{EI/\rho} \) and \( m_b = \rho l \), we obtain from Eq. (9.13)

\[
\omega_1 = \sqrt{\frac{48EI}{ml^3}} \left[ 1 - \frac{17}{35} \left( \frac{m_b}{m} \right) + \frac{376}{1617} \left( \frac{m_b}{m} \right)^2 \right]^{1/2} \tag{9.14}
\]

If the mass of the beam is zero, the frequency of the system is

\[
\omega_1 = \sqrt{\frac{48EI}{ml^3}} \tag{9.15}
\]

This last result could have been found directly. The deflection of the beam under a concentrated force \( F \) is equal to \( F = PV/48EI \). Hence, the spring constant, i.e., the force that produces a unit deflection, is \( k = 48EI/l^3 \). If we neglect the mass \( m_b \) of the beam, we may consider \( m \) as a mass attached to a spring of stiffness \( k \). Then, as shown in Chapter IV, section 2, the angular frequency of the oscillating mass \( m \) is equal to

\[
\omega_1 = \sqrt{\frac{k}{m}} = \sqrt{\frac{48EI}{ml^3}}
\]

in accordance with (9.15). Equation (9.14) indicates that in order to take into account the mass of the beam approximately, we have to increase \( m \) by the factor

\[
1 - \frac{17}{35} \frac{m_b}{m} + \frac{376}{1617} \left( \frac{m_b}{m} \right)^2 \approx 1 + 0.486 \left( \frac{m_b}{m} \right) + 0.003 \left( \frac{m_b}{m} \right)^2 + \cdots
\]

This gives the practical rule that if \( m_b \) is small compared to \( m \), one-half the mass of the beam should be added to the oscillating mass.

10. Forced Vibration of a Uniform Cantilever Beam.—A vertical beam of uniform cross section is clamped in a horizontal base. This base is given a horizontal harmonic displacement of the amplitude \( a_0 \) and of the angular frequency \( \omega \).

According to the theory of relative motion explained in Chapter III, section 5, if we use the equations of motion relative to a moving coordinate system, we have to introduce additional forces equal to the negative products of the masses and the acceleration of the system of reference. The acceleration of the base is equal to \(-a_0 \omega^2 \sin \omega t\). Hence, we must assume an additional load per unit length equal to \( \rho a_0 \omega^2 \sin \omega t \). If we assume that the deflection of the beam relative to the base is
\[ \xi = w \sin \omega t, \text{ we must modify Eq. (8.1) by writing} \]

\[ EI \frac{d^4w}{dx^4} - \rho \omega^2 w = \rho \omega^2 a_0 \quad (10.1) \]

The general solution of this equation is (with \( \beta^4 = \rho \omega^2 / EI \))

\[ w = A \cosh \beta x + B \sinh \beta x + C \cos \beta x + D \sin \beta x - a_0 \quad (10.2) \]

The boundary conditions are \( w = \frac{dw}{dx} = 0 \) at the clamped end, \( x = 0 \); \( \frac{d^2w}{dx^2} = \frac{d^3w}{dx^3} = 0 \) at the free end, \( x = l \). These conditions are satisfied by \( A + C - a_0 = 0, B + D = 0 \) and

\[ \left( \frac{d^2w}{dx^2} \right)_{x=l} = A\beta^2(\cosh \beta l + \cos \beta l) + B\beta^2(\sinh \beta l + \sin \beta l) - a_0\beta^2 \cos \beta l = 0 \]

\[ \left( \frac{d^3w}{dx^3} \right)_{x=l} = A\beta^3(\sinh \beta l - \sin \beta l) + B\beta^3(\cosh \beta l + \cos \beta l) + a_0\beta^3 \sin \beta l = 0 \quad (10.3) \]

The determinant of these two linear equations for \( A \) and \( B \) is proportional to the expression \( 1 + \cosh \beta l \cos \beta l \) of Eq. (8.9), whose roots give the frequencies of the free vibration of the beam. The values of the coefficients \( A \) and \( B \) are found by solving the two equations (10.3) thus:

\[ A = \frac{a_0 \cosh \beta l \cos \beta l + \sinh \beta l \sin \beta l + 1}{2 \cosh \beta l \cos \beta l + 1} \]

\[ B = \frac{a_0 \cosh \beta l \sin \beta l + \sinh \beta l \cos \beta l}{2 \cosh \beta l \cos \beta l + 1} \quad (10.4) \]

\( A, B \), and, therefore, \( w \) become infinite if \( \omega \) is one of the roots of Eq. (8.9), \( i.e. \), if \( \omega \) is equal to one of the frequencies of free vibration. This result is analogous to the results obtained for systems with a finite number of degrees of freedom. The beam develops resonance when the frequency of the exciting force coincides with one of the frequencies of its free vibration. The difference between the beam and the systems treated in Chapter V is that the number of such frequencies is infinite in the case of beam.

11. Buckling of a Uniform Column under Axial Load.—We consider in this and in the following sections columns loaded by axial forces and assume that the axial forces are so large that their influence on the bending must be taken into account.
We assume (Fig. 11.1) that a column is hinged at a fixed point \( x = l \) and has a support at \( x = 0 \) which prevents lateral deflection but allows free rotation and deflection of the column in the direction of the axis. The column is under the action of an axial load \( P \), which is considered positive if it produces compression. If the normal deflection at an arbitrary point is \( w(x) \),

![Diagram of a column with lateral deflection](image)

**Fig. 11.1.—Modes of buckling of an axially loaded beam.** Only the first mode can be produced without introducing additional restraint.

the bending moment \( M \) due to the force \( P \) is \( Pw \). Applying Eq. (3.4), we obtain

\[
EI \frac{d^2w}{dx^2} + Pw = 0 \tag{11.1}
\]

We are thus led to a second-order equation mathematically identical with the equation of the vibrating string [cf. Eq. (6.2)]. Moreover, the boundary conditions are identical with those of the vibrating string. The general solution of Eq. (11.1) is

\[
w = C_1 \cos \sqrt{\frac{P}{EI}} x + C_2 \sin \sqrt{\frac{P}{EI}} x \tag{11.2}
\]

In order that \( w = 0 \) at \( x = 0 \), we must have \( C_1 = 0 \), and the condition \( w = 0 \) for \( x = l \) implies
\[ \sqrt{\frac{P}{EI}} l = k \pi \]  
(11.3)

where \( k \) is an integer.

The corresponding deflection shapes known as the modes of buckling are (Fig. 11.1)

\[ w_k = C_k \sin \frac{k \pi x}{l} \]  
(11.4)

where \( C_k \) is undetermined. Each mode corresponds to a load

\[ P_k = k^2 \pi^2 \frac{EI}{l^2} \]  
(11.5)

Since for these values of the load the corresponding mode of buckling represents an equilibrium position with arbitrary amplitude, we say that under these loads the column is in neutral equilibrium. The corresponding values of the load are called critical loads. However, we must notice that the fact that the amplitude of the deflection curve is undetermined is due to the disregard of higher order terms in the equation of the elastic equilibrium, especially in the expression for the curvature. The undetermined character of the deflection is eliminated by a more exact theory.

From the engineering point of view, the first critical load is of special importance, because it is the upper limit for the stability of the undeflected equilibrium position of the column.

We can show that if the axial thrust \( P \) reaches its lowest critical value corresponding to \( k = 1 \),

\[ P_1 = \pi^2 \frac{EI}{l^2} \]  
(11.6)

the work done by the axial thrust is equal to the work required for bending the beam into the corresponding buckling mode

\[ w = C_1 \sin \frac{\pi x}{l} \]  
(11.7)
The work done by the axial thrust $P$ is equal to the product of $P$ and the axial deflection $\delta$ of the end point $x = 0$ of the column (Fig. 11.2). Now $\delta$ is equal to the length of the straight column minus the chord of the bent column. Strictly speaking, we should take a deflection curve whose length of arc is equal to $l$ and whose chord is $l - \delta$. However, we can take instead the difference between the arc and the chord corresponding to the curve (11.7); the error thus made is negligible. Hence, we put

$$\delta = \int_0^l \sqrt{1 + \left(\frac{dw}{dx}\right)^2} \, dx - l \approx \frac{1}{2} \int_0^l \left(\frac{dw}{dx}\right)^2 \, dx$$

Substituting $w$ from Eq. (11.7), we obtain

$$\delta \approx \frac{1}{2} C_1^2 \pi^2 \int_0^l \cos^2 \frac{\pi x}{l} \, dx = \frac{1}{4} C_1^2 \pi^2 \frac{l}{l}$$

Hence, the work $W_1$ done by the axial thrust is equal to

$$W_1 = \frac{P}{4} C_1^2 \pi^2 \int_0^l \left(\frac{dw}{dx}\right)^2 \, dx = \frac{EI}{2} C_1^2 \pi^4 \int_0^l \sin^2 \frac{\pi x}{l} \, dx - \frac{EI}{4} C_1^2 \pi^4 \frac{l}{l} \quad (11.8)$$

The work necessary for bending the column into the shape (11.7) is given by [cf. Eq. (3.7)]

$$W_2 = \frac{EI}{2} \int_0^l \left(\frac{d^2w}{dx^2}\right)^2 \, dx = \frac{EI}{2} C_1^2 \pi^4 \int_0^l \sin^2 \frac{\pi x}{l} \, dx - \frac{EI}{4} C_1^2 \pi^4 \frac{l}{l} \quad (11.9)$$

It is seen that

$$W_1 < W_2 \quad \text{if} \quad P < EI \frac{\pi^2}{l^2} \quad (11.10)$$

and

$$W_1 > W_2 \quad \text{if} \quad P > EI \frac{\pi^2}{l^2} \quad (11.11)$$

This result shows that if $P > P_1$, which is the first critical load, then in the bent position the total potential energy $W_2 - W_1$ of the system is smaller than in the straight position. Hence, for $P > P_1$ the column is certainly unstable in the straight equilibrium position. [To prove that it is stable for $P < P_1$, it would be necessary to show that $W_1 < W_2$, not only for the particular deflection curve (11.7), but for arbitrary variations of the straight shape.]
It can be shown in a similar way that the column is in neutral equilibrium position, \( i.e., W_1 = W_2 \) for all critical values \( P_k \). However, the higher modes of buckling can be realized only by maintaining the straight shape beyond the lowest critical value by additional restraint.

Equation (11.6), which gives the lowest critical load, is known as Euler's formula.

12. Buckling of a Tapered Column.

Buckling of a Column under Its Own Weight.—Let an axial load \( P \) be applied to a column of circular cross section with linear taper (Fig. 12.1). We take the origin of the abscissa \( x \) at the fictitious vertex \( S \) of the cone and assume that the column is clamped at its base \( x = b \). The law of variation of the moment of inertia of the cross section is

\[
I(x) = I_b \left( \frac{x}{b} \right)^4
\]

where \( I_b \) is the moment of inertia of the cross section at the base \( x = b \). The horizontal distance between the deformed beam axis and the line of action of \( P \) is denoted by \( w \). Then the equation for the deflection of the beam becomes

\[
EI_b \left( \frac{x}{b} \right)^4 \frac{d^2w}{dx^2} + Pw = 0
\]

or

\[
\frac{d^2w}{dx^2} + \frac{Pb^4}{EI_b} \frac{w}{x^4} = 0
\]

This differential equation belongs to the class represented by Eq. (7.8) of Chapter II, where \( m = 0 \) and \( n = -4 \).

The solution of Eq. (12.2) becomes [cf. Eq. (7.13) of Chapter II], with \( C = Pb^4/EI_b \),

\[
w = x^{\frac{1}{2}} Z_{-\frac{1}{2}} \left( -\sqrt{\frac{Pb^4}{EI_b}} x^{-1} \right)
\]

(12.3)
Now, according to section 4, Chapter II [Eqs. (4.6) and (4.8)], \( J_{-\lambda_4}(z) \) and \( Y_{-\lambda_4}(z) \) have the form const. \( \times \cos z/\sqrt{z} \) and const. \( \times \sin z/\sqrt{z} \), respectively. Therefore, \( w(x) \) can be written in the form:

\[
w = x \left( A \cos \frac{V}{x} + B \sin \frac{V}{x} \right) \tag{12.4}
\]

where \( V^2 = P b^4/E I_b \).

The boundary conditions are \( w = 0 \) at \( x = a \) and \( dw/dx = 0 \) at \( x = b \). Hence, we have

\[
A \cos \frac{V}{a} + B \sin \frac{V}{a} = 0 \tag{12.5}
\]

\[
A \left( \cos \frac{V}{b} + \frac{V}{b} \sin \frac{V}{b} \right) + B \left( \sin \frac{V}{b} - \frac{V}{b} \cos \frac{V}{b} \right) = 0
\]

Elimination of \( A \) and \( B \) between these two equations yields the following condition for \( V \):

\[
\tan \left( \frac{V}{a} - \frac{V}{b} \right) = -\frac{V}{b} \tag{12.6}
\]

With the notations \( b - a = l \) and \( VL/ab = \alpha \), we obtain the following transcendental equation for \( \alpha \):

\[
\tan \alpha = -\frac{a}{l} \alpha \tag{12.7}
\]

This equation has an infinite number of roots \( \alpha_k \) (see Fig. 12.2) given by the intersection of the curves \( \eta_1 = \tan \alpha \) and \( \eta_2 = -\alpha l \).

The corresponding critical loads are

\[
P_k = \alpha_k^2 \left( \frac{a}{b} \right)^2 \frac{E I_b}{l^2}
\]

The modes of buckling are given by

\[
w_k(x) = A_k x \sin \left[ \alpha_k \frac{b}{l} \left( 1 - \frac{a}{x} \right) \right] \tag{12.9}
\]

Let us assume that we remove the vertex of the cone to infinity but keep the length of the column and the cross section at the
base fixed, then the ratio $a/b \to 1$, and the column takes on a uniform cross section. Since $a/l \to \infty$, it is seen in Fig. 12.2 that the first intersection of the curves occurs when $\alpha_1 \to \pi/2$, and thus the lowest critical load is found to be

$$P_1 = \frac{\pi^2 EI_b}{4\overline{l}^2}$$

This result checks with the results obtained in the last section, for we may consider a column with one clamped and one free end as the upper half of a column with two freely supported ends.

Fig. 12.2.—Graphical construction for determining the critical buckling loads of a tapered column.

Hence, the column of length $l$ with one clamped and one free end buckles under the same load as a column of the length $2l$ with two freely supported edges. If, on the other hand, the shape is a complete cone, then $a = 0$, $\alpha_1 = \pi$, and $P_1 = 0$. Hence, according to this theory a cone would buckle at an arbitrarily small load. However, in this case we are applying the theory beyond its range of validity because the underlying assumptions of the theory of bending are not satisfied near the vertex.

Another interesting buckling problem is the stability of a vertical column loaded by its own weight. It occurs as a practical problem in the manufacture of very thin tungsten filaments for incandescent lamps.
Let us denote the cross section of the column by \( A \) and the specific weight of its material by \( \gamma \). In the buckled position (Fig. 12.3) the weight of the portion of the beam between the top and the cross section at \( x \) is in equilibrium with the resultant of the axial stresses and the shear force acting at \( x \). Therefore, if the inclination of the axis of the column to the vertical is \( \theta \), the shear force \( S \) is equal to \( S = \gamma Ax \theta \). The bending moment is \( M = -EI \frac{d\theta}{dx} \), where \( EI \) is the flexural rigidity of the column. Therefore, the shear force \( S = \frac{dM}{dx} = -EI \frac{d^2\theta}{dx^2} \). Obviously we obtain the differential equation for \( \theta \):

\[
EI \frac{d^2\theta}{dx^2} = -\gamma Ax \theta
\]

or

\[
\frac{d^2\theta}{dx^2} + \frac{\gamma A}{EI} x \theta = 0
\]  \( (12.10) \)

If we put \( x^2 \sqrt{\frac{\gamma A}{EI}} = \xi \), the differential equation (12.10) becomes

\[
\frac{d^2\theta}{d\xi^2} + \xi \theta = 0
\]  \( (12.11) \)

This equation is identical with Eq. (7.14), Chapter II, and its general solution is [Eq. (7.15), Chapter II]

\[
\theta = \xi^{\frac{3}{4}} Z_{\frac{3}{4}}(\xi^2)
\]  \( (12.12) \)

where the symbol \( Z_{\frac{3}{4}} \) means the general solution of Bessel's differential equation of one-third order. The general solution (12.12) is a linear combination of two particular solutions; one is of the form [(Eq. 7.17), Chapter II]:

\[
\theta_1 = \xi (a_0 + a_1 \xi^3 + a_2 \xi^6 + \cdots )
\]  \( (12.13) \)

and the other of the form [Eq. (7.18), Chapter II]:

\[
\theta_2 = b_0 + b_1 \xi^3 + b_2 \xi^6 + \cdots
\]  \( (12.14) \)
We have the boundary conditions \( \frac{d\theta}{dx} = \frac{d\theta}{d\xi} = 0 \) for \( \xi = 0 \) since the bending moment is zero at the top cross section and \( \theta = 0 \) for \( x = l \) or \( \xi = l\sqrt{\gamma A/ EI} \), if \( l \) is the height of the column and we assume that the column is clamped normal to a horizontal base. It follows from the first condition that \( a_0 = 0 \); therefore [cf. Eq. (7.20), Chapter II], \( \theta_1 = 0 \), and we are left with

\[
\theta = b_0 + b_1 \xi^3 + b_2 \xi^6 + \cdots \tag{12.15}
\]

where [cf. Eq. (7.19), Chapter II]

\[
b_1 = -\frac{b_0}{2 \cdot 3}, \quad b_2 = -\frac{b_1}{5 \cdot 6}, \quad \cdots
\]

and, therefore,

\[
\theta_1 = b_0 \left( 1 - \frac{\xi^3}{6} + \frac{\xi^6}{180} - \cdots \right) \tag{12.16}
\]

The second boundary condition is satisfied if the expression in the parentheses vanishes for \( \xi = l\sqrt{\gamma A/ EI} \). Therefore, we have to calculate the roots of the equation

\[
1 - \frac{\xi^3}{6} + \frac{\xi^6}{180} - \cdots = 0 \tag{12.17}
\]

The first approximation is \( \xi^3 = 6 \) or \( \xi = \sqrt[3]{6} = 1.817 \). In order to calculate the next approximations, we use Graeffe's method (Chapter V, section 8). Let us cut off (12.17) after the term with \( \xi^6 \) and substitute \( \xi^3 = 1/z \). Then we have, after dividing (12.17) by \( \xi^6 \),

\[
z^2 - \frac{z}{6} + \frac{1}{180} = 0 \tag{12.18}
\]

Then multiplying Eq. (12.18) by \( z^2 + \frac{z}{6} + \frac{1}{180} \), we obtain

\[
z^4 - z^2 \left( \frac{1}{36} - \frac{1}{90} \right) + \frac{1}{180^2} = 0
\]

Hence, the largest root \( z \) (corresponding to the smallest root \( \xi \)) is approximately

\[
z = \sqrt[4]{10}
\]
or

\[ \xi = \sqrt[3]{60} = 1.98 \]

(The exact value of the root is 1.986.) Hence, the critical length of the column is

\[ l_c = 1.98 \sqrt[3]{\frac{EI}{\gamma A}} \quad (12.19) \]

For a filament of circular cross section of radius \( r \) we have \( I/A = r^2/4 \), and, therefore,

\[ l_c = 1.98 \frac{1}{\sqrt[3]{4}} r^2 \left( \frac{E}{\gamma} \right)^{1/4} \quad (12.20) \]

The length \( L = E/\gamma \) is the length of a filament which by its own weight would produce a tensile stress equal to \( E \); obviously, \( L \) is a characteristic length of the material. Then it is seen that

\[ l_c \sim r^{3/5} L^{1/5} \quad (12.21) \]

By repeated multiplication of the two series,

\[ g(\xi) = 1 - \frac{\xi^3}{6} + \frac{\xi^6}{180} - \cdots \]

\[ g(-\xi) = 1 + \frac{\xi^3}{6} + \frac{\xi^6}{180} + \cdots \]

we can obtain approximations to the roots \( \xi_1, \xi_2, \cdots \) of \( g(\xi) = 0 \) in ascending order. These roots would give the loads corresponding to higher modes of buckling.

13. Buckling of an Elastically Supported Beam.—The differential equation for the deflection of an elastically supported uniform beam was given in section 4:

\[ EI \frac{d^4 w}{dx^4} + kw = p(x) \quad (13.1) \]

To investigate the buckling of such a beam under action of an axial force \( P \), we replace the latter by an equivalent lateral load. We have seen that the moment of an axial force \( P \) is equal to \( PW \); according to (3.3) the load corresponding to a moment distribution \( M(x) \) is equal to \( p(x) = -d^2 M/dx^2 \); hence the
transverse load which would produce a moment \( M(x) = Pw \) is equal to \( p(x) = -P \frac{d^2w}{dx^2} \). Therefore, Eq. (13.1) becomes

\[
EI \frac{d^4w}{dx^4} + kw = -P \frac{d^2w}{dx^2} \quad (13.2)
\]
or

\[
\frac{d^4w}{dx^4} + \frac{P}{EI} \frac{d^2w}{dx^2} + \frac{k}{EI} w = 0 \quad (13.3)
\]

We can also deduce this equation by considering the equilibrium of the beam in slightly curved shape. Denoting the radius of curvature by \( R \) and assuming an axial force \( P \) acting on two cross sections a unit distance apart, we obtain a resultant force normal to the axis of the beam which is equal to \( P/R \) or, with the approximation used in the beam theory, to \(-P \frac{d^2w}{dx^2}\).

The characteristic equation of the differential Eq. (13.3) is

\[
\lambda^4 + \frac{P}{EI} \lambda^2 + \frac{k}{EI} = 0 \quad (13.4)
\]
or

\[
\lambda^2 = -\frac{P}{2EI} \pm \sqrt{\frac{1}{4} \left( \frac{P}{EI} \right)^2 - \frac{k}{EI}} \quad (13.5)
\]

It is seen that the two values of \( \lambda^2 \) are real and negative when \( P > 0 \) and

\[
P^2 > 4kEI \quad (13.6)
\]

In this case all solutions are trigonometric functions.

We are especially interested in the buckling of a beam of infinite length; in this case we can limit ourselves to periodic solutions because any other solution would yield infinite deflection either at \( x = \infty \) or at \( x = -\infty \). Writing the periodic solution in the form

\[
w = C \sin \frac{2\pi(x - a)}{l} \quad (13.7)
\]

where \( a \) is an arbitrary constant and \( \lambda = 2\pi l/l \), we obtain from Eq. (13.4)

\[
\frac{16\pi^4}{l^4} - \frac{P}{EI} \frac{4\pi^2}{l^2} + \frac{k}{EI} = 0
\]
or

$$P = \frac{4\pi^2}{l^2} EI + \frac{kl^2}{4\pi^2} \quad (13.8)$$

It is seen that in the case of the infinite beam we do not obtain distinct critical loads, but a certain range for the load $P$ which is capable of holding the beam in a deflected shape. This critical range extends from a smallest value $P_{\text{min}}$ to $P = \infty$. We obtain $P_{\text{min}}$ by differentiation of the expression (13.8)

$$\frac{dP}{dl} = -\frac{8\pi^2}{l^3} EI + \frac{2kl}{4\pi^2} = 0 \quad (13.9)$$

The critical wave length, i.e., the wave length produced by the smallest axial load which causes buckling, is given by

$$\lambda = 2\pi\sqrt{\frac{EI}{k}} \quad (13.10)$$

Substituting (13.10) in (13.8), we obtain [see Eq. (13.6)]

$$P_{\text{min}} = 2\sqrt{kEI} \quad (13.11)$$

For loads $P < P_{\text{min}}$ the straight form is stable. For loads $P > P_{\text{min}}$ we obtain two different wave lengths $l_1$ and $l_2$ for each value of the buckling load (Fig. 13.1). However, these configurations do not occur unless some additional restraint is introduced, as the beam buckles when $P$ reaches the value $P_{\text{min}}$.

14. Combined Axial and Lateral Load Acting on the Spar of an Airplane Wing.—Figure 14.1 represents schematically the spar of an airplane wing hinged at the fuselage at $A(x = 0)$ and supported by a hinged strut at $B(x = l)$. The horizontal component of the strut force exerts an axial compression upon the spar, which is subjected to an additional direct bending load $p$. We denote the bending moment due to $p$ by $M_p$; then the differential equation for the deflection $w$ becomes*

$$EI \frac{d^2w}{dx^2} = -Pw - M_w$$

*The load $p$ and the deflection $w$ are here measured positive upward. The bending moment on the left is positive counterclockwise.
Differentiating twice, we obtain
\[ EI \frac{d^4w}{dx^4} + P \frac{d^2w}{dx^2} = p \]  
(14.1)

The total bending moment at the cross section \(x\) is equal to \(M = -EI \frac{d^2w}{dx^2}\). Introducing \(M\) as the unknown variable, we obtain
\[ \frac{d^2M}{dx^2} + \frac{P}{EI}M = -p \]  
(14.2)

![Figure 14.1. Wing spar of a biplane under combined axial and lateral loading.](image)

We assume that \(p\) is constant; then the general solution of equation (14.2) is
\[ M = C_1 \cos \sqrt{\frac{P}{EI}} x + C_2 \sin \sqrt{\frac{P}{EI}} x - pEI \]

The boundary conditions at the hinges are \(M = 0\) at \(x = 0\) and \(M = -M_0\) at \(x = l\), where \(l\) is the distance between the hinges and \(-M_0\) is the bending moment about the hinge point \(B\) of the lift forces acting on the cantilever part of the wing. The values of the constants \(C_1\) and \(C_2\) determined by the boundary conditions are
\[ C_1 = \frac{pEI}{P} \]
\[ C_2 = \frac{pEI}{P} \frac{1 - \cos \sqrt{(P/EI)} l - (M_0 P/pEI)}{\sin \sqrt{(P/EI)} l} \]

Hence, the total bending moment acting at an arbitrary section of the spar is
\[ M = \frac{p}{P} \left[ \cos \sqrt{\frac{P}{EI}} x ight. \\
+ \left. \left( \frac{1 - \cos \sqrt{(P/EI)}l}{\sin \sqrt{(P/EI)}l} \right) \sin \sqrt{\frac{P}{EI}} x - 1 \right] \quad (14.3) \]

It is seen that when the axial load \( P \) reaches the value \( P = \frac{\pi^2 EI}{l^2} \), so that \( \sin \sqrt{\frac{P}{EI}} l = \sin \pi = 0 \), the bending moment becomes infinite. Thus the deflection tends to infinity when the axial load approaches a critical value, as, similarly, the amplitude of a beam subjected to periodic forces tends to infinity in the case of resonance. In both cases infinite amplitudes are reached for certain critical values of a parameter (load or frequency) occurring in the equation. These values of the parameter are identical with the characteristic values of the associated homogeneous equation; i.e., the values for which this equation has solutions different from zero. For instance,

\[ \frac{d^2 M}{dx^2} + \frac{P}{EI} M = 0 \quad (14.4) \]

has solutions of the form:

\[ M_k = C_k \sin \sqrt{\frac{P_k}{EI}} x \]

for certain given values for \( P_k \). The smallest of these values is the buckling load

\[ P_1 = \frac{\pi^2 EI}{l^2} \]

For the same value of the axial thrust, the bending moment, according to (14.3), becomes infinite if the beam is subjected to a lateral load in addition to the axial force.

If, instead of compression, the beam were under tension, the equation would differ only by the sign of \( P \)

\[ \frac{d^2 M}{dx^2} - \frac{P}{EI} M = -p \quad (14.5) \]

In this case, however, the general solution involves exponential functions, and the value of \( M \) will never become infinite. The reader will remember that an equation of this type was encoun-
tered in the theory of the suspension bridge (section 5). In fact, the load condition of the bridge truss could be described as a combination of a lateral load and an axial tension.

15. Graphical Representation of the Bending Moment.—We put

\[ \tan \varphi = \frac{1 - \cos \sqrt{(P/El)}l - (M_0/p)(P/El)}{\sin \sqrt{(P/El)}l} \]  

(15.1)

Then Eq. (14.3) becomes

\[ M = \frac{pEI}{P \cos \varphi} \left[ \cos \left( \sqrt{\frac{P}{EI}}x - \varphi \right) - \cos \varphi \right] \]

or

\[ \frac{MP \cos \varphi}{pEI} = \cos \left( \sqrt{\frac{P}{EI}}x - \varphi \right) - \cos \varphi \]  

(15.2)

We draw a circle of unit diameter (Fig. 15.1) and draw the radius vector \( \overrightarrow{OA} \) so that the angle \( AOX = \varphi \). Then if we draw a circle through \( A \) with \( O \) as center, the line segments between the two circles are equal to the right side of Eq. (15.2), provided the angle \( AOX \) represents \( x \sqrt{\frac{P}{EI}} \). It is seen that if \( P \to \pi \frac{EI}{l^2} \), \( \varphi \to \) and \( M \to \infty \).

16. General Discussion of the Boundary Problems Encountered in This Chapter.—We have dealt in this chapter with differential equations of second and fourth orders. Let us consider as two representative examples of second-order equations
the equation for the deflection of a loaded string with elastic restraint \( k > 0 \), \textit{viz.,}

\[
F \frac{d^2w}{dx^2} - kw = -p(x)
\]  
(16.1)

and the equation for the harmonic vibration of a string under action of an external load [mathematically identical with Eq. (14.2)], \textit{viz.,}

\[
F \frac{d^2w}{dx^2} + \rho \omega^2 w = -p(x)
\]  
(16.2)

We treated similar equations in Chapter IV; however, the physical problems with which we were concerned in that chapter involved \textit{initial conditions} which specify the value of the unknown function and its derivative for one value of the independent variable. We found that, in general, such initial conditions determine a \textit{unique} solution of the problem.

In the problems of this chapter the conditions that we call \textit{boundary conditions} have bearing on the values of \( w \) or its derivatives for at least two values of the independent variable. For example, \( w \) was given at two points, \( x = 0 \) and \( x = l \). We find that in this case Eqs. (16.1) and (16.2) behave in a very different manner. Before we classify the different cases, let us introduce the following terminology: If a \textit{boundary condition} is satisfied by \( Cw(x) \), provided that it is satisfied by \( w(x) \), where \( C \) is an arbitrary constant, we call the boundary condition \textit{homogeneous}. For example, \( w = 0 \) and \( dw/dx = 0 \) are \textit{homogeneous boundary conditions}, whereas \( w = 1 \) is a \textit{nonhomogeneous boundary condition}. We call the problem of finding a solution of a homogeneous equation for homogeneous boundary conditions a \textit{homogeneous boundary problem}. If either the equation or at least one boundary condition is nonhomogeneous, we call the problem a \textit{nonhomogeneous boundary problem}. If we replace the right side of a nonhomogeneous equation and the nonhomogeneous boundary conditions by zero, we call the homogeneous problem obtained in this way the \textit{associated homogeneous problem}. Then we find the following results:

\textit{a.} Equation (16.1) has one, and only one, solution if two boundary conditions are to be satisfied. The only solution of the homogeneous problem is \( w = 0 \).
b. In the case of Eq. (16.2), \( w = 0 \) is the only solution of the homogeneous problem, except when the parameter \( \omega^2 \) is equal to one of the infinite number of characteristic values \( \omega_1^2, \omega_2^2, \ldots \). If \( \omega^2 \) coincides with one of these values, the homogeneous problem has solutions different from zero; these are determined only to an arbitrary multiplicative constant. A nonhomogeneous problem has one and only one solution if the associated homogeneous problem is solved by \( w = 0 \) only. If the parameter \( \omega^2 \) is equal to one of the characteristic values of the associated homogeneous problem, the nonhomogeneous problem has no finite solution. We encounter the homogeneous problem in problems of free vibration and buckling. The characteristic values represent the natural frequencies and the critical loads, respectively. The problems of forced vibration (resonance) and the problem of combined buckling and direct load lead to nonhomogeneous equations or to nonhomogeneous boundary conditions.

The mathematical reason for the different behavior of the two equations was indicated earlier in this chapter. The solution of the homogeneous equation associated with Eq. (16.2) has oscillatory character, whereas the homogeneous equation associated with (16.1) has no more than one zero point. Hence, it is important to find criteria that determine when a differential equation has oscillatory solutions.

In general, e.g., for nonlinear differential equations and linear differential equations of higher order than the second, this is a very difficult mathematical problem. In the case of a linear differential equation of second order, we can state a simple theorem which gives a criterion for a wide class of such differential equations.

A homogeneous linear differential equation of second order has the general form:

\[
\frac{d^2y}{dx^2} + a(x) \frac{dy}{dx} + b(x)y = 0 \tag{16.3}
\]

where \( a(x) \) and \( b(x) \) are given functions of \( x \). Equation (16.3) can always be reduced to the form:

\[
\frac{d^2z}{dx^2} + g(x)z = 0 \tag{16.4}
\]
In fact, if we substitute in Eq. (16.3) \( y = e^{-\int^x \frac{a(s)}{2} ds} z \), the term with \( dz/dx \) drops out, and we obtain the form (16.4).

Now, if \( g(x) > 0 \) and has a positive lower bound between \( x = a \) and \( x = \infty \), it can be proved that the solution \( z(x) \) of Eq. (16.4) takes the value \( z = 0 \) an infinite number of times in the interval between \( x = a \) and \( x = \infty \). From Eq. (16.4) it follows that \( d^2 z / dx^2 = -g(x) z \). Since \( g(x) > 0 \), this means that \( d^2 z / dx^2 \) and \( z \) have opposite signs, and, therefore, the curve \( z = z(x) \) seen from the \( x \)-axis is always concave. All inflection points must lie on the \( x \)-axis. It is easily seen that if we start from an arbitrary point \((x, z)\), the curve due to the sign and the bounded character of its curvature must intersect the \( x \)-axis; if we pass through this axis, the sign of the curvature changes, and, therefore, we obtain another intersection, and this process continues indefinitely.

Hence, if we assume that in the equation

\[
\frac{d^2 z}{dx^2} + [h(x) + \lambda g(x)] z = 0 \quad (16.5)
\]

the function \( h(x) \) is bounded and \( g(x) \) is larger than zero and has a lower bound, the solution of Eq. (16.5) has oscillatory character, provided that the parameter \( \lambda \) is positive and sufficiently large, so that \( h(x) + \lambda g(x) > 0 \). In other words, Eq. (16.5) has always positive characteristic values.

If \( g(x) \) changes its sign, it is very difficult to decide whether or not real characteristic values exist. Fortunately, this is an exceptional case in engineering problems.

As a representative example of an equation of fourth order, the equation for the deflection of beams was discussed. The most general form of this equation is the following:

\[
\frac{d^2}{dx^2} \left[ a(x) \frac{d^2 w}{dx^2} \right] + \frac{d}{dx} \left[ b(x) \frac{dw}{dx} \right] + c(x) w = p(x) \quad (16.6)
\]

An equation of such form is called a self-adjoint linear differential equation. (A linear differential equation of second order can always be written in the analogous form \( d/dx [a(x) dy/dx] + b(x) y = p(x) \) and, therefore, is always self-adjoint.) It would take us too far into the domain of the theory of differential equations to explain the origin of this term. It will be sufficient to state that all linear differential equations that are derived from a variation principle belong to this class. Hence, all equilibrium, buckling, and vibra-
tion problems that refer to conservative systems, i.e., to systems that have potential energy and are governed by linear equations, lead to equations of the self-adjoint class. The homogeneous equations of this class have the property that, if one particular solution has more than one zero point, it crosses the z-axis an infinite number of times.

It is difficult to find simple conditions that indicate whether or not the homogeneous equation associated with Eq. (16.6) has solutions of the oscillatory type, and the engineer will resort to physical reasoning.

17. Determination of Characteristic Values by the Iteration Method.—For the characteristic values of differential equations with constant coefficients we obtained transcendental equations in which trigonometric or exponential functions were involved. If the coefficients of the differential equation are variable, the determination of the characteristic values (natural frequencies, buckling loads) can cause considerable difficulty. However, in many cases the iteration method can be employed with success.

Let us assume that the differential equation has the form:

$$\frac{d}{dx} \left[ a(x) \frac{dw}{dx} \right] + \lambda b(x)w = 0 \quad (17.1)$$

We shall determine a value $\lambda$ for which $w(x)$ satisfies certain homogeneous boundary conditions.

We choose a function $w_1(x)$ in such a way that it satisfies the boundary conditions set for the solution $w(x)$, but is arbitrary otherwise. Then we can solve the equation

$$\frac{d}{dx} \left[ a(x) \frac{dw}{dx} \right] = -b(x)w_1(x) \quad (17.2)$$

by direct integration. Assume that we obtain a function $w = w_2(x)$. If $w_1(x)$ were the exact solution corresponding to a characteristic value $\lambda$ of Eq. (17.1), for example, one of the modes of oscillation or a buckling mode of the beam, etc., the function $w_2(x)$ would be proportional to $w_1(x)$, viz.,

$$w_2(x) = w_1(x) \quad (17.3)$$

where $\lambda$ is the characteristic value sought for. If the proportionality between $w_2(x)$ and $w_1(x)$ is not satisfactory, we multiply $w_2(x)$ by a constant so that for a suitably chosen value $x = x_0$
we make \( w_2(x_0) = w_1(x_0) \) and substitute the function \( w_2(x) \), reduced in this way, on the right side of Eq. (17.2):

\[
\frac{d}{dx} \left[ a(x) \frac{dw}{dx} \right] = -b(x)w_2(x)
\]  

(17.4)

We obtain in this way \( w_3(x) \). If \( w_3(x) \) and \( w_2(x) \) are proportional to each other with satisfactory accuracy, we consider \( w_3(x) \) as the solution corresponding to the characteristic value:

\[
\lambda \approx \frac{w_2(x_0)}{w_3(x_0)}
\]  

(17.5)

It can be shown that by this procedure the limiting value

\[
\lambda = \lim_{n \to \infty} \frac{w_{n-1}(x)}{w_n(x)}
\]  

(17.6)

gives us exactly the smallest characteristic value of the differential equation, \textit{i.e.}, the fundamental frequency, the lowest buckling load, or the lowest critical speed. The function \( w_n(x) \) converges to the particular solution corresponding to this characteristic value, \textit{i.e.}, the first mode of vibration or buckling. The process can be also applied to equations of fourth order.

Let us consider, for example, a shaft of constant flexural rigidity \( EI \) which carries so many masses that the mass distribution can be replaced by a continuous function \( \rho(x) \). The equation for the critical speed of the shaft is [cf. section 8, Eq. (8.15)]

\[
EI \frac{d^4w}{dx^4} = \rho(x)\omega^2 w
\]  

(17.7)

If we assume an arbitrary deflection curve \( w_1(x) \) and substitute \( w_1(x) \) on the right side of the equation, putting \( \omega^2 = 1 \),

\[
EI \frac{d^4w}{dx^4} = \rho(x)w_1
\]  

(17.8)

this means that we load the shaft by centrifugal forces corresponding to the shape of deflection determined by \( w_1(x) \) and to unit rotational speed. If \( w_1(x) \) were the correct shape of deflection and \( \omega \) the critical speed, the centrifugal force \( \rho(x)w_1\omega^2 \) would produce a deflection equal to \( w_1 \); hence, from Eq. (17.8) we would obtain
\[ w_2(x) = \frac{w_1(x)}{\omega^2} \]  \hspace{1cm} (17.9)

in accordance with Eq. (17.3).

It is seen that the method of iteration or successive approximations consists of correcting the deflection curve until the deflection produced by the centrifugal forces is exactly proportional to the assumed deflection.

Let us demonstrate this method on the example of the free oscillations of a circular membrane held under a uniform tension \( T \). We shall restrict ourselves to oscillations of axial symmetry, i.e., we assume that the deflection \( w(x) \) is only a function of the distance \( x \) from the center. We may consider a sector of the membrane as a string of variable cross section. Take, for example, a sector whose central angle is \( \Delta \alpha \). The vertical component of the tension is equal to \( T x \Delta \alpha dw/dx \), and, therefore, the equation for the equilibrium of the membrane under a vertical load \( p \) per unit area is given by

\[ \frac{d}{dx} \left( xT \Delta \alpha \frac{dw}{dx} \right) = -px\Delta \alpha \]  \hspace{1cm} (17.10)

or

\[ T \frac{d}{dx} \left( x \frac{dw}{dx} \right) = -px \]  \hspace{1cm} (17.11)

In the case of harmonic oscillations we must assume a load per unit area equal to \( \rho \omega^2 \) where \( \rho \) is the mass of the membrane per unit area. Hence, we have

\[ \frac{d}{dx} \left( x \frac{dw}{dx} \right) = -\left( \frac{\rho \omega^2}{T} \right) wx \]  \hspace{1cm} (17.12)

If the membrane is supported by a circular frame of radius \( a \), we have the boundary conditions \( dw/dx = 0 \) for \( x = 0 \) and \( w = 0 \) for \( x = a \). The solution

\[ w = \text{const.} \ J_0 \left( \sqrt{\frac{\rho}{T} \omega x} \right) \]  \hspace{1cm} (17.13)

satisfies Eq. (17.12) and the boundary condition for \( x = 0 \). The second particular solution of (17.12) is infinite for \( x = 0 \). The boundary condition for \( x = a \) is satisfied if
\[ \sqrt{\frac{\mu}{T}} \omega a = \alpha_i \quad (17.14) \]

where \( \alpha_i \) is one of the roots of \( J_0(\alpha) = 0 \).

Let us determine \( \alpha_1 \) by the iteration method. We choose

\[ w_1 = 1 - \left( \frac{x}{a} \right)^2 \quad (17.15) \]

and write

\[ \frac{d}{dx} \left( x \frac{d w_2}{d x} \right) = -k^2 \left( 1 - \frac{x^2}{a^2} \right) x \]

where \( k^2 = \rho/T \).

Then we obtain

\[ x \frac{d w_2}{d x} = -k^2 \left( \frac{x^2}{2} - \frac{x^4}{4a^2} \right) + \text{const.} \quad (17.16) \]

Since \( dw/dx = 0 \) for \( x = 0 \), the constant in (17.16) is zero and we have

\[ w_2 = k^2 a^2 \left( C - \frac{x^2}{4a^2} + \frac{x^4}{16a^4} \right) \]

For \( x = a \), we must have \( w = 0 \); therefore

\[ w_2 = k^2 a^2 \left( \frac{3}{16} - \frac{1}{4} \frac{x^2}{a^2} + \frac{1}{16} \frac{x^4}{a^4} \right) \quad (17.17) \]

We now calculate

\[ \frac{w_1(x)}{w_2(x)} = \frac{1 - (x/a)^2}{\frac{3}{16} \left( 1 - \frac{x^2}{4a^2} + \frac{1}{3}(x/a)^4 \right) k^2 a^2} \]

e.g., for \( x/a = 0 \),

\[ \lambda \approx \frac{w_1(0)}{w_2(0)} = \frac{16}{3} \frac{1}{k^2 a^2} = 5.333 \frac{1}{k^2 a^2} \quad (17.18) \]

Hence, the first approximation for \( \omega^2 \) is

\[ \omega^2 = \frac{5.333}{k^2 a^2} = 5.333 \frac{T}{\rho a^2} \]

If we continue the above procedure, we obtain as successive approximations

\[ \frac{\omega^2 \rho a^2}{T} = 5.333, 5.526, 5.756 \]
The exact value of $\alpha_1 = 2.405$; therefore, from Eq. (17.14)

$$\frac{\omega^2 \rho a^2}{T} = (2.405)^2 = 5.784$$

As was mentioned on p. 314, the iteration process leads to the fundamental frequency.

The iteration method sketched in this section is known in engineering practice as Stodola's or Vianello's method. The reader will notice that in principle it is analogous to the matrix method that we used in Chapter V for solving oscillation problems involving systems with a finite number of degrees of freedom.

Problems

1. A string of length $l$ which extends between $x = 0$ and $x = l$ is held under a tension $F$ and is loaded by a uniformly distributed load $p$ between the points $x = l/3$ and $x = l/2$. Find the location and magnitude of the maximum deflection.

2. A semi-infinite uniform beam which extends between $x = 0$ and $x = \infty$ and rests on an elastic foundation carries a concentrated load at its end $x = 0$. Find the distribution of the bending moment.

3. Find the influence function $g(x, \xi)$ which gives the deflection at the point $x$ for a load applied at $\xi$ to a beam of uniform cross section clamped at $x = 0$ and hinged at $x = l$. Verify that $g(x, \xi) = g(\xi, x)$.

*Hint: Use the general solutions of Eq. (4.1) for $p = 0$ with different coefficients for $x < \xi$ and $x > \xi$. Remember that $g$, $\frac{dg}{dx}$, and $\frac{d^2g}{dx^2}$ are continuous at $x = \xi$."

4. Two equal loads a constant distance $a$ apart travel along a beam of span $l$. Find the maximum bending moment at a given point $x = \xi$ along the beam. Show that the problem is mathematically the same as that of finding the maximum deflection at a certain point of a string of length $l$ under tension when the same loads travel on it.

5. A cantilever beam of rectangular cross section is loaded by a concentrated force $F$ at its free end. The beam has constant width; however, its height varies along the length in such a way that the maximum bending stress is constant (beam of uniform strength). Find the shape of the deflection curve.

*Hint: The maximum stress is $\sigma = \frac{6M}{bh^2}$, where $b$ is the width and $h$ the height.*

6. Find the distribution of the bending moment for the simply supported truss of a suspension bridge of span $l$. The dead load is $q$ per unit length. A live load $p$ per unit length is uniformly distributed over the half span $0 \leq x \leq l/2$. Use the method of the influence functions:

a. Calculate the moment and the deflection corresponding to a concentrated unit load applied at an arbitrary point $\xi$ of a beam under axial tension.
b. Calculate the moment distribution for the load distributed between 
\( x = 0 \) and \( x = \frac{l}{2} \) by the method of superposition.

\( \text{c. Calculate the increase of the cable tension neglecting the expansion of the cable.} \)

\textit{Solution:} First determine the solution of the equation

\[
EI \frac{d^4w_1}{dx^4} - (H + h) \frac{d^2w_1}{dx^2} = p(x)
\]

for a beam under the axial tension \((H + h)\) and a distributed load \(p(\xi)\).

We find

\[
w_1 = \int_0^l g(x, \xi)p(\xi) \, d\xi
\]

where

\[
g(x, \xi) = \begin{cases} 
\frac{(l - \xi)x}{EIl \mu^2} - \sinh \mu(l - \xi) \sinh \mu \frac{x}{H} & \text{for } x \leq \xi \\
\frac{\xi(l - x)}{EIl \mu^2} - \sinh \mu \xi \sinh \mu \frac{l - x}{H} & \text{for } x \geq \xi
\end{cases}
\]

with \( \mu = \sqrt{\frac{H + h}{EI}} \) and \( g(x, \xi) = g(\xi, x) \). Then we calculate the solution of

\[
EI \frac{d^4w_2}{dx^4} - (H + h) \frac{d^2w_2}{dx^2} = 1
\]

This solution is

\[
w_2 = \frac{1}{H + h} \left[ \frac{x(l - x)}{2} + \frac{1}{\mu^2} \left( \frac{\cosh \mu \left( x - \frac{l}{2} \right)}{\cosh \mu l/2} - 1 \right) \right]
\]

The equation for the deflection of the suspension bridge under a load \(p(x)\) is

\[
EI \frac{d^4w}{dx^4} - (H + h) \frac{d^2w}{dx^2} = p(x) - \frac{h}{H}q,
\]

and, therefore, by superposition,

\[w = w_1 - \frac{h}{H}qw_2.\]

The horizontal tension increment is still unknown. We calculate it by the condition \( \int_0^l w \, dx = 0 \), or \( \int_0^l w_1 \, dx = hq/H \int_0^l w_2 \, dx \).

This relation may be simplified as follows: We write

\[
\int_0^l w_1 \, dx = \int_0^l dx \int_0^l g(x, \xi)p(\xi) \, d\xi
\]

Because \( g(x, \xi) = g(\xi, x) \), we have

\[
\int_0^l w_1 \, dx = \int_0^l p(x) \, dx \int_0^l g(x, \xi) \, d\xi
\]
Now \( \int_0^l g(x, \xi) \, d\xi \) is the value \( w_1 \) for \( p(x) = 1 \). Hence, it is equal to \( w_2 \) and, therefore, \( \int_0^l w_1 \, dx = \int_0^l w_2 \, p \, dx \). Finally \( \frac{h}{H} = \frac{1}{q} \int_0^l p w_1 \, dx \). If we put \( h = 0 \) in the integrals, this is a first approximation for \( h \), which will be sufficiently accurate for practical purposes.

In our particular problem \( p \) is uniformly distributed over the half span. The above formulas must be applied by carrying the integration over the half span. Because of the symmetry of the function \( w_2 \) we have rigorously \( h/H = p/2q \). The deflection of the bridge is \( w = p \int_0^{l/2} g(x, \xi) \, d\xi - \frac{p}{\alpha} w_2(x) \) in which \( \mu = \sqrt{\frac{H}{EI \left( 1 + \frac{p}{2q} \right)} \). The bending moment is \( M = -EI d^2w/dx^2 \).

7. Calculate the deflection of the truss and the decrease of the cable tension for a suspension bridge loaded by a dead load \( q \) uniformly distributed over the span under the assumption that the cable temperature is raised by 55°F, and the coefficient of the linear expansion of the material is 0.0000065/°F. Neglect the elasticity of the cable.

The characteristic constants of the bridge are \( H/qL = 2.5 \) and \( Hk/EI = 16 \).

8. Find the natural frequencies and the modes of vibration for a uniform beam hinged at \( x = 0 \) and elastically supported at the end \( x = l \). The spring constant of the elastic support is equal to \( k \). Discuss the limiting cases \( k = 0 \) and \( k = \infty \). Solve the same problem for the case where the beam is clamped at \( x = 0 \).

\( \text{Hint:} \) The frequency \( \omega \) appears in the dimensionless quantity

\[
l \sqrt{\frac{\rho \omega^2}{EI}} = \beta l
\]

(cf. section 8 of this chapter). The equation for the frequencies will be found with \( \beta l \) as unknown and the quantity \( kl^3/EI \) as a dimensionless parameter. Solve the frequency equation graphically for various values of \( kl^3/EI \).

9. A cantilever shaft of length \( l \) carries at its free end \( x = l \) a disk of mass \( m \) and moment of inertia \( C \). Find the equation for the critical speeds, taking into account the gyroscopic effect of the disk. If the slope of the shaft at \( x = l \) is equal to \( \theta_1 \), the gyroscopic moment represents a restoring moment of the magnitude \( C \omega^2 \theta_1 \). Discuss the transition to the limiting case in which the mass of the beam is negligible.

10. Find the lowest critical speed of a uniform shaft running in two bearings that are restrained elastically so that an angular deflection \( \theta \) of the shaft produces a restoring moment at the bearing equal to \( k\theta \).

11. A column is tested for buckling in a testing machine between knife edges (Fig. P.11). The distance between the knife edges is \( l \). The ends of the column are clamped in shoes of the length \( a \), so that the center portion of the column of the length \( l - 2a \) is free, whereas a section of the length \( a \)
at each end may be considered as perfectly rigid. Find the correction for Euler's formula (11.6).

**Solution:** Taking \( x = 0 \) at the center of the column, the deflection may be represented by \( w \sim \cos \sqrt{\frac{P}{EI}} x \) and the boundary condition is

\[
EI \frac{d^2w}{dx^2} - P \alpha \frac{dw}{dx} = 0
\]

at \( x = \frac{l}{2} - a \). This yields for the critical load \( P \) the equation

\[
\tan \left( \frac{\pi}{2} - k + k\alpha \right) = k\alpha
\]

where \( k = l/2\sqrt{P/EI} \) and \( \alpha = 2a/l \). When \( \alpha = 0 \), the solution is \( k = \pi/2 \).

We write, therefore, \( k = \frac{\pi}{2} + \epsilon \). Hence,

\[
\tan \left[ -\epsilon + \left( \frac{\pi}{2} + \epsilon \right) \alpha \right] - \left( \frac{\pi}{2} + \epsilon \right) \alpha = 0
\]

Since \( \epsilon \) and \( \alpha \) are small, this equation is approximately

\[
-\epsilon + \frac{1}{3} \left( \frac{\pi}{2} \alpha + \epsilon (\alpha - 1) \right)^3 = 0
\]

In the first approximation, \( \epsilon = -\frac{1}{3} \left( \frac{\pi}{2} \alpha \right)^3 \). The corrected critical load is

\[
P' = 4k^2EI \frac{EI}{l^2} = \frac{\pi^2 E t}{l^2} \left( 1 + \frac{\pi^2}{6} \alpha^2 \right)
\]

12. A cantilever beam carries a mass at its free end. The mass is equal to half the total mass of the beam. Find the natural frequencies of the system.

13. A cylindrical steel vessel of length \( l \), radius \( r \), and wall thickness \( t \) is subjected to an internal pulsating pressure \( p_0 + \rho \sin \omega t \). Find the frequency for which resonance occurs between the pressure pulsation and the fundamental vibration of the walls of the vessel. Assume that the cylindrical shell is clamped at both ends to circular rigid plates. Calculate the effect of the average pressure \( p_0 \) on the lowest natural frequency.

**Method:** A longitudinal strip of unit width of the cylindrical shell behaves like a beam under an axial tension \( p_0 t^2/2\pi r = p_0 r/2 \) and resting on an elastic support \( k = Et/r^2 \). The equation for the oscillations of this beam is

\[
\frac{Et^3}{12} \frac{d^4w}{dx^4} - \frac{p_0}{2} \frac{d^2w}{dx^2} + \left( \frac{Et}{r^2} - \rho \omega^2 \right) w = 0
\]

where \( \rho \) is the mass of the shell