15. \( \frac{1}{x-2} - \frac{3}{x+2} + \frac{2x}{(x+2)^2} \).

16. \( \frac{1}{a-b} + \frac{1}{a+b} - \frac{a}{a^2-b^2} \).

17. \( \frac{a+x}{a-x} + \frac{a-x}{a+x} - \frac{a^2-x^2}{a^2+x^2} \).

18. \( \frac{1}{x+1} + \frac{2}{x+2} + \frac{1}{x+3} \).

19. \( \frac{x}{x-1} - \frac{2x}{x+1} + \frac{x}{x-2} \).

20. \( \frac{4x}{y} = \frac{x-y}{x+y} + \frac{x+y}{x-y} \).

21. \( x - \frac{x^2}{x-1} - \frac{x}{x+1} \).

22. \( x - \frac{x^2}{x+1} + \frac{x}{x-1} \).

23. \( \frac{1}{x-a} + \frac{1}{x+a} - \frac{2}{x} \).

24. \( \frac{a}{a-b} + \frac{a}{a+b} + \frac{4a^2b^2}{a^2-b^4} \).

25. \( \frac{x^2}{x^2-1} + \frac{x}{x-1} + \frac{x}{x+1} \).

26. \( \frac{a}{a-x} + \frac{3a}{a+x} - \frac{2ax}{a^2+x^2} \).

27. \( \frac{3}{2x-4} - \frac{1}{x+2} - \frac{x+10}{2x^3+8} \).

28. \( \frac{2}{x+4} - \frac{x-3}{x^2-4x+16} + \frac{x^2}{x^3+64} \).

29. \( \frac{1}{x^2-a^2} + \frac{1}{(x+a)^2} - \frac{1}{(x-a)^2} \).

T. A.
EXAMPLES. XVI.

30. \( \frac{x^2 + ax + a^2}{x^3 - a^3} - \frac{x^2 - ax + a^2}{x^3 + a^3} \).

31. \( \frac{x^2 + y^2}{xy} - \frac{x^2}{xy + y^2} - \frac{y^2}{x^2 + xy} \).

32. \( \frac{x^2 - 2x + 3}{x^3 + 1} + \frac{x - 2}{x^2 - x + 1} - \frac{1}{x + 1} \).

33. \( \frac{1}{(x - 3)(x - 4)} - \frac{2}{(x - 2)(x - 4)} + \frac{1}{(x - 2)(x - 3)} \).

34. \( \frac{1}{x(x + 1)} - \frac{2x - 3}{x(x + 1)(x + 2)} + \frac{1}{x(x + 2)} \).

35. \( \frac{1 - 2x}{3(x^2 - x + 1)} + \frac{x + 1}{2(x^2 + 1)} + \frac{1}{6(x + 1)} \).

36. \( \frac{x - y}{x^2 - xy + y^2} + \frac{1}{x + y} + \frac{xy}{x^3 + y^3} \).

37. \( \frac{1}{x - y} + \frac{x - y}{x^2 + xy + y^2} + \frac{xy - 2x^2}{x^3 - y^3} \).

38. \( \frac{x + 1}{x^2 + x + 1} + \frac{x - 1}{x^3 - x + 1} + \frac{2}{x^4 + x^3 + 1} \).

39. \( \frac{a + b}{ax + by} + \frac{a - b}{ax - by} + \frac{2(a^2x + b^2y)}{a^2x^2 + b^2y^2} \).

40. \( \frac{2x}{x^4 - x^3 + 1} - \frac{1}{x^2 - x + 1} + \frac{1}{x^2 + x + 1} \).

41. \( \frac{1}{x^2 - 7x + 12} + \frac{2}{x^2 - 4x + 3} - \frac{3}{x^2 - 5x + 4} \).

42. \( \frac{1}{x + a} - \frac{1}{x - a} + \frac{4a}{x^2 - a^2} - \frac{2a}{x^2 + a^2} \).

43. \( \frac{1}{a - b} - \frac{1}{a + b} - \frac{2b}{a^2 + b^2} - \frac{4b^3}{a^4 + b^4} \).

44. \( \frac{1}{x - 3a} - \frac{1}{x + 3a} + \frac{3}{x + a} - \frac{3}{x - a} \).
45. \[ \frac{1}{a-2b} - \frac{4}{a-b} + \frac{6}{a} - \frac{4}{a+b} + \frac{1}{a+2b} \cdot \]

46. \[ \frac{c}{(x-a)(a-b)} + \frac{c}{(x-b)(b-a)} \cdot \]

47. \[ \frac{a}{(x-a)(a-b)} + \frac{b}{(x-b)(b-a)} \cdot \]

48. \[ \frac{a^2}{(x-a)(a-b)} + \frac{b^2}{(x-b)(b-a)} \cdot \]

49. \[ \frac{1}{(x-b)(a-c)} + \frac{1}{(b-a)(b-c)} \cdot \]

50. \[ \frac{b}{(a-b)(a-c)} + \frac{a}{(b-a)(b-c)} \cdot \]

51. \[ \frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)} \cdot \]

52. \[ \frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-a)(b-c)} - \frac{1}{abc} \cdot \]

53. \[ \frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-a)(b-c)} + \frac{c^2}{(c-a)(c-b)} \cdot \]

54. \[ \frac{1}{x^2-(a+b)x+ab} + \frac{1}{x^2-(a+c)x+ac} \]
\[ + \frac{1}{x^2-(b+c)x+bc} \cdot \]

55. \[ \frac{x+c}{x^2-(a+b)x+ab} + \frac{x+b}{x^2-(a+c)x+ac} \]
\[ + \frac{x+a}{x^2-(b+c)x+bc} \cdot \]

56. \[ \frac{1}{(a-b)(a-c)(x-a)} + \frac{1}{(b-a)(b-c)(x-b)} \]
\[ + \frac{1}{(c-a)(c-b)(x-c)} \cdot \]

6-2
MULTIPLICATION OF FRACTIONS.

XVII. Multiplication of Fractions.

144. Rule for the multiplication of fractions. Multiply together the numerators for a new numerator, and the denominators for a new denominator.

145. The following is the usual demonstration of the Rule. Let \( \frac{a}{b} \) and \( \frac{c}{d} \) be two fractions which are to be multiplied together; put \( \frac{a}{b} = x \), and \( \frac{c}{d} = y \); therefore

\[
a = bx, \quad \text{and} \quad c = dy;
\]

therefore

\[
ac = bdxy;
\]

divide by \( bd \), thus

\[
\frac{ac}{bd} = xy.
\]

But

\[
xy = \frac{a}{b} \times \frac{c}{d};
\]

therefore

\[
\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.
\]

And \( ac \) is the product of the numerators, and \( bd \) the product of the denominators; this demonstrates the Rule.

Similarly the Rule may be demonstrated when more than two fractions are multiplied together.

146. We shall now give some examples. Before multiplying together the factors of the new numerator and the factors of the new denominator, it is advisable to examine if any factor occurs in both the numerator and denominator, as it may be struck out of both, and the result will thus be simplified; see Art. 137.

Multiply \( \frac{a}{c} \) by \( \frac{b}{c} \).

\[
a = \frac{a}{1}; \quad \frac{a}{1} \times \frac{b}{c} = \frac{ab}{c}.
\]

Hence \( \frac{ab}{c} \) and \( \frac{a}{c} \frac{b}{c} \) are equivalent; so, for example,

\[
\frac{4x}{5} = \frac{4x}{5}; \quad \text{and} \quad \frac{1}{4}(2x-3) = \frac{2x-3}{4}.
\]
MULTIPLICATION OF FRACTIONS.

Multiply \( \frac{x}{y} \) by \( \frac{x}{y} \).

\[
\frac{x}{y} \times \frac{x}{y} = \frac{x \times x}{y \times y} = \frac{x^2}{y^2};
\]

thus \( \left( \frac{x}{y} \right)^2 = \frac{x^2}{y^2} \).

Multiply \( \frac{3a}{4b} \) by \( \frac{8c}{9a} \).

\[
\frac{3a}{4b} \times \frac{8c}{9a} = \frac{3a \times 8c}{4b \times 9a} = \frac{2c \times 12a}{3b \times 12a} = \frac{2c}{3b}.
\]

Multiply \( \frac{3a^2}{(a+b)^2} \) by \( \frac{4(a^2 - b^2)}{3ab} \).

\[
\frac{3a^2}{(a+b)^2} \times \frac{4(a^2 - b^2)}{3ab} = \frac{4a(a - b) \times 3a(a + b)}{b(a + b) \times 3a(a + b)} = \frac{4a(a - b)}{b(a + b)}.
\]

Multiply \( \frac{a}{b} + \frac{b}{a} + 1 \) by \( \frac{a}{b} + \frac{b}{a} - 1 \).

\[
\frac{a}{b} + \frac{b}{a} + 1 = \frac{a^2}{ab} + \frac{b^2}{ab} + \frac{ab}{ab} = \frac{a^2 + b^2 + ab}{ab},
\]

\[
\frac{a}{b} + \frac{b}{a} - 1 = \frac{a^2}{ab} + \frac{b^2}{ab} - \frac{ab}{ab} = \frac{a^2 + b^2 - ab}{ab};
\]

\[
\frac{a^2 + b^2 + ab}{ab} \times \frac{a^2 + b^2 - ab}{ab} = \frac{(a^2 + b^2 + ab)(a^2 + b^2 - ab)}{a^2b^2} = \frac{a^4 + b^4 + a^2b^2}{a^2b^2}.
\]

Or we may proceed thus:

\[
\left( \frac{a}{b} + \frac{b}{a} + 1 \right) \left( \frac{a}{b} + \frac{b}{a} - 1 \right) = \left( \frac{a}{b} + \frac{b}{a} \right)^2 - 1;
\]

\[
\left( \frac{a}{b} + \frac{b}{a} \right)^2 = \left( \frac{a}{b} \right)^2 + 2 \frac{a}{b} \frac{b}{a} + \left( \frac{b}{a} \right)^2 = \frac{a^3}{b^2} + 2 + \frac{b^2}{a^2};
\]
therefore
\[
\left(\frac{a}{b} + \frac{b}{a} + 1\right) \left(\frac{a}{b} + \frac{b}{a} - 1\right) = \frac{a^2}{b^2} \frac{b^2}{a^2} + 2 + \frac{b^2}{a^2} - 1 = \frac{a^3}{b^2} + \frac{b^2}{a^2} + 1.
\]

The two results agree, for
\[
\frac{a^2}{b^2} + \frac{b^2}{a^2} + 1 = \frac{a^4 + b^4 + a^2b^2}{a^2b^2}.
\]

Multiply together \(\frac{1 - a^2}{b + b^2}, \frac{1 - b^2}{a + a^2}, \text{ and } \frac{ab}{1 - a}.
\)

We might multiply together the first two factors, and then multiply the product separately by \(b\) and by \(\frac{ab}{1 - a}\), and add the results; but it is more convenient to reduce the mixed quantity \(b + \frac{ab}{1 - a}\) to a single fraction. Thus
\[
b + \frac{ab}{1 - a} = \frac{b(1 - a) + ab}{1 - a} = \frac{b}{1 - a}.
\]

Then
\[
\frac{1 - a^2}{b + b^2} \times \frac{1 - b^2}{a + a^2} \times \frac{b}{1 - a} = \frac{(1 - a^2)(1 - b^2)b}{b(1 + b) a(1 + a)(1 - a)} = \frac{1 - b}{a}.
\]

147. As we have already done in former Chapters, we must here give some results which the student must assume to be capable of explanation, and which he must use as rules in working examples which may be proposed. See Arts. 63 and 135.

Multiply \(\frac{a}{b}\) by \(-\frac{c}{d}\).
\[
a \times -\frac{c}{d} = \frac{a}{b} \times -\frac{c}{d} = -\frac{ac}{bd} = -\frac{ac}{bd}.
\]

Multiply \(-\frac{a}{b}\) by \(\frac{c}{d}\).
\[
-\frac{a}{b} \times \frac{c}{d} = -\frac{a}{b} \times \frac{c}{d} = \frac{-ac}{bd} = -\frac{ac}{bd}.
\]

Multiply \(-\frac{a}{b}\) by \(-\frac{c}{d}\).
\[
-\frac{a}{b} \times -\frac{c}{d} = -\frac{a}{b} \times -\frac{c}{d} = \frac{ac}{bd}.
\]
EXAMPLES. XVII.

Find the value of the following:

1. \( \frac{2a}{3b} \times \frac{6bc}{5a^2} \).
2. \( \frac{a^2}{bc} \times \frac{b^2}{ac} \times \frac{c^2}{ab} \).

3. \( \frac{a^2b}{x^2y} \times \frac{b^2c}{y^2z} \times \frac{c^2a}{z^2x} \).
4. \( \frac{x+1}{x-1} \times \frac{x+2}{x^2-1} \times \frac{x-1}{(x+2)^2} \).

5. \( \frac{x^a}{x+a} \times \left( \frac{x-a}{x} \right) \).
6. \( \left( b + \frac{a^2}{b} \right) \left( a - \frac{b^2}{a} \right) \).

7. \( \left( a + \frac{ab}{a-b} \right) \left( b - \frac{ab}{a+b} \right) \).

8. \( \frac{x^4(a-x)}{a^2+2ax} \times \frac{a(a+x)}{a^2-2ax} \).

9. \( \frac{x^6-y^6}{x^4+2x^2y^2+y^4} \times \frac{x^2+y^2}{x^2-xy+y^2} \times \frac{x+y}{x^3-y^3} \).

10. \( \frac{x^2-(a+b)x+ab}{x^2-(a+c)x+ac} \times \frac{x^2-a^2}{x^2-b^2} \).

11. \( \frac{x^2+xy}{x^2+y^2} \times \left( \frac{x}{x-y} - \frac{y}{x+y} \right) \).

12. \( \left( \frac{a}{bc} - \frac{b}{ac} - \frac{c}{ab} - \frac{2}{a} \right) \times \left( 1 - \frac{2c}{a+b+c} \right) \).

13. \( \left( \frac{x^2}{a^2} + \frac{a^2}{x^2} - \frac{x-a}{a} - \frac{x-1}{x} \right) \times \left( \frac{x-a}{a} \right) \).

14. \( \left( \frac{x-a+y-b}{a-x+y-b} \right) \times \left( \frac{x-a-y+b}{a-x-y+b} \right) \).

15. \( \frac{x^2-2x+1}{x^2-5x+6} \times \frac{x^2-4x+4}{x^2-4x+3} \times \frac{x^2-6x+9}{x^2-3x+2} \).
XVIII. Division of Fractions.

148. Rule for dividing one fraction by another. Invert the divisor and proceed as in Multiplication.

149. The following is the usual demonstration of the Rule. Suppose we have to divide \( \frac{a}{b} \) by \( \frac{c}{d} \); put \( \frac{a}{b} = x \), and \( \frac{c}{d} = y \); therefore

\[
a = bx, \text{ and } c = dy;
\]

therefore

\[
ad = bd.x, \text{ and } bc = bdy;
\]

therefore

\[
\frac{ad}{bc} = \frac{bd.x}{bdy} = \frac{x}{y};
\]

But

\[
\frac{x}{y} = x \div y = \frac{a}{b} \div \frac{c}{d};
\]

therefore

\[
\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}.
\]

150. We shall now give some examples.

Divide \( \frac{a}{b} \) by \( \frac{c}{d} \).

\[
a = \frac{a}{1}; \quad \frac{a}{b} \div \frac{c}{d} = \frac{a}{1} \times \frac{d}{c} = \frac{ad}{bc};
\]

Divide \( \frac{3a}{4b} \) by \( \frac{9a}{8c} \).

\[
\frac{3a}{4b} \div \frac{9a}{8c} = \frac{3a}{4b} \times \frac{8c}{9a} = \frac{2c \times 12a}{3b \times 12a} = \frac{2c}{3b}.
\]

Divide \( \frac{ab - b^2}{(a+b)^2} \) by \( \frac{b^2}{a^2 - b^2} \).

\[
\frac{ab - b^2}{(a+b)^2} \div \frac{b^2}{a^2 - b^2} = \frac{ab - b^2}{(a+b)^2} \times \frac{a^2 - b^2}{b^2} = \frac{b(a-b)(a+b)(a+b)}{b^2(a+b)^2} = \frac{(a-b)^2}{b(a+b)}.
\]
151. Complex fractional expressions may be simplified by the aid of some or all of the rules respecting fractions which have now been given. The following are examples.

Simplify \( \frac{a+b}{a-b} + \frac{a-b}{a+b} \div \left( \frac{a+b}{a-b} - \frac{a-b}{a+b} \right) \).

\[
\frac{a+b}{a-b} + \frac{a-b}{a+b} = \frac{(a+b)^2 + (a-b)^2}{(a-b)(a+b)} = \frac{2a^2 + 2b^2}{a^2 - b^2},
\]

\[
\frac{a+b}{a-b} - \frac{a-b}{a+b} = \frac{(a+b)^2 - (a-b)^2}{(a-b)(a+b)} = \frac{4ab}{a^2 - b^2},
\]

\[
\frac{2a^2 + 2b^2}{a^2 - b^2} \div \frac{4ab}{a^2 - b^2} \times \frac{a^2 - b^2}{4ab} = \frac{a^2 + b^2}{2ab}.
\]

In this example the factors \( a-b \) and \( a+b \) are multiplied together, and the result \( a^2 - b^2 \) is used instead of \( (a+b)(a-b) \); in general however the student will find it advisable not to multiply the factors together in the course of the operation, because an opportunity may occur of striking out a common factor from the numerator and denominator of his result.

Simplify \( \frac{1}{a + \frac{1}{\frac{a+1}{3-a}}} \).

\[
1 + \frac{a+1}{3-a} = \frac{3-a + a+1}{3-a} = \frac{3-a + a+1}{3-a} = \frac{4}{3-a},
\]

\[
1 \div \frac{4}{3-a} = \frac{1}{4} \times \frac{3-a}{4} = \frac{3-a}{4},
\]

\[
a + \frac{3-a}{4} = \frac{4a}{4} + \frac{3-a}{4} = \frac{3+3a}{4},
\]

\[
1 \div \frac{3+3a}{4} = \frac{1}{4} \times \frac{4}{3+3a} = \frac{4}{3+3a}.
\]
Find the value of \( \left( \frac{2x-a}{b} \right)^2 - \frac{a-x}{b} \) when \( x = \frac{ab}{a+b} \).

\[
2x-a = \frac{2ab}{a+b} - \frac{a}{1} = \frac{2ab - a(a+b)}{a+b} = \frac{ab-a^2}{a+b};
\]

\[
2x-b = \frac{2ab}{a+b} - \frac{b}{1} = \frac{2ab-b(a+b)}{a+b} = \frac{ab-b^2}{a+b}.
\]

Therefore
\[
\frac{2x-a}{2x-b} = \frac{ab-a^2}{a+b} \div \frac{ab-b^2}{a+b} = \frac{ab-a^2}{ab-b^2} \times \frac{a+b}{a+b} = \frac{a(b-a)}{b(a-b)} = -1;
\]

therefore
\[
\left( \frac{2x-a}{2x-b} \right)^2 = \left( \frac{-a}{b} \right)^2 = \frac{a^2}{b^2}.
\]

Again,
\[
a-x = \frac{a}{1} - \frac{ab}{a+b} = \frac{a(a+b)-ab}{a+b} = \frac{a^2}{a+b};
\]

\[
b-x = \frac{b}{1} - \frac{ab}{a+b} = \frac{b(a+b)-ab}{a+b} = \frac{b^2}{a+b}.
\]

Therefore
\[
\frac{a-x}{b-x} = \frac{a^2}{a+b} \div \frac{b^2}{a+b} = \frac{a^2}{a+b} \times \frac{a+b}{b^2} = \frac{a^2}{b^2} = \frac{a^2}{b^2}.
\]

Therefore
\[
\left( \frac{2x-a}{2x-b} \right)^2 - \frac{a-x}{b-x} = \frac{a^2}{b^2} - \frac{a^2}{b^2} = 0.
\]

152. The results given in Art. 147 must be given again here in connexion with Division of Fractions.

Since \( \frac{a}{b} \times -\frac{c}{d} = -\frac{ac}{bd} \), and \( -\frac{a}{b} \times \frac{c}{d} = -\frac{ac}{bd} \);

we have \( -\frac{ac}{bd} \div -\frac{c}{d} = \frac{a}{b} \), and \( -\frac{ac}{bd} \div \frac{c}{d} = -\frac{a}{b} \).

Also since \( -\frac{a}{b} \times -\frac{c}{d} = \frac{ac}{bd} \), we have
\[
\frac{ac}{bd} \div -\frac{c}{d} = -\frac{a}{b}.
\]
EXAMPLES. XVIII.

Divide

1. \[ \frac{4a^2b}{5x^2y} \text{ by } \frac{2ab^2}{15xy^2} \]

2. \[ \frac{3a^2b^2c^4}{4x^2y^3z^4} \text{ by } \frac{4a^3b^3c^2}{3x^4y^3z^2} \]

3. \[ \frac{1}{x^2 - y^2} \text{ by } \frac{1}{x - y} \]

4. \[ \frac{6(ab - l^2)}{a(a + l)^2} \text{ by } \frac{2b^2}{a(a^2 - b^2)} \]

5. \[ \frac{a^2 - 4x^2}{a^2 + 4ax} \text{ by } \frac{a^2 - 2ax}{ax + 4x^2} \]

6. \[ \frac{8x^3}{x^3 - y^3} \text{ by } \frac{4x^2}{x^2 + xy + y^2} \]

7. \[ \frac{a^3 + 3a^2x + 3ax^2 + x^3}{x^3 + y^3} \text{ by } \frac{(a + x)^2}{x^2 - xy + y^2} \]

8. \[ \frac{x^2 + (a + c)x + ac}{x^2 + (b + c)x + bc} \text{ by } \frac{x^2 - a^2}{x^2 - b^2} \]

9. \[ \frac{a^2 + b^2 + 2ab - c^2}{c^2 - a^2 - b^2 + 2ab} \text{ by } \frac{a + b + c}{b + c - a} \]

10. \[ \frac{x^2 + xy + y^2}{x^3 + y^3} \text{ by } \frac{x}{x^2 - xy + y^2} \]

11. \[ \frac{x^2 - 3x + 2}{x^2 - 6x + 9} \text{ by } \frac{x^2 - 5x + 6}{x^2 - 2x + 1} \]

12. \[ \left(1 + \frac{x}{y}\right) \left(1 - \frac{x}{y}\right) \text{ by } \frac{y}{x^2 + y^2} \]

13. \[ 5x^2 - \frac{1}{5} \text{ by } x + \frac{1}{5} \]

14. \[ a^3 - \frac{1}{a^3} \text{ by } a - \frac{1}{a} \]

15. \[ \frac{x^4}{a^4} - \frac{a^4}{x^4} \text{ by } \frac{x}{a} - \frac{a}{x} \]
16. \( \frac{x^2}{a} - 8a + \frac{12\alpha^3}{x^2} \) by \( x - \frac{2\alpha^3}{x} \).

17. \( \frac{x^2}{y^2} - \frac{1}{x} \) by \( \frac{x}{y^2} + \frac{1}{y} + \frac{1}{x} \).

18. \( \frac{x^3}{a^3} + 1 + \frac{\alpha^2}{x^3} \) by \( \frac{x}{a} - 1 + \frac{\alpha}{x} \).

19. \( 1 + \left( \frac{a-x}{a+x} \right)^2 \) by \( 1 - \left( \frac{a-x}{a+x} \right)^3 \).

20. \( \frac{x^3}{a^3} + \frac{\alpha^3}{x^3} - 3 \left( \frac{x^2}{a^2} - \frac{\alpha^2}{x^2} \right) + \frac{x + \alpha}{a} \) by \( \frac{x + \alpha}{a} \).

Simplify the following expressions:

21. \( \frac{3x + x - 1}{2} + \frac{3}{3} \)

22. \( x - 1 + \frac{6}{x - 6} \)

23. \( \frac{3}{x + 1} - \frac{2x - 1}{x^2 + x - 2} \)

24. \( \frac{x - \alpha}{x - \frac{(x - b)(x - c)}{x + \alpha}} \)

25. \( 1 \frac{1}{1 + \frac{1}{x}} \)

26. \( 1 + \frac{x}{1 + x + \frac{2x^2}{1 - x}} \)

27. \( \frac{1}{1 - \frac{1}{1 + \frac{1}{x}}} \)

28. \( \frac{1}{1 + \frac{x}{1 + x + \frac{2x^2}{1 - x}}} \)

29. \( \left( \frac{x}{x - y} - \frac{y}{x + y} \right) \div \left( \frac{x^3}{x^2 + y^2} + \frac{y^2}{x^2 - y^2} \right) \)

30. \( \left( \frac{2x}{x + y} + \frac{y}{x - y} - \frac{y^2}{x^2 - y^2} \right) \div \left( \frac{1}{x + y} + \frac{x}{x^2 - y^2} \right) \).
EXAMPLES. XVIII.

31. \[ \frac{x + \frac{1}{y}}{x + \frac{1}{y + \frac{1}{z}}} - \frac{1}{y(xyz + x + z)}. \]

32. \[ \left( \frac{a-b}{a+b} + \frac{a+b}{a-b} \right) \div \left( \frac{a^2-b^2}{a^2+b^2} + \frac{a^2+b^2}{a^2-b^2} \right). \]

Find the values of the following expressions:

33. \[ \frac{a-x}{b-x} \text{ when } x = \frac{ab}{a+b}. \]

34. \[ \frac{x-a}{b} - \frac{x-b}{a} \text{ when } x = \frac{a^2}{a-b}. \]

35. \[ \frac{x}{a} + \frac{x}{b-a} - \frac{a}{a+b} \text{ when } x = \frac{a^2(b-a)}{b(b+a)}. \]

36. \[ \frac{a^2x + b^2y}{x+y} \text{ when } a = \frac{2}{3} \text{ and } b = \frac{2}{3}. \]

37. \[ \frac{x}{x+y} + \frac{y}{x-y} - \frac{y^2}{x^2-y^2} \text{ when } y = \frac{3x}{4}. \]

38. \[ \frac{x+2a}{2b-x} + \frac{x-2a}{2b+x} - \frac{4ab}{4b^2-x^2} \text{ when } x = \frac{ab}{a+b}. \]

39. \[ \left( \frac{x-a}{x-b} \right)^3 - \frac{x-2a+b}{x+a-2b} \text{ when } x = \frac{a+b}{2}. \]

40. \[ \frac{x+y-1}{x-y+1} \text{ when } x = \frac{a+1}{a} \text{ and } y = \frac{ab+a}{u}. \]
XIX. Simple Equations.

153. When two algebraical expressions are connected by the sign of equality the whole is called an equation. The expressions thus connected are called sides of the equation or members of the equation. The expression to the left of the sign of equality is called the first side, and the expression to the right is called the second side.

154. An identical equation is one in which the two sides are equal whatever numbers the letters represent; for example, the following are identical equations,

\[(x+a)(x-a)=x^2-a^2,\]
\[(x+a)^2=x^2+2xa+a^2;\]
\[(x+a)(x^2-xa+a^2)=x^3+a^3;\]

that is, these algebraical statements are true whatever numbers \(x\) and \(a\) may represent. The student will see that up to the present point he has been almost exclusively occupied with results of this kind, that is, with identical equations.

An identical equation is called briefly an identity.

155. An equation of condition is one which is not true whatever numbers the letters represent, but only when the letters represent some particular number or numbers. For example, \(x+1=7\) cannot be true unless \(x=6\). An equation of condition is called briefly an equation.

156. A letter to which a particular value or values must be given in order that the statement contained in an equation may be true, is called an unknown quantity. Such particular value of the unknown quantity is said to satisfy the equation, and is called a root of the equation. To solve an equation is to find the root or roots.

157. An equation involving one unknown quantity is said to be of as many dimensions as the index of the highest power of the unknown quantity. Thus, if \(x\) denote
the unknown quantity, the equation is said to be of one dimension when \( x \) occurs only in the first power; such an equation is also called a simple equation, or an equation of the first degree. If \( x^2 \) occurs, and no higher power of \( x \), the equation is said to be of two dimensions; such an equation is also called a quadratic equation, or an equation of the second degree. If \( x^3 \) occurs, and no higher power of \( x \), the equation is said to be of three dimensions; such an equation is also called a cubic equation, or an equation of the third degree. And so on.

It must be observed that these definitions suppose both members of the equation to be integral expressions so far as relates to \( x \).

158. In the present Chapter we shall shew how to solve simple equations. We have first to indicate some operations which may be performed on an equation without destroying the equality which it expresses.

159. If every term on each side of an equation be multiplied by the same number the results are equal.

The truth of this statement follows from the obvious principle, that if equals be multiplied by the same number the results are equal; and the use of this statement will be seen immediately.

Likewise if every term on each side of an equation be divided by the same number the results are equal.

160. The principal use of Art. 159 is to clear an equation of fractions; this is effected by multiplying every term by the product of all the denominators of the fractions, or, if we please, by the least common multiple of those denominators. Suppose, for example, that

\[
\frac{x}{3} + \frac{x}{4} + \frac{x}{6} = 9.
\]

Multiply every term by \( 3 \times 4 \times 6 \); thus

\[
4 \times 6 \times x + 3 \times 6 \times x + 3 \times 4 \times x = 3 \times 4 \times 6 \times 9,
\]

that is, \( 24x + 18x + 12x = 648 \);

divide every term by 6; thus

\[
4x + 3x + 2x = 108.
\]
Instead of multiplying every term by \(3 \times 4 \times 6\), we may multiply every term by 12, which is the \(\text{l.c.m.}\) of the denominators 3, 4, and 6; we should then obtain at once

\[
4x + 3x + 2x = 108;
\]

that is,

\[
9x = 108;
\]

divide both sides by 9; therefore

\[
x = \frac{108}{9} - 12.
\]

Thus 12 is the root of the proposed equation. We may verify this by putting 12 for \(x\) in the original equation. The first side becomes

\[
\frac{12}{3} + \frac{12}{4} + \frac{12}{6}, \text{ that is } 4 + 3 + 2, \text{ that is } 9;
\]

which agrees with the second side.

161. \textit{Any term may be transposed from one side of an equation to the other side by changing its sign.}

Suppose, for example, that \(x - a = b - y\).

Add \(a\) to each side; then

\[
x - a + a = b - y + a,
\]

that is

\[
x = b - y + a.
\]

Subtract \(b\) from each side; thus

\[
x - b - b + a - y - b = a - y.
\]

Here we see that \(-a\) has been removed from one side of the equation, and appears as \(+a\) on the other side; and \(+b\) has been removed from one side and appears as \(-b\) on the other side.

162. \textit{If the sign of every term of an equation be changed the equality still holds.}

This follows from Art. 161, by transposing every term. Thus suppose, for example, that \(x - a = b - y\).
By transposition \( y - b = a - x, \)
that is, \( a - x = y - b; \)
and this result is what we shall obtain if we change the sign of every term in the original equation.

163. We can now give a Rule for the solution of any simple equation with one unknown quantity. Clear the equation of fractions, if necessary; transpose all the terms which involve the unknown quantity to one side of the equation, and the known quantities to the other side; divide both sides by the coefficient, or the sum of the coefficients, of the unknown quantity, and the root required is obtained.

164. We shall now give some examples.

Solve \( 7x + 25 = 35 + 5x. \)

Here there are no fractions; by transposing we have
\[ 7x - 5x = 35 - 25; \]
that is, \( 2x - 10; \)
divide by 2; therefore \( x = \frac{10}{2} = 5. \)

We may verify this result by putting 5 for \( x \) in the original equation; then each side is equal to 60.

165. Solve \( 4(3x - 2) - 2(4x - 3) - 3(4 - x) = 0. \)

Perform the multiplications indicated; thus
\[ 12x - 8 - (8x - 6) - (12 - 3x) = 0. \]

Remove the brackets; thus
\[ 12x - 8 - 8x + 6 - 12 + 3x = 0; \]
collect the terms, \( 7x - 14 = 0; \)
transpose, \( 7x = 14; \)
divide by 7, \( x = \frac{14}{7} = 2. \)

The student will find it a useful exercise to verify the correctness of his solutions. Thus in the above example,
if we put 2 for \( x \) in the original equation we shall obtain \( 16 - 10 - 6 \), that is 0, as it should be.

166. Solve \( x - 2 - (2x - 3) = \frac{3x + 1}{2} \).

Remove the brackets; thus

\[
x - 2 - 2x + 3 = \frac{3x + 1}{2}
\]

that is,

\[
1 - x = \frac{3x + 1}{2};
\]

multiply by 2,

\[
2 - 2x - 3x + 1,
\]

transpose,

\[
2 - 1 = 2x + 3x;
\]

that is,

\[
1 = 5x, \text{ or } 5x = 1;
\]

therefore

\[
x = \frac{1}{5}.
\]

167. Solve \( \frac{5x + 4}{2} - \frac{7x + 5}{10} = \frac{53}{5} - \frac{x - 1}{2} \).

\[
5 \frac{28}{5} = \frac{28}{5}; \text{ the L.C.M. of the denominators is 10; multiply by 10;}
\]

thus

\[
5(5x + 4) - (7x + 5) = 28 \times 2 - 5(x - 1);
\]

that is,

\[
25x + 20 - 7x - 5 = 56 - 5x + 5;
\]

transpose,

\[
25x - 7x + 5x = 56 + 5 - 20 + 5;
\]

that is,

\[
23x = 46;
\]

therefore

\[
x = \frac{46}{23} = 2.
\]

The beginner is recommended to put down all the work at full, as in this example, in order to ensure accuracy. Mistakes with respect to the signs are often made in clearing an equation of fractions. In the above equation the fraction \( \frac{7x + 5}{10} \) has to be multiplied by 10, and it is advisable to put the result first in the form \( -(7x + 5) \), and afterwards in the form \( -7x - 5 \), in order to secure attention to the signs.
168. Solve \( \frac{1}{3}(5x+3) - \frac{1}{7}(16-5x) = 37-4x \).

By Art. 146 this is the same as
\[
\frac{5x+3}{3} - \frac{16-5x}{7} = 37-4x.
\]
Multiply by 21; thus \( 7(5x+3) - 3(16-5x) = 21(37-4x) \),
that is, \( 35x+21-48+15x = 777-84x \);
transpose, \( 35x+15x+84x = 777-21+48 \);
that is, \( 134x = 804 \);
therefore
\[
x = \frac{804}{134} = 6.
\]

169. Solve \( \frac{6x+15}{11} - \frac{8x-10}{7} = \frac{4x-7}{5} \).

Multiply by the product of 11, 7, and 5; thus
\( 35(6x+15)-55(8x-10)=77(4x-7) \),
that is, \( 210x+525-440x+550=308x-539 \);
transpose, \( 210x-440x-308x=-539-525-550 \);
change the signs, \( 440x+308x-210x=539+525+550 \);
that is, \( 538x=1614 \);
therefore
\[
x = \frac{1614}{538} = 3.
\]
13. $\frac{x}{3} - \frac{x}{4} + \frac{1}{6} = \frac{x}{8} + \frac{1}{12}$.

14. $\frac{4x}{3} + 24 = 2x + 6$.

15. $\frac{x}{5} + \frac{x}{3} = x - 7$.

16. $36 - \frac{4x}{9} = 8$.

17. $\frac{2x}{3} + 4 = \frac{7x}{12} + 9$.

18. $\frac{3x}{4} + 5 = \frac{5x}{6} + 2$.

19. $56 - \frac{3x}{4} = 48 - \frac{5x}{8}$.

20. $\frac{x}{6} - 4 = 24 - \frac{x}{8}$.

21. $\frac{2x}{3} + 12 = \frac{4x}{5} + 6$.

22. $\frac{2x}{3} = \frac{176 - 4x}{5}$.

23. $\frac{7x}{8} - 5 = \frac{9x}{10} - 8$.

24. $\frac{5x}{9} - 8 = 74 - \frac{7x}{12}$.

25. $\frac{3x}{4} + \frac{180 - 5x}{6} = 29$.

26. $\frac{x}{2} + \frac{x + 1}{7} = x - 2$.

27. $4(x - 3) - 7(x - 4) = 6 - x$.

28. $\frac{x}{3} - \frac{1 - x}{4} + \frac{1}{4} - \frac{x}{5} - \frac{1 - x}{6} + \frac{1}{6}$.

29. $1 + \frac{x}{2} - \frac{2x}{3} = \frac{3x}{4} - 4\frac{1}{2}$.

30. $2x - \frac{19 - 2x}{2} = \frac{2x - 11}{2}$.

31. $\frac{x + 1}{3} - \frac{3x - 1}{5} = x - 2$.

32. $x + \frac{3x - 9}{5} = 4 - \frac{5x - 12}{3}$.

33. $\frac{10x + 3}{3} - \frac{6x - 7}{2} = 10x - 10$.

34. $\frac{5x - 7}{2} - \frac{2x + 7}{3} = 3x - 14$.

35. $x - 1 - \frac{x - 2}{2} + \frac{x - 3}{3} = 0$. 
36. \( \frac{x + 3}{2} + \frac{x + 4}{3} + \frac{x + 5}{4} = 16. \)

37. \( \frac{7x + 9}{4} = 7 + x - \frac{2x - 1}{9}. \)

38. \( \frac{3x - 4}{2} - \frac{6x - 5}{8} = \frac{3x - 1}{16}. \)

39. \( \frac{2x - 5}{3} - \frac{5x - 3}{4} + 2\frac{2}{3} = 0. \)

40. \( \frac{x - 3}{4} - \frac{x - 5}{6} + \frac{x - 1}{9}. \)

41. \( \frac{x - 1}{2} - \frac{x - 3}{4} + \frac{x - 5}{6} = 4. \)

42. \( \frac{x}{3} - \frac{x}{4} + \frac{x - 2}{5} = 3. \)

43. \( \frac{7x + 5}{6} - \frac{5x + 6}{4} = \frac{8 - 5x}{12}. \)

44. \( \frac{x + 4}{3} - \frac{x - 4}{5} = 2 + \frac{3x - 1}{15}. \)

45. \( \frac{x - 1}{2} + \frac{2x + 7}{3} - \frac{x + 2}{9} = 9. \)

46. \( \frac{x - 1}{2} - \frac{x - 2}{3} + \frac{x - 3}{4} = \frac{2}{3}. \)

47. \( \frac{2x - 5}{6} + \frac{6x + 3}{4} = 5x - 17\frac{1}{2}. \)

48. \( \frac{x}{4} - \frac{5x + 8}{6} = \frac{2x - 9}{3}. \)

49. \( \frac{3x + 5}{7} - \frac{2x + 7}{3} + 10 - \frac{3x}{5} = 0. \)

50. \( \frac{1}{7} (3x - 4) + \frac{1}{3} (5x + 3) = 43 - 5x. \)

51. \( \frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \frac{x}{5} = 7\frac{8}{3}. \)

52. \( \frac{x}{2} - \frac{x}{3} + \frac{x}{4} - \frac{x}{3} = \frac{x + 3}{2} - \frac{2}{3}. \)
53. \( \frac{5-3x}{4} + \frac{5x}{3} = \frac{3}{2} - \frac{3-5x}{3} \).

54. \( \frac{1}{2} (27-2x) = \frac{9}{2} - \frac{1}{10} (7x-54) \).

55. \( 5x-[8x-3\{16-6x-(4-5x)\}] = 6 \).

56. \( \frac{1-2x}{3} - \frac{4-5x}{6} + \frac{13}{42} = 0 \).

57. \( \frac{x+1}{3} - \frac{x-1}{4} + 4x = 12 + \frac{2x-1}{6} \).

58. \( \frac{4x-7}{8} + 2\frac{2}{3} + \frac{7-4x}{4} = x + \frac{13}{24} \).

59. \( \frac{5x-1}{7} + \frac{9x-5}{11} = \frac{9x-7}{5} \).

60. \( \frac{x+3}{2} - \frac{x-2}{3} = \frac{3x-5}{12} + \frac{1}{4} \).

61. \( \frac{1}{6} (8-x) + x - 1\frac{2}{3} = \frac{1}{2} (x+6) - \frac{x}{3} \).

62. \( \frac{3x-1}{5} - \frac{13-x}{2} = \frac{7x}{3} - \frac{11}{6} (x+3) \).

63. \( \frac{2x-1}{5} + \frac{6x-4}{7} = \frac{7x+12}{11} \).

64. \( \frac{7x-4}{8} + 2\frac{2}{3} + \frac{4-7x}{4} = x - \frac{7}{12} \).

65. \( \frac{2-x}{3} + \frac{3-x}{4} + \frac{4-x}{5} + \frac{5-x}{6} + \frac{3}{4} = 0 \).

66. \( \frac{5x-3}{7} - \frac{9-x}{3} = \frac{5x}{2} + \frac{19}{6} (x-4) \).
XX. Simple Equations, continued.

170. We shall now give some examples of the solution of simple equations, which are a little more difficult than those in the preceding Chapter. The student will see that it is sometimes advantageous to clear of fractions partially, and then to effect some reductions, before we remove the remaining fractions.

171. Solve \( \frac{x + 6}{11} - \frac{2x - 18}{3} + \frac{2x + 3}{4} = 5\frac{1}{2} + \frac{3x + 4}{12} \).

Here we may conveniently multiply by 12; thus,

\[
\frac{12(x + 6)}{11} - 4(2x - 18) + 3(2x + 3) = \frac{16}{3} \times 12 + 3x + 4,
\]

that is,

\[
\frac{12(x + 6)}{11} - 8x + 72 + 6x + 9 = 64 + 3x + 4.
\]

By transposition and reduction we obtain

\[
\frac{12(x + 6)}{11} = 5x - 13.
\]

Multiply by 11; thus \( 12(x + 6) = 11(5x - 13) \),

that is, \( 12x + 72 = 55x - 143 \);

by transposition, \( 72 + 143 = 55x - 12x \),

that is, \( 43x = 215 \);

therefore \( x = \frac{215}{43} = 5 \).

172. Solve \( \frac{6x - 13\frac{1}{3}}{15 - 2x} + 2x + \frac{16x - 15}{24} = 6\frac{5}{12} - \frac{205}{3} - 8x \).

Here we may conveniently multiply by 24; thus

\[
24 \left( \frac{6x - \frac{40}{3}}{15 - 2x} \right) + 48x + 16x - 15 = 24 \times \frac{77}{12} - 8 \left( \frac{165}{8} - 8x \right);
\]
that is,
\[
\frac{144x - 320}{15 - 2x} + 48x + 16x - 15 = 154 - 165 + 64x.
\]

By transposition and reduction
\[
\frac{144x - 320}{15 - 2x} = 4;
\]
multiply by \(15 - 2x\); thus
\[
144x - 320 = 4(15 - 2x) = 60 - 8x;
\]
therefore
\[
144x + 8x = 320 + 60,
\]
that is,
\[
152x = 380;
\]
therefore
\[
x = \frac{380}{152} = 2\frac{45}{2} = 2\frac{1}{2}.
\]

173. Solve \(\frac{x - 5}{x - 7} = \frac{x + 3}{x + 9}\).

Multiply by \((x - 7)(x + 9)\); thus
\[
(x + 9)(x - 5) = (x - 7)(x + 3),
\]
that is,
\[
x^2 + 4x - 45 = x^2 - 4x - 21;
\]
subtract \(x^2\) from each side of the equation, thus
\[
4x - 45 = -4x - 21;
\]
transpose,
\[
4x + 4x = 45 - 21,
\]
that is,
\[
8x = 24;
\]
therefore
\[
x = \frac{24}{8} = 3.
\]

It will be seen that in this example \(x^2\) is found on both sides of the equation, after we have cleared of fractions; accordingly it can be removed by subtraction, and so the equation remains a simple equation.
174. Solve \[ \frac{2x+3}{x+1} - \frac{4x+5}{4x+4} = \frac{3x+3}{3x+1}. \]

Here it is convenient to multiply by 4(x+1), that is by 4(x+1);

thus \[ 4(2x+3) = 4x+5 + \frac{4(x+1)3(x+1)}{3x+1}; \]

therefore \[ 8x+12-4x-5 = \frac{12(x+1)^2}{3x+1}; \]

that is, \[ 4x+7 = \frac{12(x+1)^2}{3x+1}. \]

Multiply by 3x+1; thus \( (3x+1)(4x+7) = 12(x+1)^2; \)

that is, \[ 12x^2+25x+7 = 12x^2+24x+12. \]

Subtract 12x^2 from each side, and transpose; thus \[ 25x-24x = 12-7, \]

that is, \[ x = 5. \]

175. Solve \[ \frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{x-4}{x-5} - \frac{x-5}{x-6}. \]

We have \[ \frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{(x-1)(x-3)-(x-2)^2}{(x-2)(x-3)} \]

\[ = \frac{x^2-4x+3-(x^2-4x+4)}{(x-2)(x-3)} = -1 \]

And \[ \frac{x-4}{x-5} - \frac{x-5}{x-6} = \frac{(x-4)(x-6)-(x-5)^2}{(x-5)(x-6)} \]

\[ = \frac{x^2-10x+24-(x^2-10x+25)}{(x-5)(x-6)} = -1 \]

Thus the proposed equation becomes

\[ \frac{1}{(x-2)(x-3)} = \frac{1}{(x-5)(x-6)}. \]
Change the signs; thus \( \frac{1}{(x-2)(x-3)} = \frac{1}{(x-5)(x-6)} \).

Clear of fractions; thus \((x-5)(x-6) = (x-2)(x-3)\);

that is, \(x^2 - 11x + 30 = x^2 - 5x + 6\);

therefore \(-11x + 5x = 6 - 30\);

that is, \(-6x = -24\);

therefore \(6x = 24\);

therefore \(x = 4\).

176. Solve \(5x + \frac{45x - 75}{6} = \frac{12}{2} - \frac{3x - 6}{9}\).

To ensure accuracy it is advisable to express all the decimals as common fractions; thus

\[
\frac{5x}{10} + \frac{10}{6} \left( \frac{45x}{100} - \frac{75}{100} \right) = \frac{10}{2} \times \frac{12}{10} - \frac{10}{9} \left( \frac{3x}{10} - \frac{6}{10} \right).
\]

Simplifying,

\[
\frac{x}{2} + \frac{5}{3} \left( \frac{9x}{20} - \frac{3}{4} \right) = 6 - \left( \frac{x}{3} - \frac{2}{3} \right);
\]

that is, \(\frac{x}{2} + \frac{3x}{4} - \frac{5}{4} = 6 - \frac{x}{3} + \frac{2}{3}\).

Multiply by 12,

\(6x + 9x - 15 = 72 - 4x + 8\);

transpose,

\(19x = 72 + 8 + 15 = 95\);

therefore \(x = \frac{95}{19} = 5\).

177. Equations may be proposed in which \textit{letters} are used to represent known quantities; we shall continue to represent the unknown quantity by \(x\), and any other letter will be supposed to represent a known quantity. We will solve three such equations.
178. Solve \( \frac{x}{a} + \frac{x}{b} = c \).

Multiply by \( ab \); thus \( bx + ax = abc \);
that is, \( (a + b)x = abc \);
divide by \( a + b \); thus \( x = \frac{abc}{a + b} \).

179. Solve \( (a + x)(b + x) = a(b + c) + \frac{a^2c}{b} + x^2 \).

Here \( ab + ax + bx + x^2 = ab + ac + \frac{a^2c}{b} + x^2 \);
therefore \( ax + bx = ac + \frac{a^2c}{b} \);
that is, \( (a + b)x = ac \left(1 + \frac{a}{b}\right) = \frac{ac(a + b)}{b} \);
divide by \( a + b \); thus \( x = \frac{ac}{b} \).

180. Solve \( \frac{x - a}{x - b} = \frac{(2x - a)^2}{(2x - b)^2} \).

Clear of fractions; thus
\( (x - a)(2x - b)^2 = (x - b)(2x - a)^2 \);
that is, \( (x - a)(4x^2 - 4xb + b^2) = (x - b)(4x^2 - 4xa + a^2) \).

Multiplying out we obtain
\[ 4x^3 - 4x^2(a + b) + x(4ab + b^2) - ab^2 = 4x^3 - 4x^2(a + b) + x(4ab + a^2) - a^2b \];
therefore \( xb^2 - ab^2 = xa^2 - a^2b \) ;
therefore \( x(a^2 - b^2) = a^2b - ab^2 = ab(a - b) \);
therefore \( x = \frac{ab(a - b)}{a^2 - b^2} = \frac{ab}{a + b} \).
181. Although the following equation does not strictly belong to the present Chapter we give it as there will be no difficulty in following the steps of the solution, and it will serve as a model for similar examples. The equation resembles those already solved, in the circumstance that we obtain only a single value solved, in the circumstance that we obtain only a single value of the unknown quantity.

Solve \( \sqrt{x} + \sqrt{x-16} = 8. \)

By transposition, \( \sqrt{x-16} = 8 - \sqrt{x} \); square both sides; thus \( x-16 = (8 - \sqrt{x})^2 = 64 - 16 \sqrt{x} + x; \) therefore \(-16 = 64 - 16 \sqrt{x};\)

transpose, \( 16 \sqrt{x} = 64 + 16 = 80; \)

therefore \( \sqrt{x} = 5; \)

therefore \( x = 25. \)

---

**Examples. XX.**

1. \( \frac{12}{x} + \frac{1}{12x} = \frac{29}{24}. \)

2. \( \frac{42}{x-2} = \frac{35}{x-3}. \)

3. \( \frac{128}{3x-4} = \frac{216}{5x-6}. \)

4. \( \frac{45}{2x+3} = \frac{57}{4x-5}. \)

5. \( \frac{3x-1}{2} - \frac{2x-5}{3} + \frac{x-3}{4} - \frac{x}{6} = x+1. \)

6. \( \frac{1}{2} \frac{x-3}{5} + \frac{3}{4} \frac{x-10}{2} + \frac{4-x}{4} = \frac{10-x}{6}. \)

7. \( \frac{5}{6} \left( \frac{x}{3} - \frac{1}{3} \right) + \frac{7}{6} \left( \frac{x}{5} - \frac{1}{7} \right) = 49. \)

8. \( x + \frac{5x-8}{3} = 6 - \frac{3x-8}{5}. \)

9. \( \frac{x-2}{4} + \frac{1}{3} = x - \frac{2x-1}{3}. \)
10. \(\frac{x + 1 - \frac{x^2 + 3}{x + 2}}{x - 2} = 2\).  
11. \(\frac{x - 1}{x - 2} = \frac{7x - 21}{7x - 26}\).

12. \(\frac{7x - 4}{x - 1} = \frac{7x - 26}{x - 3}\).  
13. \(\frac{x - 3}{2} + \frac{71}{7} = \frac{3x + 1}{2} + 1\frac{1}{4}\).

14. \(\frac{2x - 6}{3x - 8} = \frac{2x - 5}{3x - 7}\).

15. \(x - 3 - (3 - x)(x + 1) = x(x - 3) + 8\).

16. \(3 - x - 2(x - 1)(x + 2) = (x - 3)(5 - 2x)\).

17. \(\frac{7 + 9x}{4} - 1 + \frac{2 - x}{9} = 7x\).  
18. \((x + 7)(x + 1) = (x + 3)^2\).

19. \(\frac{1}{3}(2x - 10) - \frac{1}{11}(3x - 40) = 15 - \frac{1}{5}(57 - x)\).

20. \(\frac{6x + 8}{2x + 1} - \frac{2x + 3}{x + 12} = 1\).

21. \(\frac{x - 1}{4} - \frac{x - 5}{32} + \frac{15 - 2x}{40} = 9 - \frac{x}{2} - \frac{7}{8}\).

22. \(\frac{4x + 17}{x + 3} + \frac{3x - 10}{x - 4} = 7\).

23. \(\frac{x + 1}{7} + x(x - 2) = (x - 1)^2\).

24. \(\frac{x - 4}{3} + (x - 1)(x - 2) = x^2 - 2x - 4\).

25. \(\frac{3x^2 - 2x - 8}{5} = \frac{(7x - 2)(3x - 6)}{35}\).

26. \(\frac{x + 10}{3} - \frac{2}{5}(3x - 4) + \frac{(3x - 2)(2x - 3)}{6} = x^2 - \frac{2}{15}\).

27. \(\frac{3x - 1}{2x - 1} - \frac{4x - 2}{3x - 2} = \frac{1}{6}\).

28. \(\frac{2}{2x - 3} + \frac{1}{x - 2} - \frac{6}{3x + 2}\).
29. \[ \frac{x-4}{x-5} - \frac{x-5}{x-6} = \frac{x-7}{x-8} - \frac{x-8}{x-9}. \]

30. \[ \frac{x}{x-2} + \frac{x-9}{x-7} = \frac{x+1}{x-1} + \frac{x-8}{x-6}. \]

31. \[ \frac{3-2x}{1-2x} - \frac{2x-5}{2x-7} = 1 - \frac{4x^2-1}{7-16x+4x^2}. \]

32. \[ \frac{3+x}{3-x} - \frac{2+x}{2-x} - \frac{1+x}{1-x} = 1. \]

33. \[ \frac{x-5}{7} + \frac{x^2+6}{3} = \frac{x^2-2}{2} - \frac{x^2-x+1}{6} + 3. \]

34. \[ (x+1)(x+2)(x+3) = (x-1)(x-2)(x-3) + 3(4x-2)(x+1). \]

35. \[ (x-9)(x-7)(x-5)(x-1) = (x-2)(x-4)(x-6)(x-10). \]

36. \[ (8x-3)^2(x-1) = (4x-1)^2(4x-5). \]

37. \[ \frac{x^2-x+1}{x-1} + \frac{x^2+x+1}{x+1} = 2x. \]

38. \[ .5x-2 = .25x + .2x - 1. \]

39. \[ .5x + .6x - .8 = .75x + .25. \]

40. \[ .15x + \frac{.135x - .225}{.6} = \frac{.36}{.2} - \frac{.09x - .18}{.9}. \]

41. \[ a \frac{a-x}{b} - b \frac{b+x}{a} = x. \]

42. \[ a \frac{x-a}{b} + b \frac{x-b}{a} = x. \]

43. \[ \frac{x^2-a^2}{bx} - \frac{a-x}{b} = \frac{2x}{b} - \frac{a}{x}. \]

44. \[ x(x-a) + x(x-b) = 2(x-a)(x-b). \]

45. \[ (x-a)(x-b)(x+2a+2b) = (x+2a)(x+2b)(x-a-b). \]
46. \((x-a)(x-b) = (x-a-b)^2\).

47. \(\frac{a}{x-a} - \frac{b}{x-b} = \frac{a-b}{x-a}\).

48. \(\frac{a}{x+a} + \frac{b}{x+b} = \frac{a+b}{x+c}\).

49. \(\frac{1}{x-a} - \frac{1}{x-b} = \frac{a-b}{x^2-ab}\).

50. \(\frac{1}{x-a} - \frac{1}{x-a+c} = \frac{1}{x-b-c} - \frac{1}{x-b}\).

51. \(\frac{mx-a-b}{nx-c-d} = \frac{mx-a-c}{nx-b-d}\).

52. \((a-b)(x-c) - (b-c)(x-a) - (c-a)(x-b) = 0\).

53. \(\frac{x-a}{a-b} - \frac{x+a}{a+b} = \frac{2ax}{a^2-b^2}\).

54. \((a-x)(b-x) = (p+x)(q+x)\).

55. \(\frac{x-a}{x-a-1} - \frac{x-a-1}{x-a-2} = \frac{x-b}{x-b-1} - \frac{x-b-1}{x-b-2}\).

56. \((x+a)(2x+b+c)^2 = (x+b)(2x+a+c)^2\).

57. \((x+2a)(x-a)^2 = (x+2b)(x-b)^2\).

58. \((x-a)^3(x+a-2b) = (x-b)^3(x-2a+b)\).

59. \(\sqrt{4x} + \sqrt{4x-7} = 7\).

60. \(\sqrt{x+14} + \sqrt{x-14} = 14\).

61. \(\sqrt{x+11} + \sqrt{x-9} = 10\).

62. \(\sqrt{9x+4} + \sqrt{9x-1} = 3\).

63. \(\sqrt{x+4ab} = 2a - \sqrt{x}\).

64. \(\sqrt{x-a} + \sqrt{x-b} = \sqrt{a-b}\).
XXI. Problems.

182. We shall now apply the methods explained in the preceding two Chapters to the solution of some problems, and thus exhibit to the student specimens of the use of Algebra. In these problems certain quantities are given, and another, which has some assigned relations to these, has to be found; the quantity which has to be found is called the unknown quantity. The relations are usually expressed in ordinary language in the enunciation of the problem, and the method of solving the problem may be thus described in general terms: denote the unknown quantity by the letter \( x \), and express in algebraical language the relations which hold between the unknown quantity and the given quantities; an equation will thus be obtained from which the value of the unknown quantity may be found.

183. The sum of two numbers is 85, and their difference is 27: find the numbers.

Let \( x \) denote the less number; then, since the difference of the numbers is 27, the greater number will be denoted by \( x + 27 \); and since the sum of the numbers is 85 we have

\[
x + x + 27 = 85;
\]

that is,

\[
2x + 27 = 85;
\]

therefore

\[
2x = 85 - 27 = 58;
\]

therefore

\[
x = \frac{58}{2} = 29.
\]

Thus the less number is 29; and the greater number is 29 + 27, that is 56.

184. Divide £2. 10s. among \( A \), \( B \), and \( C \), so that \( B \) may have 5s. more than \( A \), and \( C \) may have as much as \( A \) and \( B \) together.

Let \( x \) denote the number of shillings in \( A \)'s share, then \( x + 5 \) will denote the number of shillings in \( B \)'s share, and \( 2x + 5 \) will denote the number of shillings in \( C \)'s share.
The whole number of shillings is 50; therefore
\[ x + x + 5 + 2x + 5 = 50; \]
that is,
\[ 4x + 10 = 50; \]
therefore
\[ 4x - 50 - 10 = 40; \]
therefore
\[ x = 10. \]

Thus A's share is 10 shillings, B's share is 15 shillings, and C's share is 25 shillings.

185. A certain sum of money was divided between A, B, and C; A and B together received £17. 15s.; A and C together received £15. 15s.; B and C together received £12. 10s.; find the sum received by each.

Let \( x \) denote the number of pounds which A received, then B received \( 17\frac{3}{4} - x \) pounds, because A and B together received \( 17\frac{3}{4} \) pounds; and C received \( 15\frac{3}{4} - x \) pounds, because A and C together received \( 15\frac{3}{4} \) pounds. Also B and C together received \( 12\frac{1}{2} \) pounds; therefore

\[ 12\frac{1}{2} = 17\frac{3}{4} - x + 15\frac{3}{4} - x; \]
that is,
\[ 12\frac{1}{2} = 33\frac{1}{2} - 2x; \]
therefore
\[ 2x = 33\frac{1}{2} - 12\frac{1}{2} = 21; \]
therefore
\[ x = \frac{21}{2} = 10\frac{1}{2}. \]

Thus A received £10. 10s., B received £7. 5s., and C received £5. 5s.

186. A grocer has some tea worth 2s. a lb., and some worth 3s. 6d. a lb.: how many lbs. must he take of each sort to produce 100lbs. of a mixture worth 2s. 6d. a lb.?

Let \( x \) denote the number of lbs. of the first sort; then \( 100 - x \) will denote the number of lbs. of the second sort. The value of the \( x \) lbs. is \( 2x \) shillings; and the value of the
100 - \(x\) lbs. is \(\frac{7}{2}(100-x)\) shillings. And the whole value is to be \(\frac{5}{2} \times 100\) shillings; therefore

\[\frac{5}{2} \times 100 = 2x + \frac{7}{2}(100-x)\];

multiply by 2, thus \(500 = 4x + 700 - 7x\);
therefore \(7x - 4x = 700 - 500\);
that is, \(3x = 200\);
therefore \(x = \frac{200}{3} = 66\frac{2}{3}\).

Thus there must be \(66\frac{2}{3}\) lbs. of the first sort, and \(33\frac{1}{3}\) lbs. of the second sort.

187. A line is 2 feet 4 inches long; it is required to divide it into two parts, such that one part may be three-fourths of the other part.

Let \(x\) denote the number of inches in the larger part; then \(\frac{3x}{4}\) will denote the number of inches in the other part.

The number of inches in the whole line is 28; therefore

\[x + \frac{3x}{4} = 28\];

therefore \(4x + 3x = 112\);
that is, \(7x = 112\);
therefore \(x = 16\).

Thus one part is 16 inches long, and the other part 12 inches long.

188. A person had £1000, part of which he lent at 4 per cent., and the rest at 5 per cent.; the whole annual interest received was £44: how much was lent at 4 per cent.?
PROBLEMS.

Let $x$ denote the number of pounds lent at 4 per cent.; then $1000 - x$ will denote the number of pounds lent at 5 per cent. The annual interest obtained from the former is $\frac{4x}{100}$, and from the latter $\frac{5(1000 - x)}{100}$; therefore $\frac{4x}{100} + \frac{5(1000 - x)}{100}$; therefore $4400 = 4x + 5(1000 - x)$; that is, $4400 = 4x + 5000 - 5x$; therefore $x = 5000 - 4400 = 600$.

Thus £600 was lent at 4 per cent.

189. The student will find that the only difficulty in solving a problem consists in translating statements expressed in ordinary language into Algebraical language; and he should not be discouraged, if he is sometimes a little perplexed, since nothing but practice can give him readiness and certainty in this process. One remark may be made, which is very important for beginners; what is called the unknown quantity is really an unknown number, and this should be distinctly noticed in forming the equation. Thus, for example, in the second problem which we have solved, we begin by saying, let $x$ denote the number of shillings in $A$'s share; beginners often say, let $x = A$'s money, which is not definite, because $A$'s money may be expressed in various ways, in pounds, or in shillings, or as a fraction of the whole sum. Again, in the fifth problem which we have solved, we begin by saying, let $x$ denote the number of inches in the longer part; beginners often say, let $x =$ the longer part, or, let $x =$ a part, and to these phrases the same objection applies as to that already noticed.

190. Beginners often find a difficulty in translating a problem from ordinary language into Algebraical language, because they do not understand what is meant by the ordinary language. If no consistent meaning can be assigned to the words, it is of course impossible to translate them; but it often happens that the words are not ab-
solately unintelligible, but appear to be susceptible of more than one meaning. The student should then select one meaning, express that meaning in Algebraical symbols, and deduce from it the result to which it will lead. If the result be inadmissible, or absurd, the student should try another meaning of the words. But if the result is satisfactory he may infer that he has probably understood the words correctly; though it may still be interesting to try the other possible meanings, in order to see if the enunciation really is susceptible of more than one meaning.

191. A student in solving the problems which are given for exercise, may find some which he can readily solve by Arithmetic, or by a process of guess and trial; and he may be thus inclined to undervalue the power of Algebra, and look on its aid as unnecessary. But we may remark that by Algebra the student is enabled to solve all these problems, without any uncertainty; and moreover, he will find as he proceeds, that by Algebra he can solve problems which would be extremely difficult or altogether impracticable, if he relied on Arithmetic alone.

Examples. XXI.

1. Find the number which exceeds its fifth part by 24.

2. A father is 30 years old, and his son is 2 years old: in how many years will the father be eight times as old as the son?

3. The difference of two numbers is 7, and their sum is 33: find the numbers.

4. The sum of £155 was raised by A, B, and C together; B contributed £15 more than A, and C £20 more than B: how much did each contribute?

5. The difference of two numbers is 14, and their sum is 48: find the numbers.

6. A is twice as old as B, and seven years ago their united ages amounted to as many years as now represent the age of A: find the ages of A and B.
7. If 56 be added to a certain number, the result is treble that number: find the number.

8. A child is born in November, and on the tenth day of December he is as many days old as the month was on the day of his birth: when was he born?

9. Find that number the double of which increased by 24 exceeds 80 as much as the number itself is below 100.

10. There is a certain fish, the head of which is 9 inches long; the tail is as long as the head and half the back; and the back is as long as the head and tail together: what is the length of the back and of the tail?

11. Divide the number 84 into two parts such that three times one part may be equal to four times the other.

12. The sum of £76 was raised by $A$, $B$, and $C$ together; $B$ contributed as much as $A$ and £10 more, and $C$ as much as $A$ and $B$ together: how much did each contribute?

13. Divide the number 60 into two parts such that a seventh of one part may be equal to an eighth of the other part.

14. After 34 gallons had been drawn out of one of two equal casks, and 80 gallons out of the other, there remained just three times as much in one cask as in the other: what did each cask contain when full?

15. Divide the number 75 into two parts such that 3 times the greater may exceed 7 times the less by 15.

16. A person distributes 20 shillings among 20 persons, giving sixpence each to some, and sixteen pence each to the rest: how many persons received sixpence each?

17. Divide the number 20 into two parts such that the sum of three times one part, and five times the other part, may be 84.

18. The price of a work which comes out in parts is £2. 16s. 8d.; but if the price of each part were 13 pence more than it is, the price of the work would be £3. 7s. 6d.: how many parts were there?

19. Divide 45 into two parts such that the first divided by 2 shall be equal to the second multiplied by 2.
20. A father is three times as old as his son; four years ago the father was four times as old as his son then was: what is the age of each?

21. Divide 188 into two parts such that the fourth of one part may exceed the eighth of the other by 14.

22. A person meeting a company of beggars gave four pence to each, and had sixteen pence left; he found that he should have required a shilling more to enable him to give the beggars sixpence each: how many beggars were there?

23. Divide 100 into two parts such that if a third of one part be subtracted from a fourth of the other the remainder may be 11.

24. Two persons, A and B, engage at play; A has £72 and B has £52 when they begin, and after a certain number of games have been won and lost between them, A has three times as much money as B: how much did A win?

25. Divide 60 into two parts such that the difference between the greater and 64 may be equal to twice the difference between the less and 38.

26. The sum of £276 was raised by A, B, and C together; B contributed twice as much as A and £12 more; and C three times as much as B and £12 more: how much did each contribute?

27. Find a number such that the sum of its fifth and its seventh shall exceed the sum of its eighth and its twelfth by 113.

28. An army in a defeat loses one-sixth of its number in killed and wounded, and 4000 prisoners; it is reinforced by 3000 men, but retreats, losing one-fourth of its number in doing so; there remain 18000 men: what was the original force?

29. Find a number such that the sum of its fifth and its seventh shall exceed the difference of its fourth and its seventh by 99.

30. One-half of a certain number of persons received eighteen-pence each, one-third received two shillings each, and the rest received half a crown each; the whole sum distributed was £2. 4s.: how many persons were there?
31. A person had £900; part of it he lent at the rate of 4 per cent., and part at the rate of 5 per cent., and he received equal sums as interest from the two parts: how much did he lend at 4 per cent.?

32. A father has six sons, each of whom is four years older than his next younger brother; and the eldest is three times as old as the youngest: find their respective ages.

33. Divide the number 92 into four such parts that the first may exceed the second by 10, the third by 18, and the fourth by 24.

34. A gentleman left £550 to be divided among four servants A, B, C, D; of whom B was to have twice as much as A, C as much as A and B together, and D as much as C and B together; how much had each?

35. Find two consecutive numbers such that the half and the fifth of the first taken together shall be equal to the third and the fourth of the second taken together.

36. A sum of money is to be distributed among three persons A, B, and C; the shares of A and B together amount to £60; those of A and C to £80; and those of B and C to £92: find the share of each person.

37. Two persons A and B are travelling together; A has £100, and B has £48; they are met by robbers who take twice as much from A as from B, and leave to A three times as much as to B; how much was taken from each?

38. The sum of £500 was divided among four persons, so that the first and second together received £280, the first and third together £260, and the first and fourth together £220: find the share of each.

39. After A has received £10 from B he has as much money as B and £6 more; and between them they have £40: what money had each at first?

40. A wine merchant has two sorts of wines, one sort worth 2 shillings a quart, and the other worth 3s. 4d. a quart; from these he wants to make a mixture of 100 quarts worth 2s. 4d. a quart: how many quarts must he take from each sort?
41. In a mixture of wine and water the wine composed 25 gallons more than half of the mixture, and the water 5 gallons less than a third of the mixture: how many gallons were there of each?

42. In a lottery consisting of 10000 tickets, half the number of prizes added to one-third the number of blanks was 3500: how many prizes were there in the lottery?

43. In a certain weight of gunpowder the saltpetre composed 6 lbs. more than a half of the weight, the sulphur 5 lbs. less than a third, and the charcoal 3 lbs. less than a fourth: how many lbs. were there of each of the three ingredients?

44. A general, after having lost a battle, found that he had left fit for action 3600 men more than half of his army; 600 men more than one-eighth of his army were wounded; and the remainder, forming one-fifth of the army, were slain, taken prisoners, or missing: what was the number of the army?

45. How many sheep must a person buy at £7 each that after paying one shilling a score for folding them at night he may gain £79. 16s. by selling them at £8 each?

46. A certain sum of money was shared among five persons $A, B, C, D,$ and $E$; $B$ received £10 less than $A$; $C$ received £16 more than $B$; $D$ received £5 less than $C$; and $E$ received £15 more than $D$; and it was found that $E$ received as much as $A$ and $B$ together: how much did each receive?

47. A tradesman starts with a certain sum of money; at the end of the first year he had doubled his original stock, all but £100; also at the end of the second year he had doubled the stock at the beginning of the second year, all but £100; also in like manner at the end of the third year; and at the end of the third year he was three times as rich as at first: find his original stock.

48. A person went to a tavern with a certain sum of money; there he borrowed as much as he had about him, and spent a shilling out of the whole; with the remainder he went to a second tavern, where he borrowed as much as he had left, and also spent a shilling; and he then went to a third tavern, borrowing and spending as before, after which he had nothing left: how much had he at first?
XXII. Problems, continued.

192. We shall now give some examples in which the process of translation from ordinary language to algebraical language is rather more difficult than in the examples of the preceding Chapter.

193. It is required to divide the number 80 into four such parts, that the first increased by 3, the second diminished by 3, the third multiplied by 3, and the fourth divided by 3 may all be equal.

Let the number \(x\) denote the first part; then if it be increased by 3 we obtain \(x + 3\), and this is to be equal to the second part diminished by 3, so that the second part must be \(x + 6\); again, \(x + 3\) is to be equal to the third part multiplied by 3, so that the third part must be \(\frac{x + 3}{3}\); and \(x + 3\) is to be equal to the fourth part divided by 3, so that the fourth part must be \(3(x + 3)\). And the sum of the parts is to be equal to 80.

Therefore \[x + x + 6 + \frac{x + 3}{3} + 3(x + 3) = 80,\]

that is, \[2x + 6 + \frac{x + 3}{3} + 3x + 9 = 80,\]

that is, \[5x + \frac{x + 3}{3} = 80 - 15 = 65;\]

multiply by 3; thus \(15x + x + 3 = 195,\)

that is, \(16x = 192;\)

therefore \[x = \frac{192}{16} = 12.\]

Thus the parts are 12, 18, 5, 45.
194. A alone can perform a piece of work in 9 days, and B alone can perform it in 12 days: in what time will they perform it if they work together?

Let \( x \) denote the required number of days. In one day \( A \) can perform \( \frac{1}{9} \) th of the work; therefore in \( x \) days he can perform \( \frac{x}{9} \) ths of the work. In one day \( B \) can perform \( \frac{1}{12} \) th of the work; therefore in \( x \) days he can perform \( \frac{x}{12} \) ths of the work. And since in \( x \) days \( A \) and \( B \) together perform the whole work, the sum of the fractions of the work must be equal to unity; that is,

\[
\frac{x}{9} + \frac{x}{12} = 1.
\]

Multiply by 36; thus \( 4x + 3x = 36 \),

that is, \( 7x = 36 \);

therefore \( x = \frac{36}{7} = 5\frac{1}{7} \).

195. A cistern could be filled with water by means of one pipe alone in 6 hours, and by means of another pipe alone in 8 hours; and it could be emptied by a tap in 12 hours if the two pipes were closed: in what time will the cistern be filled if the pipes and the tap are all open?

Let \( x \) denote the required number of hours. In one hour the first pipe fills \( \frac{1}{6} \) th of the cistern; therefore in \( x \) hours it fills \( \frac{x}{6} \) ths of the cistern. In one hour the second pipe fills \( \frac{1}{8} \) th of the cistern; therefore in \( x \) hours it fills \( \frac{x}{8} \) ths of the cistern. In one hour the tap empties \( \frac{1}{12} \) th
of the cistern; therefore in \( x \) hours it empties \( \frac{x}{12} \) ths of the cistern. And since in \( x \) hours the whole cistern is filled, we have

\[
\frac{x}{6} + \frac{x}{8} - \frac{x}{12} = 1.
\]

Multiply by 24; thus \( 4x + 3x - 2x = 24 \),
that is, \( 5x = 24 \);
therefore \( x = \frac{24}{5} = 4\frac{4}{5} \).

196. It is sometimes convenient to denote by \( x \), not the unknown quantity which is explicitly required, but some other quantity from which that can be easily deduced; this will be illustrated in the next two problems.

197. A colonel on attempting to draw up his regiment in the form of a solid square finds that he has 31 men over, and that he would require 24 men more in his regiment in order to increase the side of the square by one man: how many men were there in the regiment?

Let \( x \) denote the number of men in the side of the first square; then the number of men in the square is \( x^2 \) and the number of men in the regiment is \( x^2 + 31 \). If there were \( x + 1 \) men in a side of the square, the number of men in the square would be \( (x + 1)^2 \); thus the number of men in the regiment is \( (x + 1)^2 - 24 \).

Therefore \( (x + 1)^2 - 24 = x^2 + 31 \),
that is, \( x^2 + 2x + 1 - 24 = x^2 + 31 \).

From these two equal expressions we can remove \( x^2 \) which occurs in both; thus

\[
2x + 1 - 24 = 31;
\]

therefore \( 2x = 31 - 1 + 24 = 54 \);
therefore \( x = \frac{54}{2} = 27 \).

Hence the number of men in the regiment is \( (27)^2 + 31 \),
that is, \( 729 + 31 \), that is, 760.
198. \( A \) starts from a certain place, and travels at the rate of 7 miles in 5 hours; \( B \) starts from the same place 8 hours after \( A \), and travels in the same direction at the rate of 5 miles in 3 hours: how far will \( A \) travel before he is overtaken by \( B \)?

Let \( x \) represent the number of hours which \( A \) travels before he is overtaken; therefore \( B \) travels \( x - 8 \) hours. Now since \( A \) travels 7 miles in 5 hours, he travels \( \frac{7}{5} \) of a mile in one hour; and therefore in \( x \) hours he travels \( \frac{7x}{5} \) miles. Similarly \( B \) travels \( \frac{5}{3} \) of a mile in one hour, and therefore in \( x - 8 \) hours he travels \( \frac{5}{3} \) \((x - 8)\) miles. And when \( B \) overtakes \( A \) they have travelled the same number of miles. Therefore

\[
\frac{5}{3} (x - 8) = \frac{7x}{5};
\]

multiply by 15; thus \( 25(x - 8) = 21x \);

that is, \( 25x - 200 = 21x \);

therefore \( 25x - 21x = 200 \);

that is, \( 4x = 200 \);

therefore \( x = \frac{200}{4} = 50 \).

Therefore \( \frac{7x}{5} = \frac{7}{5} \times 50 = 70 \); so that \( A \) travelled 70 miles before he was overtaken.

199. Problems are sometimes given which suppose the student to have obtained from Arithmetic a knowledge of
the meaning of proportion; this will be illustrated in the next two problems. After them we shall conclude the Chapter with three problems of a more difficult character than those hitherto given.

200. It is required to divide the number 56 into two parts such that one may be to the other as 3 to 4.

Let the number \( x \) denote the first part; then the other part must be \( 56 - x \); and since \( x \) is to be to \( 56 - x \) as 3 to 4 we have

\[
\frac{x}{56 - x} = \frac{3}{4}
\]

Clear of fractions; thus

\[4x = 3(56 - x);\]

that is,

\[4x = 168 - 3x;\]

therefore

\[7x = 168;\]

therefore

\[x = \frac{168}{7} = 24.\]

Thus the first part is 24 and the other part is \( 56 - 24 \), that is 32.

The preceding method of solution is the most natural for a beginner; the following however is much shorter.

Let the number \( 3x \) denote the first part; then the second part must be \( 4x \), because the first part is to the second as 3 to 4. Then the sum of the two parts is equal to 56; thus

\[3x + 4x = 56,\]

that is,

\[7x = 56;\]

therefore

\[x = 8.\]

Thus the first part is \( 3 \times 8 \), that is 24; and the second part is \( 4 \times 8 \), that is 32.
201. A cask, \( A \), contains 12 gallons of wine and 18 gallons of water; and another cask, \( B \), contains 9 gallons of wine and 3 gallons of water: how many gallons must be drawn from each cask so as to produce by their mixture 7 gallons of wine and 7 gallons of water?

Let \( x \) denote the number of gallons to be drawn from \( A \); then since the mixture is to consist of 14 gallons, \( 14 - x \) will denote the number of gallons to be drawn from \( B \). Now the number of gallons in \( A \) is 30, of which 12 are wine; that is, the wine is \( \frac{12}{30} \) of the whole. Therefore the \( x \) gallons drawn from \( A \) contain \( \frac{12x}{30} \) gallons of wine. Similarly the \( 14 - x \) gallons drawn from \( B \) contain \( \frac{9(14 - x)}{12} \) gallons of wine. And the mixture is to contain 7 gallons of wine; therefore

\[
\frac{12x}{30} + \frac{9(14 - x)}{12} = 7;
\]

that is,

\[
\frac{2x}{5} + \frac{3(14 - x)}{4} = 7;
\]

therefore

\[
8x + 15(14 - x) = 140,
\]

that is,

\[
8x + 210 - 15x = 140;
\]

therefore

\[
7x = 70;
\]

therefore

\[
x = 10.
\]

Thus 10 gallons must be drawn from \( A \), and 4 from \( B \).

202. At what time between 2 o'clock and 3 o'clock is one hand of a watch exactly over the other?

Let \( x \) denote the required number of minutes after 2 o'clock. In \( x \) minutes the long hand will move over \( x \) divisions of the watch face; and as the long hand moves twelve times as fast as the short hand, the short hand will move over \( \frac{x}{12} \) divisions in \( x \) minutes. At 2 o'clock the
short hand is 10 divisions in advance of the long hand; so that in the \( x \) minutes the long hand must pass over 10 more divisions than the short hand; therefore

\[
x = \frac{x}{12} + 10;
\]

therefore

\[12x = x + 120;\]

therefore

\[11x = 120;\]

therefore

\[x = \frac{120}{11} = 10\frac{10}{11}.\]

203. A hare takes four leaps to a greyhound’s three, but two of the greyhound’s leaps are equivalent to three of the hare’s; the hare has a start of fifty leaps: how many leaps must the greyhound take to catch the hare?

Suppose that \( 3x \) denote the number of leaps taken by the greyhound; then \( 4x \) will denote the number of leaps taken by the hare in the same time. Let \( a \) denote the number of inches in one leap of the hare; then \( 3a \) denotes the number of inches in three leaps of the hare, and therefore also the number of inches in two leaps of the greyhound; therefore \( \frac{3a}{2} \) denotes the number of inches in one leap of the greyhound. Then \( 3x \) leaps of the greyhound will contain \( 3x \times \frac{3a}{2} \) inches. And \( 50 + 4x \) leaps of the hare will contain \( (50 + 4x)a \) inches; therefore

\[
\frac{9xa}{2} = (50 + 4x)a.
\]

Divide by \( a \); thus

\[
\frac{9x}{2} = 50 + 4x;
\]

therefore

\[9x = 100 + 8x;\]

therefore

\[x = 100.\]

Thus the greyhound must take 300 leaps.

The student will see that we have introduced an auxiliary symbol \( a \), to enable us to form the equation easily; and that we can remove it by division when the equation is formed.
EXAMPLES. XXII.

204. Four gamasters, $A$, $B$, $C$, $D$, each with a different stock of money, sit down to play; $A$ wins half of $B$'s first stock, $B$ wins a third part of $C$'s, $C$ wins a fourth part of $D$'s, and $D$ wins a fifth part of $A$'s; and then each of the gamasters has £23. Find the stock of each at first.

Let $x$ denote the number of pounds which $D$ won from $A$; then $5x$ will denote the number in $A$'s first stock. Thus $4x$, together with what $A$ won from $B$, make up 23; therefore $23 - 4x$ denotes the number of pounds which $A$ won from $B$. And, since $A$ won half of $B$'s stock, $23 - 4x$ also denotes what was left with $B$ after his loss to $A$.

Again, $23 - 4x$, together with what $B$ won from $C$, make up 23; therefore $4x$ denotes the number of pounds which $B$ won from $C$. And, since $B$ won a third of $C$'s first stock, $12x$ denotes $C$'s first stock; and therefore $8x$ denotes what was left with $C$ after his loss to $B$.

Again, $8x$, together with what $C$ won from $D$, make up 23; therefore $23 - 8x$ denotes the number of pounds which $C$ won from $D$. And, since $C$ won a fourth of $D$'s first stock, $4(23 - 8x)$ denotes $D$'s first stock; and therefore $3(23 - 8x)$ denotes what was left with $D$ after his loss to $C$.

Finally, $3(23 - 8x)$, together with $x$, which $D$ won from $A$, make up 23; thus

$$23 = 3(23 - 8x) + x;$$

therefore

$$23x = 46;$$

therefore

$$x = 2.$$

Thus the stocks at first were 10, 30, 24, 28.

EXAMPLES. XXII.

1. A privateer running at the rate of 10 miles an hour discovers a ship 18 miles off, running at the rate of 8 miles an hour: how many miles can the ship run before it is overtaken?

2. Divide the number 50 into two parts such that if three-fourths of one part be added to five-sixths of the other part the sum may be 40.
3. Suppose the distance between London and Edinburgh is 360 miles, and that one traveller starts from Edinburgh and travels at the rate of 10 miles an hour, while another starts at the same time from London and travels at the rate of 8 miles an hour: it is required to know where they will meet.

4. Find two numbers whose difference is 4, and the difference of their squares 112.

5. A sum of 24 shillings is received from 24 people; some contribute 9d. each, and some 13½d. each: how many contributors were there of each kind?

6. Divide the number 48 into two parts such that the excess of one part over 20 may be three times the excess of 20 over the other part.

7. A person has £98; part of it he lent at the rate of 5 per cent. simple interest, and the rest at the rate of 6 per cent. simple interest; and the interest of the whole in 15 years amounted to £81: how much was lent at 5 per cent.?

8. A person lent a certain sum of money at 6 per cent. simple interest; in 10 years the interest amounted to £12 less than the sum lent: what was the sum lent?

9. A person rents 25 acres of land for £7. 12s.; the land consists of two sorts, the better sort he rents at 8s. per acre, and the worse at 5s. per acre: how many acres are there of each sort?

10. A cistern could be filled in 12 minutes by two pipes which run into it; and it would be filled in 20 minutes by one alone: in what time could it be filled by the other alone?

11. Divide the number 90 into four parts such that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2 may all be equal.

12. A person bought 30 lbs. of sugar of two different sorts, and paid for the whole 19s.; the better sort cost 10d. per lb., and the worse 7d. per lb.: how many lbs. were there of each sort?
13. Divide the number 88 into four parts such that the first increased by 2, the second diminished by 3, the third multiplied by 4, and the fourth divided by 5, may all be equal.

14. If 20 men, 40 women, and 50 children receive £50 among them for a week's work, and 2 men receive as much as 3 women or 5 children, what does each woman receive for a week's work?

15. Divide 100 into two parts such that the difference of their squares may be 1000.

16. There are two places 154 miles apart, from which two persons start at the same time with a design to meet; one travels at the rate of 3 miles in two hours, and the other at the rate of 5 miles in four hours: when will they meet?

17. Divide 44 into two parts such that the greater increased by 5 may be to the less increased by 7, as 4 is to 3.

18. A can do half as much work as B, B can do half as much as C, and together they can complete a piece of work in 24 days: in what time could each alone complete the work?

19. Divide the number 90 into four parts such that if the first be increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the results shall all be equal.

20. Three persons can together complete a piece of work in 60 days; and it is found that the first does three-fourths of what the second does, and the second four-fifths of what the third does: in what time could each one alone complete the work?

21. Divide the number 36 into two parts such that one part may be five-sevenths of the other.

22. A general on attempting to draw up his army in the form of a solid square finds that he has 60 men over, and that he would require 41 men more in his army in order to increase the side of the square by one man: how many men were there in the army?
23. Divide the number 90 into two parts such that one part may be two-thirds of the other.

24. A person bought a certain number of eggs, half of them at 2 a penny, and half of them at 3 a penny; he sold them again at the rate of 5 for two pence, and lost a penny by the bargain: what was the number of eggs?

25. A and B are at present of the same age; if A’s age be increased by 36 years, and B’s by 52 years, their ages will be as 3 to 4: what is the present age of each?

26. For 1 lb. of tea and 9 lbs. of sugar the charge is 8s. 6d.; for 1 lb. of tea and 15 lbs. of sugar the charge is 12s. 6d.: what is the price of 1 lb. of sugar?

27. A prize of £2000 was divided between A and B, so that their shares were in the proportion of 7 to 9: what was the share of each?

28. A workman was hired for 40 days at 3s. 4d. per day, for every day he worked; but with this condition that for every day he did not work he was to forfeit 1s. 4d.; and on the whole he had £3. 3s. 4d. to receive: how many days out of the 40 did he work?

29. A at play first won £5 from B, and had then as much money as B; but B, on winning back his own money and £5 more, had five times as much money as A: what money had each at first?

30. Divide 100 into two parts, such that the square of their difference may exceed the square of twice the less part by 2000.

31. A cistern has two supply pipes, which will singly fill it in 4½ hours and 6 hours respectively; and it has also a leak by which it would be emptied in 5 hours: in how many hours will it be filled when all are working together?

32. A farmer would mix wheat at 4s. a bushel with rye at 2s. 6d. a bushel, so that the whole mixture may consist of 90 bushels, and be worth 3s. 2d. a bushel: how many bushels must be taken of each?
33. A bill of £3. 1s. 6d. was paid in half-crowns, and florins, and the whole number of coins was 28: how many coins were there of each kind?

34. A grocer with 56 lbs. of fine tea at 5s. a lb. would mix a coarser sort at 3s. 6d. a lb., so as to sell the whole together at 4s. 6d. a lb.: what quantity of the latter sort must he take?

35. A person hired a labourer to do a certain work on the agreement that for every day he worked he should receive 2s., but that for every day he was absent he should lose 9d.; he worked twice as many days as he was absent, and on the whole received £1. 19s.: find how many days he worked.

36. A regiment was drawn up in a solid square; when some time after it was again drawn up in a solid square it was found that there were 5 men fewer in a side; in the interval 295 men had been removed from the field: what was the original number of men in the regiment?

37. A sum of money was divided between A and B, so that the share of A was to that of B as 5 to 3; also the share of A exceeded five-ninths of the whole sum by £50: what was the share of each person?

38. A gentleman left his whole estate among his four sons. The share of the eldest was £800 less than half of the estate; the share of the second was £120 more than one-fourth of the estate; the third had half as much as the eldest; and the youngest had two-thirds of what the second had. How much did each son receive?

39. A and B began to play together with equal sums of money; A first won £20, but afterwards lost half of all he then had, and then his money was half as much as that of B: what money had each at first?

40. A lady gave a guinca in charity among a number of poor, consisting of men, women, and children; each man had 12d., each woman 6d., and each child 3d. The number of women was two less than twice the number of men; and the number of children four less than three times the number of women. How many persons were there relieved?
EXAMPlES. XXII.

41. A draper bought a piece of cloth at 3s. 2d. per yard. He sold one-third of it at 4s. per yard, one-fourth of it at 3s. 8d. per yard, and the remainder at 3s. 4d. per yard; and his gain on the whole was 14s. 2d. How many yards did the piece contain?

42. A grazier spent £33. 7s. 6d. in buying sheep of different sorts. For the first sort, which formed one-third of the whole, he paid 9s. 6d. each. For the second sort, which formed one-fourth of the whole, he paid 11s. each. For the rest he paid 12s. 6d. each. What number of sheep did he buy?

43. A market woman bought a certain number of eggs, at the rate of 5 for twopence; she sold half of them at 2 a penny, and half of them at 3 a penny, and gained 4d. by so doing: what was the number of eggs?

44. A pudding consists of 2 parts of flour, 3 parts of raisins, and 4 parts of suet; flour costs 3d. a lb., raisins, 6d., and suet 8d. Find the cost of the several ingredients of the pudding, when the whole cost is 2s. 4d.

45. Two persons, A and B, were employed together for 50 days, at 5s. per day each. During this time A, by spending 6d. per day less than B, saved twice as much as B, besides the expenses of two days over. How much did A spend per day?

46. Two persons, A and B, have the same income. A lays by one-fifth of his; but B by spending £60 per annum more than A, at the end of three years finds himself £100 in debt. What is the income of each?

47. A and B shoot by turns at a target. A puts 7 bullets out of 12 into the bull’s eye, and B puts in 9 out of 12; between them they put in 32 bullets. How many shots did each fire?

48. Two casks, A and B, contain mixtures of wine and water; in A the quantity of wine is to the quantity of water as 4 to 3; in B the like proportion is that of 2 to 3. If A contain 84 gallons, what must B contain, so that when the two are put together, the new mixture may be half wine and half water?
49. The squire of a parish bequeaths a sum equal to one-hundredth part of his estate towards the restoration of the church; £200 less than this towards the endowment of the school; and £200 less than this latter sum towards the County Hospital. After deducting these legacies, of the estate remain to the heir. What was the value of the estate?

50. How many minutes does it want to 4 o'clock, if three-quarters of an hour ago it was twice as many minutes past two o'clock?

51. Two casks, A and B, are filled with two kinds of sherry, mixed in the cask A in the proportion of 2 to 7, and in the cask B in the proportion of 2 to 5: what quantity must be taken from each to form a mixture which shall consist of 2 gallons of the first kind and 6 of the second kind?

52. An officer can form the men of his regiment into a hollow square 12 deep. The number of men in the regiment is 1296. Find the number of men in the front of the hollow square.

53. A person buys a piece of land at £30 an acre, and by selling it in allotments finds the value increased threefold, so that he clears £150, and retains 25 acres for himself: how many acres were there?

54. The national debt of a country was increased by one-fourth in a time of war. During a long peace which followed £25000000 was paid off, and at the end of that time the rate of interest was reduced from 4½ to 4 per cent. It was then found that the amount of annual interest was the same as before the war. What was the amount of the debt before the war?

55. A and B play at a game, agreeing that the loser shall always pay to the winner one shilling less than half the money the loser has; they commence with equal quantities of money, and after B has lost the first game and won the second, he has two shillings more than A: how much had each at the commencement?
56. A clock has two hands turning on the same centre; the swifter makes a revolution every twelve hours, and the slower every sixteen hours: in what time will the swifter gain just one complete revolution on the slower?

57. At what time between 3 o'clock and 4 o'clock is one hand of a watch exactly in the direction of the other hand produced?

58. The hands of a watch are at right angles to each other at 3 o'clock: when are they next at right angles?

59. A certain sum of money lent at simple interest amounted to £297. 12s. in eight months; and in seven more months it amounted to £306: what was the sum?

60. A watch gains as much as a clock loses; and 1799 hours by the clock are equivalent to 1801 hours by the watch: find how much the watch gains and the clock loses per hour.

61. It is between 11 and 12 o'clock, and it is observed that the number of minute spaces between the hands is two-thirds of what it was ten minutes previously: find the time.

62. A and B made a joint stock of £500 by which they gained £160, of which A had for his share £32 more than B: what did each contribute to the stock?

63. A distiller has 51 gallons of French brandy, which cost him 8 shillings a gallon; he wishes to buy some English brandy at 3 shillings a gallon to mix with the French, and sell the whole at 9 shillings a gallon. How many gallons of the English must he take, so that he may gain 30 per cent. on what he gave for the brandy of both kinds?

64. An officer can form his men into a hollow square 4 deep, and also into a hollow square 8 deep; the front in the latter formation contains 16 men fewer than in the former formation: find the number of men.
XXIII. Simultaneous equations of the first degree with two unknown quantities.

205. Suppose we have an equation containing two unknown quantities $x$ and $y$, for example $3x - 7y = 8$. For every value which we please to assign to one of the unknown quantities we can determine the corresponding value of the other; and thus we can find as many pairs of values as we please which satisfy the given equation. Thus, for example, if $y = 1$ we find $3x = 15$, and therefore $x = 5$; if $y = 2$ we find $3x = 22$, and therefore $x = 7\frac{1}{3}$; and so on.

Also, suppose that there is another equation of the same kind, as for example $2x + 5y = 44$; then we can also find as many pairs of values as we please which satisfy this equation.

But suppose we ask for values of $x$ and $y$ which satisfy both equations; we shall find that there is only one value of $x$ and one value of $y$. For multiply the first equation by 5; thus

$$15x - 35y = 40;$$

and multiply the second equation by 7; thus

$$14x + 35y = 308.$$

Therefore, by addition,

$$15x - 35y + 14x + 35y = 40 + 308;$$

that is,

$$29x = 348;$$

therefore

$$x = \frac{348}{29} = 12.$$

Thus if both equations are to be satisfied $x$ must equal 12. Put this value of $x$ in either of the two given equations, for example in the second; thus we obtain

$$24 + 5y = 44;$$

therefore

$$5y = 20;$$

therefore

$$y = 4.$$
SIMULTANEOUS SIMPLE EQUATIONS. 137

206. Two or more equations which are to be satisfied by the same values of the unknown quantities are called simultaneous equations. In the present Chapter we treat of simultaneous equations involving two unknown quantities, where each unknown quantity occurs only in the first degree, and the product of the unknown quantities does not occur.

207. There are three methods which are usually given for solving these equations. There is one principle common to all the methods; namely, from two given equations containing two unknown quantities a single equation is deduced containing only one of the unknown quantities. By this process we are said to eliminate the unknown quantity which does not appear in the single equation. The single equation containing only one unknown quantity can be solved by the method of Chapter XIX; and when the value of one of the unknown quantities has thus been determined, we can substitute this value in either of the given equations, and then determine the value of the other unknown quantity.

208. First method. Multiply the equations by such numbers as will make the coefficient of one of the unknown quantities the same in the resulting equations; then by addition or subtraction we can form an equation containing only the other unknown quantity.

This method we used in Art. 205; for another example, suppose

\[8x + 7y = 100,\]
\[12x - 5y = 88.\]

If we wish to eliminate \(y\) we multiply the first equation by 5, which is the coefficient of \(y\) in the second equation, and we multiply the second equation by 7, which is the coefficient of \(y\) in the first equation. Thus we obtain

\[40x + 35y = 500,\]
\[84x - 35y = 616;\]

therefore, by addition,

\[40x + 84x = 500 + 616;\]

that is,

\[124x = 1116;\]

therefore

\[x = 9.\]
Then put this value of $x$ in either of the given equations, for example in the second; thus

$$108 - 5y = 88;$$
therefore
$$20 = 5y;$$
therefore
$$y = 4.$$

Suppose, however, that in solving these equations we wish to begin by eliminating $x$. If we multiply the first equation by 12, and the second by 8, we obtain

$$96x + 84y = 1200,$$
$$96x - 40y = 704.$$

Therefore, by subtraction,

$$84y + 40y = 1200 - 704;$$
that is,
$$124y = 496;$$
therefore
$$y = 4.$$

Or we may render the process more simple; for we may multiply the first equation by 3, and the second by 2; thus

$$24x + 21y = 300,$$
$$24x - 10y = 176.$$

Therefore, by subtraction,

$$21y + 10y = 300 - 176;$$
that is,
$$31y = 124;$$
therefore
$$y = 4.$$

209. Second method. Express one of the unknown quantities in terms of the other from either equation, and substitute this value in the other equation.

Thus, taking the example given in the preceding Article, we have from the first equation

$$8x = 100 - 7y;$$
therefore
$$x = \frac{100 - 7y}{8}.$$
SIMULTANEOUS SIMPLE EQUATIONS. 139

Substitute this value of $x$ in the second equation, and we obtain

\[
\frac{12(100-7y)}{8} - 5y = 88;
\]

that is,

\[
\frac{3(100-7y)}{2} - 5y = 88;
\]

therefore

\[
3(100-7y) - 10y = 176;
\]

that is,

\[
300 - 21y - 10y = 176;
\]

therefore

\[
300 - 176 = 21y + 10y;
\]

that is,

\[
31y = 124;
\]

therefore

\[
y = 4.
\]

Then substitute this value of $y$ in either of the given equations, and we shall obtain $x = 9$.

Or thus: from the first equation we have

\[
7y = 100 - 8x;
\]

therefore

\[
y = \frac{100 - 8x}{7}.
\]

Substitute this value of $y$ in the second equation, and we obtain

\[
12x - \frac{5(100 - 8x)}{7} = 88;
\]

therefore

\[
84x - 5(100 - 8x) = 616;
\]

that is,

\[
84x - 500 + 40x = 616;
\]

therefore

\[
124x - 500 + 616 = 1116;
\]

therefore

\[
x = 9.
\]

210. Third method. **Express the same unknown quantity in terms of the other from each equation, and equate the expressions thus obtained.**

Thus, taking again the same example, from the first equation $x = \frac{100 - 7y}{8}$, and from the second equation

\[
x = \frac{88 + 5y}{12}.
\]
Therefore \[ \frac{100-7y}{8} = \frac{88+5y}{12} \].

Clear of fractions, by multiplying by 24; thus

\[ 3(100-7y) = 2(88+5y) \];

that is, \[ 300-21y = 176+10y \];

therefore \[ 300-176 = 21y+10y \];

that is, \[ 31y = 124 \];

therefore \[ y = 4 \].

Then, as before, we can deduce \( x = 9 \).

Or thus: from the first equation \[ y = \frac{100-8x}{7} \], and from the second equation \[ y = \frac{12x-88}{5} \]; therefore

\[ \frac{100-8x}{7} = \frac{12x-88}{5} \].

From this equation we shall obtain \( x = 9 \); and then, as before, we can deduce \( y = 4 \).

211. Solve \( 19x - 21y = 100, \ 21x - 19y = 140 \).

These equations may be solved by the methods already explained; we shall use them however to shew that these methods may be sometimes abbreviated.

Here, by addition, we obtain

\[ 19x - 21y + 21x - 19y = 100 + 140 \];

that is, \[ 40x - 40y = 240 \];

therefore \[ x - y = 6 \].

Again, from the original equations, by subtraction, we obtain

\[ 21x - 19y - 19x + 21y = 140 - 100 \];

that is, \[ 2x + 2y = 40 \];

therefore \[ x + y = 20 \].
Then since \( x - y = 6 \) and \( x + y = 20 \), we obtain by addition \( 2x = 26 \), and by subtraction \( 2y = 14 \); therefore \( x = 13 \), and \( y = 7 \).

212. The student will find as he proceeds that in all parts of Algebra, particular examples may be treated by methods which are shorter than the general rules; but such abbreviations can only be suggested by experience and practice, and the beginner should not waste his time in seeking for them.

213. Solve \( \frac{12}{x} + \frac{8}{y} = 8 \), \( \frac{27}{x} - \frac{12}{y} = 3 \).

If we cleared these equations of fractions they would involve the product \( xy \) of the unknown quantities; and thus strictly they do not belong to the present Chapter. But they may be solved by the methods already given, as we shall now shew. For multiply the first equation by 3 and the second by 2, and add; thus

\[
\frac{36}{x} + \frac{24}{y} + \frac{54}{x} - \frac{24}{y} = 24 + 6;
\]

that is,

\[
\frac{36}{x} + \frac{54}{x} = 30;
\]

that is,

\[
\frac{90}{x} = 30;
\]

therefore

\[
x = 3.
\]

Substitute the value of \( x \) in the first equation; thus

\[
\frac{12}{3} + \frac{8}{y} = 8;
\]

therefore

\[
\frac{8}{y} = 8 - 4 = 4;
\]

therefore

\[
y = 4y;
\]

therefore

\[
y = 2.
\]
214. Solve \( a^2x + b^2y - c^2, \ ax + by - c. \)

Here \( x \) and \( y \) are supposed to denote unknown quantities, while the other letters are supposed to denote known quantities.

Multiply the second equation by \( b \), and subtract it from the first; thus 

\[
a^2x + b^2y - abx - b^2y = c^2 - bc;
\]

that is, 

\[
a(a - b)x = c(c - b);
\]

therefore 

\[
x = \frac{c(c - b)}{a(a - b)}.
\]

Substitute this value of \( x \) in the second equation; thus

\[
\frac{ac(c - b)}{a(a - b)} + by = c;
\]

therefore 

\[
by = c - \frac{c(c - b)}{a - b} = \frac{c(a - b) - c(c - b)}{a - b} = \frac{c(a - c)}{a - b};
\]

therefore 

\[
y = \frac{c(a - c)}{b(a - b)} = \frac{c(c - a)}{b(b - a)}.
\]

Or the value of \( y \) might be found in the same way as that of \( x \) was found.

Examples. XXIII.

1. \( 3x - 4y = 2, \quad 7x - 9y = 7. \)
2. \( 7x - 5y = 24, \quad 4x - 3y = 11. \)
3. \( 3x + 2y = 32, \quad 20x - 3y = 1. \)
4. \( 11x - 7y = 37, \quad 8x + 9y = 41. \)
5. \( 7x + 5y = 60, \quad 13x - 11y = 10. \)
6. \( 6x - 7y = 42, \quad 7x - 6y = 75. \)
7. \( 10x + 9y = 290, \quad 12x - 11y = 130. \)
8. \( 3x - 4y = 18, \quad 3x + 2y = 0. \)
9. \( 4x - \frac{y}{2} = 11, \quad 2x - 3y = 0. \)
10. \( \frac{x}{3} + 3y = 7 \), \( \frac{4x - 2}{5} = 3y - 4 \).

11. \( 6x - 5y = 1 \), \( 7x - 4y = 8 \frac{1}{2} \).

12. \( 2x + \frac{y - 2}{5} = 21 \), \( 4y + \frac{x - 4}{6} = 29 \).

13. \( \frac{3x}{19} + 5y = 13 \), \( 2x + \frac{4 - 7y}{2} = 33 \).

14. \( \frac{x}{7} + \frac{y}{14} = 10 \frac{1}{2} \), \( 2x - y = 7 \).

15. \( \frac{x + y}{3} + \frac{y - x}{2} = 9 \), \( \frac{x}{2} + \frac{x + y}{9} = 5 \).

16. \( \frac{3x}{4} - \frac{2y}{3} = 1 \), \( \frac{7x}{3} + \frac{5y}{6} = 6 \).

17. \( \frac{x + y}{3} + x = 15 \), \( \frac{x - y}{5} + y = 6 \).

18. \( \frac{7x}{6} + \frac{5y}{3} = 34 \), \( \frac{7x}{8} + \frac{3y}{4} = \frac{5y}{8} + 12 \).

19. \( \frac{x + y}{8} + \frac{x - y}{6} = 5 \), \( \frac{x + y}{4} - \frac{x - y}{3} = 10 \).

20. \( \frac{2x}{3} + \frac{3y}{2} = 16 \frac{1}{6} \), \( \frac{3x}{2} - \frac{2y}{3} = 16 \frac{1}{6} \).

21. \( \frac{x - 1}{8} + \frac{y - 2}{5} = 2 \), \( 2x + \frac{2y - 5}{3} = 21 \).

22. \( \frac{7x}{4} + \frac{5y}{8} = 20 \), \( \frac{3x}{5} + \frac{7y}{4} = 2x - 7 \).

23. \( \frac{2x + 3y}{5} = 10 - \frac{y}{3} \), \( \frac{4y - 3x}{6} = \frac{3x}{4} + 1 \).

24. \( \frac{1 - 3x}{7} + \frac{3y - 1}{5} = 2 \), \( \frac{3x + y}{11} + y = 9 \).

25. \( 2(2x + 3y) = 3(2x - 3y) + 10 \),
\( 4x - 3y = 4(6y - 2x) + 3 \).
26. $3x + 9y = 24$,  
   $21x - 0.06y = 0.3$.
27. $3x + 12.5y = x - 6$,  
   $3x - 5y = 28 - 25y$.
28. $0.08x - 21y = 33$,  
   $72x + 7y = 354$.
29. $\frac{9}{x} - \frac{4}{y} = 1$,  
   $\frac{18}{x} + \frac{20}{y} = 16$.
30. $x - 4y = 7$,  
   $\frac{x}{3y} + \frac{11}{10} = \frac{4x - 5y}{5y}$.
31. $\frac{x + 1}{y - 1} - \frac{x - 1}{y} = \frac{6}{y}$,  
   $x - y = 1$.
32. $4x + y = 11$,  
   $\frac{y}{5x} = \frac{7x - y}{3x} - \frac{23}{15}$.
   
   $\frac{x + y}{2} - 3 = 0$,  
   $\frac{3y - 10(x - 1)}{6} + \frac{x - y}{4} + 1 = 0$.
34. $\frac{x}{a} + \frac{y}{b} = 2$,  
   $bx - ay = 0$.
35. $x + y = a + b$,  
   $bx + ay = 2ab$.
36. $\frac{x}{a} + \frac{y}{b} = 1$,  
   $\frac{x}{b} + \frac{y}{a} = 1$.
37. $(a + c)x - by = bc$,  
   $x + y = a + b$.
38. $\frac{x}{a} + \frac{y}{b} = c$,  
   $\frac{x}{b} - \frac{y}{a} = 0$.
39. $x + y = c$,  
   $ax - by = c(a - b)$.
40. $a(x + y) + b(x - y) = 1$,  
   $a(x - y) + b(x + y) = 1$.
41. $\frac{x - a}{b} + \frac{y - b}{a} = 0$,  
   $\frac{x + y - b}{a} + \frac{x - y - a}{b} = 0$.
42. $(a + b)x - (a - b)y = 4ab$,  
   $(a - b)x + (a + b)y = 2a^2 - 2b^2$.
43. $\frac{x}{a + b} + \frac{y}{a - b} = 2a$,  
   $\frac{x - y}{2ab} = \frac{x + y}{a^2 + b^2}$.
44. $(a + k)x + (b - h)y = c$,  
   $(b + k)x + (a - k)y = c$. 
XXIV. Simultaneous equations of the first degree with more than two unknown quantities.

215. If there be three simple equations containing three unknown quantities, we can deduce from two of the equations an equation which contains only two of the unknown quantities, by the methods of the preceding Chapter; then from the third given equation, and either of the former two, we can deduce another equation which contains the same two unknown quantities. We have thus two equations containing two unknown quantities, and therefore the values of these unknown quantities may be found by the methods of the preceding Chapter. By substituting these values in one of the given equations, the value of the remaining unknown quantity may be found.

216. Solve \(7x + 3y - 2z = 16\) \(\ldots \ldots (1)\), \(2x + 5y + 3z = 39\) \(\ldots \ldots (2)\), \(5x - y + 5z = 31\) \(\ldots \ldots (3)\).

For convenience of reference the equations are numbered (1), (2), (3); and this numbering is continued as we proceed with the solution.

Multiply (1) by 3, and multiply (2) by 2; thus
\[21x + 9y - 6z = 48,\]
\[4x + 10y + 6z = 78;\]
therefore, by addition,
\[25x + 19y = 126 \ldots \ldots (4).\]

Multiply (1) by 5, and multiply (3) by 2; thus
\[35x + 15y - 10z = 80,\]
\[10x - 2y + 10z = 62;\]
therefore, by addition,
\[45x + 13y = 142 \ldots \ldots (5).\]
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We have now to find the values of \( x \) and \( y \) from (4) and (5).

Multiply (4) by 9, and multiply (5) by 5; thus
\[
225x + 171y = 1134, \\
225x + 65y = 710;
\]
therefore, by subtraction,
\[
106y = 424;
\]
therefore
\[
y = 4.
\]

Substitute the value of \( y \) in (4); thus
\[
25x + 76 = 126;
\]
therefore
\[
25x = 126 - 76 = 50;
\]
therefore
\[
x = 2.
\]

Substitute the values of \( x \) and \( y \) in (1); thus
\[
14 + 12 - 2z = 16;
\]
therefore
\[
10 = 2z;
\]
therefore
\[
z = 5.
\]

217. Solve \[
\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1 \quad \ldots \ldots \ldots (1),
\]
\[
\frac{5}{x} + \frac{4}{y} + \frac{6}{z} = 24 \quad \ldots \ldots \ldots (2),
\]
\[
\frac{7}{x} - \frac{8}{y} + \frac{9}{z} = 14 \quad \ldots \ldots \ldots (3).
\]

Multiply (1) by 2, and add the result to (2); thus
\[
\frac{2}{x} + \frac{4}{y} - \frac{6}{z} + \frac{5}{x} + \frac{4}{y} + \frac{6}{z} = 2 + 24;
\]
that is,
\[
\frac{7}{x} + \frac{8}{y} = 26 \quad \ldots \ldots \ldots (4).
\]
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Multiply (1) by 3, and add the result to (3); thus

\[ \frac{3}{x} + \frac{\frac{6}{y} - \frac{9}{z}}{x} + \frac{\frac{7}{y} + \frac{8}{z}}{y} = 3 + 14; \]

that is,

\[ \frac{10}{x} - \frac{2}{y} = 17 \quad \ldots \ldots \quad (5). \]

Multiply (5) by 4, and add the result to (4); thus

\[ \frac{40}{x} - \frac{8}{y} + \frac{7}{x} + \frac{8}{y} = 68 + 26; \]

that is,

\[ \frac{47}{x} = 94; \]

therefore

\[ 47 = 94x; \]

therefore

\[ x = \frac{47}{94} = \frac{1}{2}. \]

Substitute the value of \( x \) in (5); thus

\[ 20 - \frac{2}{y} = 17; \]

therefore

\[ \frac{2}{y} = 20 - 17 = 3; \]

therefore

\[ y = \frac{3}{3} = 1. \]

Substitute the values of \( x \) and \( y \) in (1); thus

\[ 2 + 3 - \frac{3}{z} = 1; \]

therefore

\[ \frac{3}{z} = 4; \]

therefore

\[ z = \frac{3}{4}. \]

10—2
218. Solve
\[ \frac{x}{a} + \frac{y}{b} = 3 \quad \cdots \quad (1), \]
\[ \frac{y}{b} + \frac{z}{c} = 5 \quad \cdots \quad (2), \]
\[ \frac{x}{a} + \frac{z}{c} = 4 \quad \cdots \quad (3). \]

Subtract (1) from (2); thus
\[ \frac{y}{b} + \frac{z}{c} - \frac{x}{a} - \frac{y}{b} = 5 - 3; \]
that is,
\[ \frac{z}{c} - \frac{x}{a} = 2 \quad \cdots \quad (4). \]

By subtracting (4) from (3) we obtain
\[ \frac{2x}{a} = 2; \]
therefore \( \frac{x}{a} = 1; \) therefore \( x = a. \)

By adding (4) to (3) we obtain
\[ \frac{2z}{c} = 6; \]
therefore \( \frac{z}{c} = 3; \) therefore \( z = 3c. \)

By substituting the value of \( x \) in (1) we find that \( y = 2b. \)

219. In a similar manner we may proceed if the number of equations and unknown quantities should exceed three.
Examples. XXIV.

1. \(x + 3y + 2z = 11,\quad 2x + y + 3z = 14,\quad 3x + 2y + z = 11.\)
2. \(5x - 6y + 4z = 15,\quad 7x + 4y - 3z = 19,\quad 2x + y + 6z = 46.\)
3. \(4x - 5y + z = 6,\quad 7x - 11y + 2z = 9,\quad x + y + 3z = 12.\)
4. \(7x - 3y = 30,\quad 9y - 5z = 34,\quad x + y + z = 33.\)
5. \(3x - y + z = 17,\quad 5x + 3y - 2z = 10,\quad 7x + 4y - 5z = 3.\)
6. \(x + y + z = 5,\quad 3x - 5y + 7z = 75,\quad 9x - 11z + 10 = 0.\)
7. \(x + 2y + 3z = 6,\quad 2x + 4y + 2z = 8,\quad 3x + 2y + 8z = 101.\)
8. \(\frac{6y - 4x}{3z - 7} = 1,\quad \frac{5z - x}{2y - 3z} = 1,\quad \frac{y - 2z}{3y - 2x} = 1.\)
9. \(\frac{x + 2y}{7} = \frac{3y + 4z}{8} = \frac{5x + 6z}{9},\quad x + y - z = 126.\)
10. \(\frac{1}{x} - \frac{1}{y} = \frac{1}{6},\quad \frac{1}{y} + \frac{1}{z} = \frac{5}{6},\quad \frac{4}{x} + \frac{3}{y} = 4.\)
11. \(y + z = a,\quad z + x = b,\quad x + y = c.\)
12. \(x + y + z = a + b + c,\quad x + a = y + b = z + c.\)
13. \(y + z - x = a,\quad z + x - y = b,\quad x + y - z = c.\)
14. \(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,\quad \frac{x}{a} + \frac{y}{c} + \frac{z}{b} = 1,\quad \frac{x}{b} + \frac{y}{a} + \frac{z}{c} = 1.\)
15. \(\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 3,\quad \frac{a}{x} + \frac{b}{y} - \frac{c}{z} = 1,\quad \frac{2a}{x} - \frac{b}{y} - \frac{c}{z} = 0.\)
16. \(v + x + y + z = 14,\)
\[2v + x = 2y + z - 2,\]
\[3v - x + 2y + 2z = 19,\]
\[v + \frac{x}{3} + \frac{y}{4} + \frac{z}{2} = 4.\]
XXV. Problems which lead to simultaneous equations of the first degree with more than one unknown quantity.

220. We shall now solve some problems which lead to simultaneous equations of the first degree with more than one unknown quantity.

Find the fraction which becomes equal to \(\frac{2}{3}\) when the numerator is increased by 2, and equal to \(\frac{4}{7}\) when the denominator is increased by 4.

Let \(x\) denote the numerator, and \(y\) the denominator of the required fraction; then, by supposition,

\[
\frac{x+2}{y} = \frac{2}{3} \quad \frac{x}{y+4} = \frac{4}{7}.
\]

Clear the equations of fractions; thus we obtain

\[
3x - 2y = -6 \quad \quad (1),
\]
\[
7x - 4y = 16 \quad \quad (2).
\]

Multiply (1) by 2, and subtract it from (2); thus

\[
7x - 4y - 6x + 4y = 16 + 12;
\]

that is,

\[
x = 28.
\]

Substitute the value of \(x\) in (1); thus

\[
84 - 2y = -6;
\]

therefore \(2y = 90\); therefore \(y = 45\).

Hence the required fraction is \(\frac{28}{45}\).

221. A sum of money was divided equally among a certain number of persons; if there had been six more, each would have received two shillings less than he did; and if there had been three fewer, each would have received two shillings more than he did: find the number of persons, and what each received.
Let $x$ denote the number of persons, and $y$ the number of shillings which each received. Then $xy$ is the number of shillings in the sum of money which is divided; and, by supposition,
\begin{align*}
(x+6)(y-2) &= xy \quad \ldots \ldots \quad (1),
(x-3)(y+2) &= xy \quad \ldots \ldots \quad (2).
\end{align*}

From (1) we obtain
\[ xy + 6y - 2x - 12 = xy; \]
therefore
\[ 6y - 2x = 12 \quad \ldots \ldots \quad (3). \]

From (2) we obtain
\[ xy + 2x - 3y - 6 = xy; \]
therefore
\[ 2x - 3y = 6 \quad \ldots \ldots \quad (4). \]

From (3) and (4), by addition, $3y = 18$; therefore $y = 6$.

Substitute the value of $y$ in (4); thus
\[ 2x - 18 = 6; \]
therefore $2x = 24$; therefore $x = 12$.

Thus there were 12 persons, and each received 6 shillings.

222. A certain number of two digits is equal to five times the sum of its digits; and if nine be added to the number the digits are reversed: find the number.

Let $x$ denote the digit in the tens' place, and $y$ the digit in the units' place. Then the number is $10x + y$; and, by supposition, the number is equal to five times the sum of its digits; therefore
\[ 10x + y = 5(x+y) \quad \ldots \ldots \quad (1). \]

If nine be added to the number its digits are reversed, that is, we obtain the number $10y + x$; therefore
\[ 10x + y + 9 = 10y + x \quad \ldots \ldots \quad (2). \]

From (1) we obtain
\[ 5x = 4y \quad \ldots \ldots \quad (3). \]

From (2) we obtain $9x + 9 = 9y$; therefore $x + 1 = y$. 
Substitute for \( y \) in (3); thus
\[
5x = 4x + 4;
\]
therefore
\[
x = 4.
\]

Then from (3) we obtain \( y = 5 \).

Hence the required number is 45.

223. A railway train after travelling an hour is detained 24 minutes, after which it proceeds at six-fifths of its former rate, and arrives 15 minutes late. If the detention had taken place 5 miles further on, the train would have arrived 2 minutes later than it did. Find the original rate of the train, and the distance travelled.

Let \( 5x \) denote the number of miles per hour at which the train originally travelled, and let \( y \) denote the number of miles in the whole distance travelled. Then \( y - 5x \) will denote the number of miles which remain to be travelled after the detention. At the original rate of the train this distance would be travelled in \( \frac{y - 5x}{5x} \) hours; at the increased rate it will be travelled in \( \frac{y - 5x}{6x} \) hours. Since the train is detained 24 minutes, and yet is only 15 minutes late at its arrival, it follows that the remainder of the journey is performed in 9 minutes less than it would have been if the rate had not been increased. And 9 minutes is \( \frac{9}{60} \) of an hour; therefore

\[
\frac{y - 5x}{6x} = \frac{y - 5x}{5x} - \frac{9}{60} \quad \cdots \cdots \cdots \cdots \cdots (1).
\]

If the detention had taken place 5 miles further on, there would have been \( y - 5x - 5 \) miles left to be travelled. Thus we shall find that

\[
\frac{y - 5x - 5}{6x} = \frac{y - 5x - 5}{5x} - \frac{7}{60} \quad \cdots \cdots \cdots \cdots \cdots (2).
\]
Subtract (2) from (1); thus
\[
\frac{5}{6x} = \frac{5}{5x} - \frac{2}{60};
\]
therefore
\[
50 = 60 - 2x;
\]
therefore \(2x = 10\); therefore \(x = 5\).

Substitute this value of \(x\) in (1), and it will be found by solving the equation that \(y = 47\frac{1}{2}\).

224. \(A, B,\) and \(C\) can together perform a piece of work in 30 days; \(A\) and \(B\) can together perform it in 32 days; and \(B\) and \(C\) can together perform it in 120 days; find the time in which each alone could perform the work.

Let \(x\) denote the number of days in which \(A\) alone could perform it, \(y\) the number of days in which \(B\) alone could perform it, \(z\) the number of days in which \(C\) alone could perform it. Then we have

\[
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{30} \quad \ldots \ldots \quad (1),
\]

\[
\frac{1}{x} + \frac{1}{y} = \frac{1}{32} \quad \ldots \ldots \quad (2),
\]

\[
\frac{1}{y} + \frac{1}{z} = \frac{1}{120} \quad \ldots \ldots \quad (3).
\]

Subtract (2) from (1); thus
\[
\frac{1}{z} = \frac{1}{30} - \frac{1}{32} = \frac{1}{480}.
\]

Subtract (3) from (1); thus
\[
\frac{1}{x} = \frac{1}{30} - \frac{1}{120} = \frac{1}{40}.
\]

Therefore \(x = 40\), and \(z = 480\); and by substitution in any of the given equations we shall find that \(y = 160\).

225. We may observe that a problem may often be solved in various ways, and with the aid of more or fewer letters to represent the unknown quantities. Thus, to take a very simple example, suppose we have to find two
numbers such that one is two-thirds of the other, and their sum is 100.

We may proceed thus. Let \( x \) denote the greater number, and \( y \) the less number; then we have

\[
y = \frac{2x}{3}, \quad x + y = 100.
\]

Or we may proceed thus. Let \( x \) denote the greater number, then \( 100 - x \) will denote the less number; therefore

\[
100 - x = \frac{2x}{3}.
\]

Or we may proceed thus. Let \( 3x \) denote the greater number, then \( 2x \) will denote the less number; therefore

\[
2x + 3x = 100.
\]

By completing any of these processes we shall find that the required numbers are 60 and 40.

The student may accordingly find that he can solve some of the examples at the end of the present Chapter, with the aid of only one letter to denote an unknown quantity; and, on the other hand, some of the examples at the end of Chapter xxii. may appear to him most naturally solved with the aid of two letters. As a general rule it may be stated that the employment of a larger number of unknown quantities renders the work longer, but at the same time allows the successive steps to be more readily followed; and thus is more suitable for beginners.

The beginner will find it a good exercise to solve the example given in Art. 204 with the aid of four letters to represent the four unknown quantities which are required.

**Examples. XXV.**

1. If \( A \)’s money were increased by 36 shillings he would have three times as much as \( B \); and if \( B \)’s money were diminished by 5 shillings he would have half as much as \( A \); find the sum possessed by each.

2. Find two numbers such that the first with half the second may make 20, and also that the second with a third of the first may make 20.
3. If $B$ were to give £25 to $A$ they would have equal sums of money; if $A$ were to give £22 to $B$ the money of $B$ would be double that of $A$; find the money which each actually has.

4. Find two numbers such that half the first with a third of the second may make 32, and that a fourth of the first with a fifth of the second may make 18.

5. A person buys 8 lbs. of tea and 3 lbs. of sugar for £1. 2s.; and at another time he buys 5 lbs. of tea and 4 lbs. of sugar for 15s. 2d.: find the price of tea and sugar per lb.

6. Seven years ago $A$ was three times as old as $B$ was; and seven years hence $A$ will be twice as old as $B$ will be: find their present ages.

7. Find the fraction which becomes equal to $\frac{1}{3}$ when the numerator is increased by 1, and equal to $\frac{1}{4}$ when the denominator is increased by 1.

8. A certain fishing rod consists of two parts; the length of the upper part is to the length of the lower as 5 to 7; and 9 times the upper part together with 13 times the lower part exceed 11 times the whole rod by 36 inches: find the lengths of the two parts.

9. A person spends half-a-crown in apples and pears, buying the apples at 4 a penny, and the pears at 5 a penny; he sells half his apples and one-third of his pears for 13 pence, which was the price at which he bought them: find how many apples and how many pears he bought.

10. A wine merchant has two sorts of wine, a better and a worse; if he mixes them in the proportion of two quarts of the better sort with three of the worse, the mixture will be worth 1s. 9d. a quart; but if he mixes them in the proportion of seven quarts of the better sort with eight of the worse, the mixture will be worth 1s. 10d. a quart: find the price of a quart of each sort.

11. A farmer sold to one person 30 bushels of wheat, and 40 bushels of barley for £13. 10s.; to another person he sold 50 bushels of wheat and 30 bushels of barley for £17: find the price of wheat and barley per bushel.
12. A farmer has 28 bushels of barley at 2s. 4d. a bushel; with these he wishes to mix rye at 3s. a bushel, and wheat at 4s. a bushel, so that the mixture may consist of 100 bushels, and be worth 3s. 4d. a bushel: find how many bushels of rye and wheat he must take.

13. $A$ and $B$ lay a wager of 10 shillings; if $A$ loses he will have as much as $B$ will then have; if $B$ loses he will have half of what $A$ will then have: find the money of each.

14. If the numerator of a certain fraction be increased by 1, and the denominator be diminished by 1, the value will be 1; if the numerator be increased by the denominator, and the denominator diminished by the numerator, the value will be 4: find the fraction.

15. A number of posts are placed at equal distances in a straight line. If to twice the number of them we add the distance between two consecutive posts, expressed in feet, the sum is 68. If from four times the distance between two consecutive posts, expressed in feet, we subtract half the number of posts, the remainder is 68. Find the distance between the extreme posts.

16. A gentleman distributing money among some poor men found that he wanted 10 shillings, in order to be able to give 5 shillings to each man; therefore he gives to each man 4 shillings only, and finds that he has 5 shillings left: find the number of poor men and of shillings.

17. A certain company in a tavern found, when they came to pay their bill, that if there had been three more persons to pay the same bill, they would have paid one shilling each less than they did; and if there had been two fewer persons they would have paid one shilling each more than they did: find the number of persons and the number of shillings each paid.

18. There is a certain rectangular floor, such that if it had been two feet broader, and three feet longer, it would have been sixty-four square feet larger; but if it had been three feet broader, and two feet longer, it would have been sixty-eight square feet larger: find the length and breadth of the floor.

19. A certain number of two digits is equal to four
times the sum of its digits; and if 18 be added to the number the digits are reversed: find the number.

20. Two digits which form a number change places on the addition of 9; and the sum of the two numbers is 33: find the digits.

21. When a certain number of two digits is doubled, and increased by 36, the result is the same as if the number had been reversed, and doubled, and then diminished by 36; also the number itself exceeds four times the sum of its digits by 3: find the number.

22. Two passengers have together 5 cwt. of luggage, and are charged for the excess above the weight allowed 5s. 2d. and 9s. 10d. respectively; if the luggage had all belonged to one of them he would have been charged 19s. 2d.: find how much luggage each passenger is allowed without charge.

23. A and B ran a race which lasted 5 minutes; B had a start of 20 yards; but A ran 3 yards while B was running 2, and won by 30 yards: find the length of the course and the speed of each.

24. A and B have each a certain number of counters; A gives to B as many as B has already, and B returns back again to A as many as A has left; A gives to B as many as B has left, and B returns to A as many as A has left; each of them has now sixteen counters: find how many each had at first.

25. A and B can together perform a certain work in 30 days; at the end of 18 days however B is called off and A finishes it alone in 20 more days: find the time in which each could perform the work alone.

26. A, B, and C can drink a cask of beer in 15 days; A and B together drink four-thirds of what C does; and C drinks twice as much as A: find the time in which each alone could drink the cask of beer.

27. A cistern holding 1200 gallons is filled by three pipes A, B, C together in 24 minutes. The pipe A requires 30 minutes more than C to fill the cistern; and 10 gallons less run through C per minute than through A and B together. Find the time in which each pipe alone would fill the cistern.
28. A and B run a mile. At the first heat A gives B a start of 20 yards, and beats him by 30 seconds. At the second heat A gives B a start of 32 seconds, and beats him by $9\frac{5}{11}$ yards. Find the rate per hour at which A runs.

29. A and B are two towns situated 24 miles apart, on the same bank of a river. A man goes from A to B in 7 hours, by rowing the first half of the distance, and walking the second half. In returning he walks the first half at three-fourths of his former rate, but the stream being with him he rows at double his rate in going; and he accomplishes the whole distance in 6 hours. Find his rates of walking and rowing.

30. A railway train after travelling an hour is detained 15 minutes, after which it proceeds at three-fourths of its former rate, and arrives 24 minutes late. If the detention had taken place 5 miles further on, the train would have arrived 3 minutes sooner than it did. Find the original rate of the train and the distance travelled.

31. The time which an express train takes to travel a journey of 120 miles is to that taken by an ordinary train as 9 is to 14. The ordinary train loses as much time in stoppages as it would take to travel 20 miles without stopping. The express train only loses half as much time in stoppages as the ordinary train, and it also travels 15 miles an hour quicker. Find the rate of each train.

32. Two trains, 92 feet long and 84 feet long respectively, are moving with uniform velocities on parallel rails; when they move in opposite directions they are observed to pass each other in one second and a half; but when they move in the same direction the faster train is observed to pass the other in six seconds; find the rate at which each train moves.

33. A railroad runs from A to C. A goods' train starts from A at 12 o'clock, and a passenger train at 1 o'clock. After going two-thirds of the distance the goods' train breaks down, and can only travel at three-fourths of its former rate. At 40 minutes past 2 o'clock a collision occurs, 10 miles from C. The rate of the passenger train is double the diminished rate of the goods' train. Find the distance from A to C, and the rates of the trains.
34. A certain sum of money was divided between A, B, and C, so that A's share exceeded four-sevenths of the shares of B and C by £30; also B's share exceeded three-eighths of the shares of A and C by £30; and C's share exceeded two-ninths of the shares of A and B by £30. Find the share of each person.

35. A and B working together can earn 40 shillings in 6 days; A and C together can earn 54 shillings in 9 days; and B and C together can earn 80 shillings in 15 days: find what each man can earn alone per day.

36. A certain number of sovereigns, shillings, and sixpences amount to £8. 6s. 6d. The amount of the shillings is a guinea less than that of the sovereigns, and a guinea and a half more than that of the sixpences. Find the number of each coin.

37. A and B can perform a piece of work together in 48 days; A and C in 30 days; and B and C in 26 $\frac{2}{3}$ days: find the time in which each could perform the work alone.

38. There is a certain number of three digits which is equal to 48 times the sum of its digits, and if 198 be subtracted from the number the digits will be reversed; also the sum of the extreme digits is equal to twice the middle digit: find the number.

39. A man bought 10 bullocks, 120 sheep, and 46 lambs. The price of 3 sheep is equal to that of 5 lambs. A bullock, a sheep, and a lamb together cost a number of shillings greater by 300 than the whole number of animals bought; and the whole sum spent was £468. 6s. Find the price of a bullock, a sheep, and a lamb respectively.

40. A farmer sold at a market 100 head of stock consisting of horses, oxen, and sheep, so that the whole realised £2. 7s. per head; while a horse, an ox, and a sheep were sold for £22, £12. 10s., and £1. 10s., respectively. Had he sold one-fourth the number of oxen, and 25 more sheep than he did, the amount received would have been still the same. Find the number of horses, oxen, and sheep, respectively which were sold.
XXVI. Quadratic Equations.

226. A quadratic equation is an equation which contains the square of the unknown quantity, but no higher power.

227. A pure quadratic equation is one which contains only the square of the unknown quantity. An affected quadratic equation is one which contains the first power of the unknown quantity as well as its square. Thus, for example, \(2x^2 = 50\) is a pure quadratic equation; and \(2x^2 - 7x + 3 = 0\) is an affected quadratic equation.

228. The following is the Rule for solving a pure quadratic equation. Find the value of the square of the unknown quantity by the Rule for solving a simple equation; then, by extracting the square root, the values of the unknown quantity are found.

For example, solve \(\frac{x^2 - 13}{3} + \frac{x^2 - 5}{10} = 6\).

Clear of fractions by multiplying by 30; thus

\[10(x^2 - 13) + 3(x^2 - 5) = 180;\]

therefore

\[13x^2 = 180 + 130 + 15 = 325;\]

therefore

\[x^2 = \frac{325}{13} = 25;\]

extract the square root, thus \(x = \pm 5\).

In this example, we find by the Rule for solving a simple equation, that \(x^2\) is equal to 25; therefore \(x\) must be such a number, that if multiplied into itself the product is 25. That is to say, \(x\) must be a square root of 25. In Arithmetic 5 is the square root of 25; in Algebra we may consider either 5 or \(-5\) as a square root of 25, since, by the Rule of Signs \(-5 \times -5 = 5 \times 5\). Hence \(x\) may have either of the values 5 or \(-5\), and the equation will be satisfied. This we denote thus, \(x = \pm 5\).