constant ratio to the base; by which we mean that the number which represents the area bears a constant ratio to the number which represents the base.

These remarks are intended to explain the notation and phraseology which are used in the present Chapter. When we say that $A$ varies as $B$, we mean that $A$ represents the numerical value of any one of a certain series of quantities, and $B$ the numerical value of the corresponding quantity in a certain other series, and that $A = mB$, where $m$ is some number which remains constant for every corresponding pair of quantities.

It will be convenient to give a formal demonstration of the relation $A = mB$, deduced from the definition in Art. 376.

378. If $A$ vary as $B$, then $A$ is equal to $B$ multiplied by some constant number.

Let $a$ and $b$ denote one pair of corresponding values of the two quantities, and let $A$ and $B$ denote any other pair; then $\frac{A}{a} = \frac{B}{b}$, by definition. Hence $A = \frac{a}{b}B = mB$, where $m$ is equal to the constant $\frac{a}{b}$.

379. The symbol $\alpha$ is used to express variation; thus $A \propto B$ stands for $A$ varies as $B$.

380. One quantity is said to vary inversely as another, when the first varies as the reciprocal of the second. See Art. 323.

Or if $A = \frac{m}{B}$, where $m$ is constant, $A$ is said to vary inversely as $B$.

381. One quantity is said to vary as two others jointly, when, if the former is changed in any manner, the product of the other two is changed in the same proportion.

Or if $A = mBC$, where $m$ is constant, $A$ is said to vary jointly as $B$ and $C$. 

T. A.
382. One quantity is said to vary directly as a second and inversely as a third, when it varies jointly as the second and the reciprocal of the third.

Or if \( A = \frac{mB}{C} \), where \( m \) is constant, \( A \) is said to vary directly as \( B \) and inversely as \( C \).

383. If \( A \propto B \), and \( B \propto C \), then \( A \propto C \).

For let \( A = mB \), and \( B = nC \), where \( m \) and \( n \) are constants; then \( A = mnC \); and, as \( mn \) is constant, \( A \propto C \).

384. If \( A \propto C \), and \( B \propto C \), then \( A + B \propto C \), and \( \sqrt{(AB)} \propto C \).

For let \( A = mC \), and \( B = nC \), where \( m \) and \( n \) are constants; then \( A + B = (m + n)C \); therefore \( A + B \propto C \).

Also \( \sqrt{(AB)} = \sqrt{(mnC^2)} = C \sqrt{(mn)} \); therefore \( \sqrt{(AB)} \propto C \).

385. If \( A \propto BC \), then \( B \propto \frac{A}{C} \), and \( C \propto \frac{A}{B} \).

For let \( A = mBC \), then \( B = \frac{1}{m} \frac{A}{C} \); therefore \( B \propto \frac{A}{C} \).

Similarly, \( C \propto \frac{A}{B} \).

386. If \( A \propto B \), and \( C \propto D \), then \( AC \propto BD \).

For let \( A = mB \), and \( C = nD \); then \( AC = mnBD \); therefore \( AC \propto BD \).

Similarly, if \( A \propto B \), and \( C \propto D \), and \( E \propto F \), then \( ACE \propto BDF \); and so on.

387. If \( A \propto B \), then \( A^n \propto B^n \).

For let \( A = mB \), then \( A^n = m^nB^n \); therefore \( A^n \propto B^n \).
388. If \( A \propto B \), then \( AP \propto BP \), where \( P \) is any quantity variable or invariable.

For let \( A = mB \), then \( AP = mBP \); therefore \( AP \propto BP \).

389. If \( A \propto B \) when \( C \) is invariable, and \( A \propto C \) when \( B \) is invariable, then \( A \propto BC \) when both \( B \) and \( C \) are variable.

The variation of \( A \) depends on the variations of the two quantities \( B \) and \( C \); let the variations of the latter quantities take place separately. When \( B \) is changed to \( b \) let \( A \) be changed to \( a' \); then, by supposition, \( \frac{A}{a'} = \frac{B}{b} \). Now let \( C \) be changed to \( c \), and in consequence let \( a' \) be changed to \( a \); then, by supposition, \( \frac{a'}{a} = \frac{C}{c} \). Therefore

\[
\frac{A}{a'} \times \frac{a'}{a} = \frac{B}{b} \times \frac{C}{c};
\]

that is, \( \frac{A}{a} = \frac{BC}{bc} \); therefore \( A \propto BC \).

A very good example of this proposition is furnished in Geometry. It can be shewn that the area of a triangle varies as the base when the height is invariable, and that the area varies as the height when the base is invariable. Hence when both the base and the height vary, the area varies as the product of the numbers which represent the base and the height.

Other examples of this proposition are supplied by the questions which occur in Arithmetick under the head of the Double Rule of Three. For instance suppose that the quantity of a work which can be accomplished varies as the number of workmen when the time is given, and varies as the time when the number of workmen is given; then the quantity of the work will vary as the product of the number of workmen and the time when both vary.

390. In the same manner, if there be any number of quantities \( B, C, D, \ldots \) each of which varies as another quantity \( A \) when the rest are constant, when they all vary \( A \) varies as their product.
Examples. XXXVII.

1. \( A \) varies as \( B \), and \( A = 2 \) when \( B = 1 \); find the value of \( A \) when \( B = 2 \).

2. If \( A^2 + B^2 \) varies as \( A^2 - B^2 \), shew that \( A + B \) varies as \( A - B \).

3. \( 3A + 5B \) varies as \( 5A + 3B \), and \( A = 5 \) when \( B = 2 \); find the ratio \( A : B \).

4. \( A \) varies as \( nB + C \); and \( A = 4 \) when \( B = 1 \), and \( C = 2 \); and \( A = 7 \) when \( B = 2 \), and \( C = 3 \); find \( n \).

5. \( A \) varies as \( B \) and \( C \) jointly; and \( A = 1 \) when \( B = 1 \), and \( C = 1 \); find the value of \( A \) when \( B = 2 \) and \( C = 2 \).

6. \( A \) varies as \( B \) and \( C \) jointly; and \( A = 8 \) when \( B = 2 \), and \( C = 2 \); find the value of \( BC \) when \( A = 10 \).

7. \( A \) varies as \( B \) and \( C \) jointly; and \( A = 12 \) when \( B = 2 \), and \( C = 3 \); find the value of \( A : B \) when \( C = 4 \).

8. \( A \) varies as \( B \) and \( C \) jointly; and \( A = a \) when \( B = b \), and \( C = c \); find the value of \( A \) when \( B = b^2 \) and \( C = c^2 \).

9. \( A \) varies as \( B \) directly and as \( C \) inversely; and \( A = a \) when \( B = b \), and \( C = c \); find the value of \( A \) when \( B = c \) and \( C = b \).

10. The expenses of a Charitable Institution are partly constant, and partly vary as the number of inmates. When the inmates are 960 and 3000 the expenses are respectively £112 and £180. Find the expenses for 1000 inmates.

11. The wages of 5 men for 7 weeks being £17. 10s. find how many men can be hired to work 4 weeks for £30.

12. If the cost of making an embankment vary as the length if the area of the transverse section and height be constant, as the height if the area of the transverse section and length be constant, and as the area of the transverse section if the length and height be constant, and an embankment 1 mile long, 10 feet high, and 12 feet broad cost £9600 find the cost of an embankment half a mile long, 16 feet high, and 15 feet broad.
XXXVIII. *Arithmetical Progression.*

391. Quantities are said to be in Arithmetical Progression when they increase or decrease by a common difference.

Thus the following series are in Arithmetical Progression,

\[ 2, 5, 8, 11, 14, \ldots \]
\[ 20, 18, 16, 14, 12, \ldots \]
\[ a, a+b, a+2b, a+3b, a+4b \ldots \]

The common difference is found by subtracting any term from that which immediately follows it. In the first series the common difference is 3; in the second series it is \(-2\); in the third series it is \(b\).

392. Let \(a\) denote the first term of an Arithmetical Progression, \(b\) the common difference; then the second term is \(a+b\), the third term is \(a+2b\), the fourth term is \(a+3b\), and so on. Thus the \(n^{th}\) term is \(a+(n-1)b\).

393. To find the sum of a given number of terms of an Arithmetical Progression, the first term and the common difference being supposed known.

Let \(a\) denote the first term, \(b\) the common difference, \(n\) the number of terms, \(l\) the last term, \(s\) the sum of the terms. Then

\[ s = a + (a+b) + (a+2b) + \ldots \ldots + l. \]

And, by writing the series in the reverse order, we have also

\[ s = l + (l-b) + (l-2b) + \ldots \ldots + a. \]

Therefore, by addition,

\[ 2s = (l+a) + (l+a) + \ldots \ldots \text{to } n \text{ terms} \]

\[ = n(l+a); \]

therefore

\[ s = \frac{n}{2} (l+a) \ldots \ldots \ldots \ldots (1). \]
Also \[ l = a + (n - 1)b \] ............(2),
thus \[ s = \frac{n}{2} \{2a + (n - 1)b\} \] ...........(3).

The equation (3) gives the value of \( s \) in terms of the quantities which were supposed known. Equation (1) also gives a convenient expression for \( s \), and furnishes the following rule: the sum of any number of terms in Arithmetical Progression is equal to the product of the number of the terms into half the sum of the first and last terms.

We shall now apply the equations in the present Article to solve some examples relating to Arithmetical Progression.

394. Find the sum of 20 terms of the series 1, 2, 3, 4,...
Here \( a = 1 \), \( b = 1 \), \( n = 20 \); therefore
\[ s = \frac{20}{2}(2 + 19) = 10 \times 21 = 210. \]

395. Find the sum of 20 terms of the series, 1, 3, 5, 7,...
Here \( a = 1 \), \( b = 2 \), \( n = 20 \); therefore,
\[ s = \frac{20}{2}(2 + 19 \times 2) = \frac{20}{2} \times 40 = (20)^2 = 400. \]

396. Find the sum of 12 terms of the series 20, 18, 16,...
Here \( a = 20 \), \( b = -2 \), \( n = 12 \); therefore
\[ s = \frac{12}{2}(40 - 2 \times 11) = 6(40 - 22) = 6 \times 18 = 108. \]

397. Find the sum of 8 terms of the series \( \frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \ldots \)
Here \( a = \frac{1}{12} \), \( b = \frac{1}{12} \), \( n = 8 \); therefore
\[ s = 8 \left( \frac{2}{12} + \frac{7}{12} \right) = 4 \times \frac{9}{12} = 3. \]
398. How many terms must be taken of the series 15, 12, 9, ... that the sum may be 42?

Here \( s = 42, \ a = 15, \ b = -3; \) therefore

\[
42 = \frac{n}{2} \left( 30 - 3(n - 1) \right) = \frac{n}{2} (33 - 3n).
\]

We have to find \( n \) from this quadratic equation; by solving it we shall obtain \( n = 4 \) or 7. The series is 15, 12, 9, 6, 3, 0, -3, ......; and thus it will be found that we obtain 42 as the sum of the first 4 terms, or as the sum of the first 7 terms.

399. Insert five Arithmetical means between 11 and 23.

Here we have to obtain an Arithmetical Progression consisting of seven terms, beginning with 11 and ending with 23. Thus \( a = 11, l = 23, n = 7; \) therefore by equation (2) of Art. 393,

\[
23 = 11 + 6b,
\]

therefore \( b = 2. \)

Thus the whole series is 11, 13, 15, 17, 19, 21, 23.

EXAMPLES. XXXVIII.

Sum the following series:

1. 100, 101, 102, ........... to 9 terms.
2. 1, 2\frac{1}{2}, 4, .............. to 10 terms.
3. 1, 2\frac{2}{3}, 4\frac{1}{3}, ............. to 9 terms.
4. 2, 3\frac{3}{4}, 5\frac{1}{2}, .............. to 12 terms.
5. \frac{2}{3}, 5, 6, 1, ............... to 18 terms.
6. \frac{1}{2}, -\frac{2}{3}, -\frac{11}{6}, ........... to 15 terms.
8. Insert 5 Arithmetical means between 14 and 16.
9. Insert 7 Arithmetical means between 8 and \(-4\).

10. Insert 8 Arithmetical means between \(-1\) and 5.

11. The first term of an Arithmetical Progression is 13, the second term is 11, the sum is 40: find the number of terms.

12. The first term of an Arithmetical Progression is 5, and the fifth term is 11: find the sum of 8 terms.

13. The sum of four terms in Arithmetical Progression is 44, and the last term is 17: find the terms.

14. The sum of three numbers in Arithmetical Progression is 21, and the sum of their squares is 155: find the numbers.

15. The sum of five numbers in Arithmetical Progression is 15, and the sum of their squares is 55: find the numbers.

16. The seventh term of an Arithmetical Progression is 12, and the twelfth term is 7; the sum of the series is 171: find the number of terms.

17. A traveller has a journey of 140 miles to perform. He goes 26 miles the first day, 24 the second, 22 the third, and so on. In how many days does he perform the journey?

18. \(A\) sets out from a place and travels 2½ miles an hour. \(B\) sets out 3 hours after \(A\), and travels in the same direction, 3 miles the first hour, 3½ miles the second, 4 miles the third, and so on. In how many hours will \(B\) overtake \(A\) ?

19. The sum of three numbers in Arithmetical Progression is 12; and the sum of their squares is 66: find the numbers.

20. If the sum of \(n\) terms of an Arithmetical Progression is always equal to \(n^2\), find the first term and the common difference.
XXXIX. Geometrical Progression.

400. Quantities are said to be in Geometrical Progression when each is equal to the product of the preceding and some constant factor. The constant factor is called the common ratio of the series, or more shortly, the ratio.

Thus the following series are in Geometrical Progression.

\[ 1, 3, 9, 27, 81, \ldots \]
\[ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \]
\[ a, ar, ar^2, ar^3, ar^4, \ldots \]

The common ratio is found by dividing any term by that which immediately precedes it. In the first example the common ratio is 3, in the second it is \( \frac{1}{2} \), in the third it is \( r \).

401. Let \( a \) denote the first term of a Geometrical Progression, \( r \) the common ratio; then the second term is \( ar \), the third term is \( ar^2 \), the fourth term is \( ar^3 \), and so on. Thus the \( n^{th} \) term is \( ar^{n-1} \).

402. To find the sum of a given number of terms of a Geometrical Progression, the first term and the common ratio being supposed known.

Let \( a \) denote the first term, \( r \) the common ratio, \( n \) the number of terms, \( s \) the sum of the terms. Then

\[ s = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}; \]

therefore \[ sr = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n. \]

Therefore, by subtraction,

\[ sr - s = ar^n - a, \]

therefore \[ s = \frac{a(r^n - 1)}{r - 1} \] \[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1) \]
GEOMETRICAL PROGRESSION.

If \( l \) denote the last term we have
\[
l = ar^{n-1} \quad \text{.........(2),}
\]
therefore
\[
s = \frac{rl - a}{r - 1} \quad \text{.........(3)}.
\]

Equation (1) gives the value of \( s \) in terms of the quantities which were supposed known. Equation (3) is sometimes a convenient form.

We shall now apply these equations to solve some examples relating to Geometrical Progression.

403. Find the sum of 6 terms of the series 1, 3, 9, 27, ...
Here \( a = 1, r = 3, n = 6 \); therefore
\[
s = \frac{3^6 - 1}{3 - 1} = \frac{729 - 1}{3 - 1} = 364.
\]

404. Find the sum of 6 terms of the series 1, -3, 9, -27, ...
Here \( a = 1, r = -3, n = 6 \); therefore
\[
s = \frac{(-3)^6 - 1}{-3 - 1} = \frac{729 - 1}{-4} = -182.
\]

405. Find the sum of 8 terms of the series 4, 2, 1, \( \frac{1}{2} \), ...
Here \( a = 4, r = \frac{1}{2}, n = 8 \); therefore
\[
s = \frac{4 \left( \frac{1}{2^8} - 1 \right)}{\frac{1}{2} - 1} = \frac{4 \left( 1 - \frac{1}{2^8} \right)}{1 - \frac{1}{2}} = \frac{255}{64} \times \frac{2}{1} = \frac{255}{32}.
\]

406. Find the sum of 7 terms of the series, 8, -4, 2, -1, \( \frac{1}{2} \), ...
Here \( a = 8, r = -\frac{1}{2}, n = 7 \); therefore
\[
s = \frac{8 \left\{ \left( -\frac{1}{2} \right)^7 - 1 \right\}}{-\frac{1}{2} - 1} = \frac{8 \left( -\frac{1}{128} - 1 \right)}{-\frac{1}{2} - 1} = \frac{129 \times 2}{16 \times 3} = \frac{43}{8}.
\]
407. Insert three Geometrical means between 2 and 32.
Here we have to obtain a Geometrical Progression consisting of five terms, beginning with 2 and ending with 32. Thus \( a = 2 \), \( l = 32 \), \( n = 5 \); therefore, by equation (2) of Art. 402,
\[
32 = 2r^4;
\]
that is
\[
r^4 = 16 = 2^4;
\]
therefore
\[
r = 2.
\]
Thus the whole series is 2, 4, 8, 16, 32.

408. We may write the value of \( s \), given in Art. 402, thus
\[
s = \frac{a(1 - r^n)}{1 - r}.
\]
Now suppose that \( r \) is less than unity; then the larger \( n \) is, the smaller will \( r^n \) be, and by taking \( n \) large enough \( r^n \) can be made as small as we please. If we neglect \( r^n \) we obtain
\[
s = \frac{a}{1 - r},
\]
and we may enunciate the result thus. *In a Geometrical Progression in which the common ratio is numerically less than unity, by taking a sufficient number of terms the sum can be made to differ as little as we please from \( \frac{a}{1 - r} \).*

409. For example, take the series 1, \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \)
Here \( a = 1 \), \( r = \frac{1}{2} \); therefore \( \frac{a}{1 - r} = 2 \). Thus by taking a sufficient number of terms the sum can be made to differ as little as we please from 2. In fact if we take four terms the sum is \( 2 - \frac{1}{8} \), if we take five terms the sum is \( 2 - \frac{1}{16} \), if we take six terms the sum is \( 2 - \frac{1}{32} \), and so on.

The result is sometimes expressed thus for shortness, *the sum of an infinite number of terms of this series is 2; or thus, the sum to infinity is 2.*
410. Recurring decimals are examples of what are called infinite Geometrical Progression. Thus for example \(0.3242424\ldots\) denotes \(\frac{3}{10} + \frac{24}{10^3} + \frac{24}{10^5} + \frac{24}{10^7} + \ldots\).

Here the terms after \(\frac{3}{10}\) form a Geometrical Progression, of which the first term is \(\frac{24}{10^3}\), and the common ratio is \(\frac{1}{10^2}\). Hence we may say that the sum of an infinite number of terms of this series is \(\frac{24}{10^3} \div \left(1 - \frac{1}{10^2}\right)\), that is \(\frac{24}{990}\). Therefore the value of the recurring decimal is \(\frac{3}{10} + \frac{24}{990}\).

The value of the recurring decimal may be found practically thus:
Let \(s = 0.32424\ldots\);
then \(10s = 3.2424\ldots\),
and \(1000s = 324.2424\ldots\).

Hence, by subtraction, \((1000 - 10)s = 324 - 3 = 321\);
therefore \(s = \frac{321}{990}\).

And any other example may be treated in a similar manner.

**Examples. XXXIX.**

Sum the following series:
1. 1, 4, 16, ............. to 6 terms.
2. 9, 3, 1, ............. to 5 terms.
3. 25, 10, 4, ............. to 4 terms.
4. 1, \(\sqrt{2}, \sqrt{2}, \ldots\), to 12 terms.
5. $\frac{3}{8}, \frac{1}{4}, \frac{1}{6}, \ldots \ldots$ to 6 terms.

6. $\frac{2}{3}, -1, \frac{3}{2}, \ldots \ldots$ to 7 terms.

7. $1, -\frac{1}{3}, \frac{1}{9}, \ldots \ldots$ to infinity.

8. $1, \frac{1}{4}, \frac{1}{16}, \ldots \ldots$ to infinity.

9. $1, -\frac{1}{2}, \frac{1}{4}, \ldots \ldots$ to infinity.

10. $6, -2, \frac{2}{3}, \ldots \ldots$ to infinity.

Find the value of the following recurring decimals:

11. $\cdot 151515\ldots$ 12. $\cdot 123123123\ldots$

13. $\cdot 4282828\ldots$ 14. $\cdot 28131313\ldots$

15. Insert 3 Geometrical means between 1 and 256.

16. Insert 4 Geometrical means between $5\frac{1}{3}$ and $40\frac{1}{2}$.

17. Insert 4 Geometrical means between 3 and $-729$.

18. The sum of three terms in Geometrical Progression is 63, and the difference of the first and third terms is 45; find the terms.

19. The sum of the first four terms of a Geometrical Progression is 40, and the sum of the first eight terms is 3280: find the Progression.

20. The sum of three terms in Geometrical Progression is 21, and the sum of their squares is 189: find the terms.
XL. Harmonical Progression.

411. Three quantities $A$, $B$, $C$ are said to be in Harmonical Progression when $A : C :: A - B : B - C$.

Any number of quantities are said to be in Harmonical Progression when every three consecutive quantities are in Harmonical Progression.

412. The reciprocals of quantities in Harmonical Progression are in Arithmetical Progression.

Let $A$, $B$, $C$ be in Harmonical Progression; then $A : C :: A - B : B - C$.

Therefore $A (B - C) = C (A - B)$.

Divide by $ABC$; thus $\frac{1}{C} - \frac{1}{B} = \frac{1}{B} - \frac{1}{A}$.

This demonstrates the proposition.

413. The property established in the preceding Article will enable us to solve some questions relating to Harmonical Progression. For example, insert five Harmonical means between $\frac{2}{3}$ and $\frac{8}{15}$. Here we have to insert five Arithmetical means between $\frac{3}{2}$ and $\frac{15}{8}$. Hence, by equation (2) of Art. 393,

$$\frac{15}{8} = \frac{3}{2} + 6b,$$

therefore $6b = \frac{3}{8}$, therefore $b = \frac{1}{16}$.

Hence the Arithmetical Progression is $\frac{3}{2}$, $\frac{25}{16}$, $\frac{26}{16}$, $\frac{27}{16}$, $\frac{28}{16}$, $\frac{29}{16}$; and therefore the Harmonical Progression is $\frac{2}{3}$, $\frac{16}{25}$, $\frac{16}{26}$, $\frac{16}{27}$, $\frac{16}{28}$, $\frac{16}{29}$, $\frac{8}{15}$. 
414. Let \(a\) and \(c\) be any two quantities; let \(A\) be their Arithmetical mean, \(G\) their Geometrical mean, \(H\) their Harmonical mean. Then

\[
A - a = c - A ; \text{ therefore } A = \frac{1}{2} (a + c).
\]

\[
a : G :: G : c; \text{ therefore } G = \sqrt{ac}.
\]

\[
a : c :: a - H : H - c; \text{ therefore } H = \frac{2ac}{a + c}.
\]

**Examples. XL.**

1. Continue the Harmonical Progression 6, 3, 2 for three terms.

2. Continue the Harmonical Progression 8, 2, 1\(\frac{1}{2}\) for three terms.

3. Insert 2 Harmonical means between 4 and 2.

4. Insert 3 Harmonical means between \(\frac{1}{3}\) and \(\frac{1}{21}\).

5. The Arithmetical mean of two numbers is 9, and the Harmonical mean is 8: find the numbers.

6. The Geometrical mean of two numbers is 48, and the Harmonical mean is 46\(\frac{2}{5}\): find the numbers.

7. Find two numbers such that the sum of their Arithmetical, Geometrical, and Harmonical means is 9\(\frac{1}{5}\), and the product of these means is 27.

8. Find two numbers such that the product of their Arithmetical and Harmonical means is 27, and the excess of the Arithmetical mean above the Harmonical mean is 1\(\frac{1}{2}\).

9. If \(a, b, c\) are in Harmonical Progression, shew that

\[
a + c - 2b : a - c :: a - c : a + c.
\]

10. If three numbers are in Geometrical Progression, and each of them is increased by the middle number, shew that the results are in Harmonical Progression.
XLI. Permutations, and Combinations.

415. The different orders in which a set of things can be arranged are called their permutations.

Thus the permutations of the three letters \(a, b, c\), taken two at a time, are \(ab, ba, ac, ca, bc, cb\).

416. The combinations of a set of things are the different collections which can be formed out of them, without regarding the order in which the things are placed.

Thus the combinations of the three letters \(a, b, c\), taken two at a time, are \(ab, ac, bc\); \(ab\) and \(ba\), though different permutations, form the same combination, so also do \(ac\) and \(ca\), and \(bc\) and \(cb\).

417. The number of permutations of \(n\) things taken \(r\) at a time is \(n(n-1)(n-2)\ldots(n-r+1)\).

Let there be \(n\) letters \(a, b, c, d, \ldots\); we shall first find the number of permutations of them taken two at a time. Put \(a\) before each of the other letters; we thus obtain \(n-1\) permutations in which \(a\) stands first. Put \(b\) before each of the other letters; we thus obtain \(n-1\) permutations in which \(b\) stands first. Similarly there are \(n-1\) permutations in which \(c\) stands first. And so on. Thus, on the whole, there are \(n(n-1)\) permutations of \(n\) letters taken two at a time. We shall next find the number of permutations of \(n\) letters taken three at a time. It has just been shown that out of \(n\) letters we can form \(n(n-1)\) permutations, each of two letters; hence out of the \(n-1\) letters \(b, c, d, \ldots\) we can form \((n-1)(n-2)\) permutations, each of two letters: put \(a\) before each of these, and we have \((n-1)(n-2)\) permutations, each of three letters, in which \(a\) stands first. Similarly there are \((n-1)(n-2)\) permutations, each of three letters, in which \(b\) stands first. Similarly there are as many in which \(c\) stands first. And so on. Thus, on the whole, there are \(n(n-1)(n-2)\) permutations of \(n\) letters taken three at a time.
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From considering these cases it might be conjectured that the number of permutations of \( n \) letters taken \( r \) at a time is \( n(n-1)(n-2) \ldots (n-r+1) \); and we shall shew that this is the case. For suppose it known that the number of permutations of \( n \) letters taken \( r-1 \) at a time is \( n(n-1)(n-2) \ldots \{n-(r-1)+1\} \), we shall shew that a similar formula will give the number of permutations of \( n \) letters, taken \( r \) at a time. For out of the \( n-1 \) letters \( b, c, d, \ldots \) we can form \( (n-1)(n-2) \ldots \{n-1-(r-1)+1\} \) permutations, each of \( r-1 \) letters; put \( a \) before each of these, and we obtain as many permutations, each of \( r \) letters, in which \( a \) stands first. Similarly there are as many permutations, each of \( r \) letters, in which \( b \) stands first. Similarly there are as many permutations, each of \( r \) letters, in which \( c \) stands first. And so on. Thus on the whole there are \( n(n-1)(n-2) \ldots (n-r+1) \) permutations of \( n \) letters taken \( r \) at a time.

If then the formula holds when the letters are taken \( r-1 \) at a time it will hold when they are taken \( r \) at a time. But it has been shewn to hold when they are taken three at a time, therefore it holds when they are taken four at a time, and therefore it holds when they are taken five at a time, and so on: thus it holds universally.

418. Hence the number of permutations of \( n \) things taken all together is \( n(n-1)(n-2) \ldots 1 \).

419. For the sake of brevity \( n(n-1)(n-2) \ldots 1 \) is often denoted by \( [n] \); thus \( [n] \) denotes the product of the natural numbers from 1 to \( n \) inclusive. The symbol \( [n] \) may be read, factorial \( n \).

420. Any combination of \( r \) things will produce \( [r] \) permutations.

For by Art. 418 the \( r \) things which form the given combination can be arranged in \( [r] \) different orders.

421. The number of combinations of \( n \) things taken \( r \) at a time is \( \frac{n(n-1)(n-2) \ldots (n-r+1)}{r} \).
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For the number of permutations of \( n \) things taken \( r \) at a time is \( n(n-1)(n-2)\ldots(n-r+1) \) by Art. 417; and each combination produces \( \binom{n}{r} \) permutations by Art. 420; hence the number of combinations must be

\[
\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}
\]

If we multiply both numerator and denominator of this expression by \( n-r \) it takes the form \( \binom{n}{r} \binom{n-r}{n-r} \), the value of course being unchanged.

422. To find the number of permutations of \( n \) things taken all together which are not all different.

Let there be \( n \) letters; and suppose \( p \) of them to be \( a \), \( q \) of them to be \( b \), \( r \) of them to be \( c \), and the rest of them to be the letters \( d, e, \ldots \), each occurring singly: then the number of permutations of them taken all together will be

\[
\binom{n}{p, q, r}
\]

For suppose \( N \) to represent the required number of permutations. If in any one of the permutations the \( p \) letters \( a \) were changed into \( p \) new and different letters, then, without changing the situation of any of the other letters, we could from the single permutation produce \( p \) different permutations: and thus if the \( p \) letters \( a \) were changed into \( p \) new and different letters the whole number of permutations would be \( N \times \binom{p}{p} \). Similarly if the \( q \) letters \( b \) were also changed into \( q \) new and different letters the whole number of permutations we could now obtain would be \( N \times \binom{p}{p} \times \binom{q}{q} \). And if the \( r \) letters \( c \) were also changed into \( r \) new and different letters the whole number of permutations would be \( N \times \binom{p}{p} \times \binom{q}{q} \times \binom{r}{r} \). But this number must be equal to the number of permutations of \( n \) different letters taken all together, that is to \( \binom{n}{n} \).

Thus \( N \times \binom{p}{p} \times \binom{q}{q} \times \binom{r}{r} = \binom{n}{n} \); therefore \( N = \frac{\binom{n}{n}}{\binom{p}{p} \binom{q}{q} \binom{r}{r}} \).

And similarly any other case may be treated.
423. The student should notice the peculiar method of demonstration which is employed in Art. 417. This is called mathematical induction, and may be thus described: We shew that if a theorem is true in one case, whatever that case may be, it is also true in another case so related to the former that it may be called the next case; we also shew in some manner that the theorem is true in a certain case; hence it is true in the next case, and hence in the next to that, and so on; thus finally the theorem must be true in every case after that with which we began.

The method of mathematical induction is frequently used in the higher parts of mathematics.

**Examples. XLI.**

1. Find how many parties of 6 men each can be formed from a company of 24 men.

2. Find how many permutations can be formed of the letters in the word company, taken all together.

3. Find how many combinations can be formed of the letters in the word longitude, taken four at a time.

4. Find how many permutations can be formed of the letters in the word consonant, taken all together.

5. The number of the combinations of a set of things taken four at a time is twice as great as the number taken three at a time: find how many things there are in the set.

6. Find how many words each containing two consonants and one vowel can be formed from 20 consonants and 5 vowels, the vowel being the middle letter of the word.

7. Five persons are to be chosen by lot out of twenty: find in how many ways this can be done. Find also how often an assigned person would be chosen.

8. A boat's crew consisting of eight rowers and a steersman is to be formed out of twelve persons, nine of whom can row but cannot steer, while the other three can steer but cannot row: find in how many ways the crew can be formed. Find also in how many ways the crew could be formed if one of the three were able both to row and to steer.
XLII. Binomial Theorem.

424. We have already seen that \((x + a)^2 = x^2 + 2xa + a^2\), and that \((x + a)^3 = x^3 + 3x^2a + 3xa^2 + a^3\); the object of the present Chapter is to find an expression for \((x + a)^n\) where \(n\) is any positive integer.

425. By actual multiplication we obtain
\[
(x + a)(x + b) = x^2 + (a + b)x + ab,
\]
\[
(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc,
\]
\[
(x + a)(x + b)(x + c)(x + d) = x^4 + (a + b + c + d)x^3
\]
\[
\quad\quad + (ab + ac + ad + bc + bd + cd)x^2
\]
\[
\quad\quad + (abc + bcd + cda + dab)x + abcd.
\]

Now in these results we see that the following laws hold:

I. The number of terms on the right-hand side is one more than the number of binomial factors which are multiplied together.

II. The exponent of \(x\) in the first term is the same as the number of binomial factors, and in the other terms each exponent is less than that of the preceding term by unity.

III. The coefficient of the first term is unity; the coefficient of the second term is the sum of the second letters of the binomial factors; the coefficient of the third term is the sum of the products of the second letters of the binomial factors taken two at a time; the coefficient of the fourth term is the sum of the products of the second letters of the binomial factors taken three at a time; and so on; the last term is the product of all the second letters of the binomial factors.

We shall shew that these laws always hold, whatever be the number of binomial factors. Suppose the laws to hold when \(n - 1\) factors are multiplied together; that is,
suppose there are \( n-1 \) factors \( x + a, x + b, x + c, \ldots x + k \),
and that
\[
(x + a)(x + b) \ldots (x + k) = x^{n-1} + px^{n-2} + qx^{n-3} + rx^{n-4} + \ldots + u,
\]
where \( p \) = the sum of the letters \( a, b, c, \ldots k \),
\( q \) = the sum of the products of these letters taken two at a time,
\( r \) = the sum of the products of these letters taken three at a time,
\( u \) = the product of all these letters.

Multiply both sides of this identity by another factor \( x + l \), and arrange the product on the right hand according to powers of \( x \); thus
\[
(x + a)(x + b)(x + c) \ldots (x + k) (x + l) = x^n + (p + l)x^{n-1} + (q + pl)x^{n-2} + (r + ql)x^{n-3} + \ldots + ul.
\]

Now \( p + l = a + b + c + \ldots + k + l \)

= the sum of all the letters \( a, b, c, \ldots k, l \);

\( q + pl = q + l(a + b + c + \ldots + k) \)

= the sum of the products taken two at a time of all the letters \( a, b, c, \ldots k, l \);

\( r + ql = r + l(ab + ac + bc + \ldots) \)

= the sum of the products taken three at a time of all the letters \( a, b, c, \ldots k, l \);

\( ul \) = the product of all the letters.

Hence, if the laws hold when \( n-1 \) factors are multiplied together, they hold when \( n \) factors are multiplied together; but they have been shewn to hold when four factors are multiplied together, therefore they hold when five factors are multiplied together, and so on: thus they hold universally.
BINOMIAL THEOREM.

We shall write the result for the multiplication of \( n \) factors thus for abbreviation:
\[
(x + a)(x + b)\ldots(x + k)(x + l) = x^n + P x^{n-1} + Q x^{n-2} + R x^{n-3} + \ldots + V.
\]

Now \( P \) is the sum of the letters \( a, b, c, \ldots k, l \), which are \( n \) in number; \( Q \) is the sum of the products of these letters two and two, so that there are \( \frac{n(n-1)}{1\cdot2} \) of these products; \( R \) is the sum of \( \frac{n(n-1)(n-2)}{1\cdot2\cdot3} \) products; and so on. See Art. 421.

Suppose \( b, c, \ldots k, l \) each equal to \( a \). Then \( P \) becomes \( n a \), \( Q \) becomes \( \frac{n(n-1)}{1\cdot2} a^2 \), \( R \) becomes \( \frac{n(n-1)(n-2)}{1\cdot2\cdot3} a^3 \); and so on. Thus finally
\[
(x + a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1\cdot2} a^2 x^{n-2} + \frac{n(n-1)(n-2)}{1\cdot2\cdot3} a^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4} a^4 x^{n-4} + \ldots + a^n.
\]

426. The formula just obtained is called the Binomial Theorem; the series on the right-hand side is called the expansion of \( (x + a)^n \), and when we put this series instead of \( (x + a)^n \) we are said to expand \( (x + a)^n \). The theorem was discovered by Newton.

It will be seen that we have demonstrated the theorem in the case in which the exponent \( n \) is a positive integer; and that we have used in this demonstration the method of mathematical induction.

427. Take for example \( (x + a)^6 \). Here \( n = 6 \),
\[
\frac{n(n-1)}{1\cdot2} = 6.5 = 15, \quad \frac{n(n-1)(n-2)}{1\cdot2\cdot3} = 6.5.4 = 20,
\]
\[
\frac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4} = 6.5.4.3 = 15,
\]
\[
\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot2\cdot3\cdot4\cdot5} = 6.5.4.3.2 = 6;
\]
Bino\v{m}ial Theorem.

\[(x + a)^6 = x^6 + 6ax^5 + 15a^2x^4 + 20a^3x^3 + 15a^4x^2 + 6a^5x + a^6.\]

Again, suppose we require the expansion of \((b^2 + cy)^6\); we have only to put \(b^2\) for \(x\) and \(cy\) for \(a\) in the preceding identity; thus
\[
(b^2 + cy)^6 = (b^2)^6 + 6cy(b^2)^5 + 15(cy)^2(b^2)^4 + 20(cy)^3(b^2)^3 + 15(cy)^4(b^2)^2 + (cy)^6 = b^{12} + 6cby^{10} + 15c^2y^5b^8 + 20c^3y^3b^6 + 15c^4y^2b^4 + 6c^5y^5b^2 + c^6y^6.
\]

Again, suppose we require the expansion of \((x - c)^n\); we must put \(-c\) for \(a\) in the result of Art. 425; thus
\[
(x - c)^n = x^n - ncx^{n-1} + \frac{n(n-1)}{1 \cdot 2} c^2x^{n-2} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} c^3x^{n-3} + \ldots
\]

Again, in the expansion of \((x + a)^n\) put 1 for \(x\); thus
\[
(1 + a)^n = 1 + na + \frac{n(n-1)}{1 \cdot 2} a^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^3 + \ldots
\]
and as this is true for all values of \(a\) we may put \(x\) for \(a\); thus
\[
(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \ldots
\]

428. We may apply the Binomial Theorem to expand expressions containing more than two terms. For example, required to expand \((1 + 2x - x^2)^4\). Put \(y\) for \(2x - x^2\); then we have \((1 + 2x - x^2)^4 = (1 + y)^4 = 1 + 4y + 6y^2 + 4y^3 + y^4 = 1 + 4(2x - x^2) + 6(2x - x^2)^2 + 4(2x - x^2)^3 + (2x - x^2)^4.\)

Also \((2x - x^2)^2 = (2x)^2 - 2(2x)x^3 + (x^2)^2 = 4x^2 - 4x^3 + x^4,\)
\[
(2x - x^2)^3 = (2x)^3 - 3(2x)^2x^2 + 3(2x)(x^2)^2 - (x^2)^3 = 8x^3 - 12x^4 + 6x^5 - x^6,
\]
\[
(2x - x^2)^4 = (2x)^4 - 4(2x)^3x^2 + 6(2x)^2(x^2)^2 - 4(2x)(x^2)^3 + (x^2)^4 = 16x^4 - 32x^5 + 24x^6 - 8x^7 + x^8.
\]
Hence, collecting the terms, we obtain
\[(1 + 2x - x^3)^4 = 1 + 8x + 20x^2 + 8x^3 - 26x^4 - 8x^5 + 20x^6 - 8x^7 + x^8.\]

429. *In the expansion of \((1 + x)^n\) the coefficients of terms equally distant from the beginning and the end are the same.*

The coefficient of the \(r^{th}\) term from the beginning is
\[\frac{n(n-1)(n-2)\ldots(n-r+2)}{r!};\]
by multiplying both numerator and denominator by \(n-r+1\) this becomes
\[\frac{n}{r!} \cdot \frac{n-r+1}{n-r+1}.\]

The \(r^{th}\) term from the end is the \((n-r+2)^{th}\) term from the beginning, and its coefficient is
\[\frac{n(n-1)\ldots\{n-(n-r+2)+2\}}{n-r+1},\] that is \[\frac{n(n-1)\ldots r}{n-r+1};\]
by multiplying both numerator and denominator by \(r!\) this also becomes
\[\frac{n}{r!} \cdot \frac{n-r+1}{n-r+1}.\]

430. *Hitherto in speaking of the expansion of \((x+a)^n\) we have assumed that \(n\) denotes some positive integer. But the Binomial Theorem is also applied to expand \((x+a)^n\) when \(n\) is a positive fraction, or a negative quantity whole or fractional.* For a discussion of the Binomial Theorem with any exponent the student is referred to the larger Algebra; it will however be a useful exercise to obtain various particular cases from the general formula. Thus the student will assume for the present that whatever be the values of \(x, a,\) and \(n,\)
\[(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1 \cdot 2}a^2x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^3x^{n-3}
+ \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}a^4x^{n-4} + \ldots.\]

*If \(n\) is not a positive integer the series never ends.*
431. As an example take $(1+y)^{\frac{1}{2}}$. Here in the formula of Art. 430 we put 1 for $x$, $y$ for $a$, and $\frac{1}{2}$ for $n$.

\[
\frac{n(n-1)}{1 \cdot 2} = \frac{\frac{1}{2}(1-1)}{1 \cdot 2} = -\frac{1}{8},
\]

\[
\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} = \frac{\frac{1}{2}(1-1)(1-2)}{1 \cdot 2 \cdot 3} = \frac{1}{16},
\]

\[
\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{\frac{1}{2}(1-1)(1-2)(1-3)}{1 \cdot 2 \cdot 3 \cdot 4} = -\frac{5}{128},
\]

and so on. Thus

\[(1+y)^{\frac{1}{2}} = 1 + \frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 - \frac{5}{128}y^4 + \ldots.
\]

As another example take $(1+y)^{-\frac{1}{2}}$. Here we put 1 for $x$, $y$ for $a$, and $-\frac{1}{2}$ for $n$.

\[
n = -\frac{1}{2}, \quad \frac{n(n-1)}{1 \cdot 2} = \frac{3}{8}, \quad \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} = -\frac{5}{16},
\]

\[
\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{35}{128}, \text{ and so on. Thus}
\]

\[(1+y)^{-\frac{1}{2}} = 1 - \frac{1}{2}y + \frac{3}{8}y^2 - \frac{5}{16}y^3 + \frac{35}{128}y^4 - \ldots.
\]

Again, expand $(1+y)^{-m}$. Here we put 1 for $x$, $y$ for $a$, and $-m$ for $n$.

\[
n = -m, \quad \frac{n(n-1)}{1 \cdot 2} = \frac{m(m+1)}{1 \cdot 2},
\]

\[
\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} = -\frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3},
\]

\[
\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{m(m+1)(m+2)(m+3)}{1 \cdot 2 \cdot 3 \cdot 4}, \text{ and so on.}
\]
**EXAMPLES. XLII.**

Thus \((1 + y)^{-m} = 1 - my + \frac{m(m+1)}{1 \cdot 2} y^2 - \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3} y^3 + \frac{m(m+1)(m+2)(m+3)}{1 \cdot 2 \cdot 3 \cdot 4} y^4 - \ldots\)

As a particular case suppose \(m = 1\); thus
\[(1 + y)^{-1} = 1 - y + y^2 - y^3 + y^4 - \ldots\]

This may be verified by dividing 1 by \(1 + y\).

Again, expand \((1 + 2x - x^2)^{\frac{1}{2}}\) in powers of \(x\). Put \(y\) for \(2x - x^2\); thus we have \((1 + 2x - x^2)^{\frac{1}{2}} = (1 + y)^{\frac{1}{2}}\)

\[
= 1 + \frac{1}{2} y - \frac{1}{8} y^2 + \frac{1}{16} y^3 - \frac{5}{128} y^4 + \ldots
\]

\[
= 1 + \frac{1}{2} (2x - x^2) - \frac{1}{8} (2x - x^2)^2 + \frac{1}{16} (2x - x^2)^3 - \frac{5}{128} (2x - x^2)^4 + \ldots
\]

Now expand \((2x - x^2)^2\), \((2x - x^2)^3\), \ldots and collect the terms; thus we shall obtain

\[
(1 + 2x - x^2)^{\frac{1}{2}} = 1 + x - x^2 + x^3 - \frac{3}{2} x^4 + \ldots
\]

**EXAMPLES. XLII.**

1. Write down the first three and the last three terms of \((a - x)^{13}\).
2. Write down the expansion of \((3 - 2x^2)^5\).
3. Expand \((1 - 2y)^7\).
4. Write down the first four terms in the expansion of \((x + 2y)^n\).
5. Expand \((1 + x - x^2)^4\).
6. Expand \((1 + x + x^2)^6\).
7. Expand \((1 - 2x + x^2)^4\).

8. Find the coefficient of \(x^5\) in the expansion of \((1 + 2x + 3x^2)^7\).

9. Find the coefficient of \(x^5\) in the expansion of \((1 - 2x + 3x^2)^5\).

10. If the second term in the expansion of \((x + y)^n\) be 240, the third term 720, and the fourth term 1080, find \(x, y,\) and \(n\).

11. If the sixth, seventh, and eighth terms in the expansion of \((x + y)^n\) be respectively 112, 7, and \(\frac{1}{4}\), find \(x, y,\) and \(n\).

12. Write down the first five terms of the expansion of \((a - 2x)^\frac{1}{3}\).

13. Expand to four terms \(\left(1 - \frac{5}{6}x\right)^{-\frac{3}{5}}\).

14. Expand \((1 - 2x)^{-1}\).

15. Write down the coefficient of \(x^r\) in the expansion of \((1 - x)^{-2}\).

16. Write down the sixth term in the expansion of \((3x - y)^{-\frac{3}{4}}\).

17. Expand to five terms \((a - 3b)^{-\frac{1}{5}}\): shew that if \(a = 1\) and \(b = \frac{1}{5}\) the fourth term is greater than either the third or the fifth.

18. Write down the coefficient of \(x^r\) in the expansion of \((1 - x)^{-4}\).

19. Expand \((1 + x + x^2)^\frac{1}{3}\) to four terms in powers of \(x\).

20. Expand \((1 - x + x^2)^{-\frac{1}{2}}\) to four terms in powers of \(x\).
XLIII. Scales of Notation.

432. The student will of course have learned from Arithmetic that in the ordinary method of expressing whole numbers by figures, the number represented by each figure is always some multiple of some power of ten. Thus in 523 the 5 represents 5 hundreds, that is 5 times 10²; the 2 represents 2 tens, that is 2 times 10¹; and the 3, which represents 3 units, may be said to represent 3 times 10⁰; see Art. 324.

This mode of expressing whole numbers is called the common scale of notation, and ten is said to be the base or radix of the common scale.

433. We shall now shew that any positive integer greater than unity may be used instead of 10 for the radix; and then explain how a given whole number may be expressed in any proposed scale.

The figures by means of which a number is expressed are called digits. When we speak in future of any radix we shall always mean that this radix is some positive integer greater than unity.

434. To shew that any whole number may be expressed in terms of any radix.

Let \( N \) denote the whole number, \( r \) the radix. Suppose that \( r^n \) is the highest power of \( r \) which is not greater than \( N \); divide \( N \) by \( r^n \); let the quotient be \( a \), and the remainder \( P \); thus

\[
N = ar^n + P.
\]

Here, by supposition, \( a \) is less than \( r \), and \( P \) is less than \( r^n \). Divide \( P \) by \( r^{n-1} \); let the quotient be \( b \), and the remainder \( Q \); thus

\[
P = br^{n-1} + Q.
\]

Proceed in this way until the remainder is less than \( r \): thus we find \( N \) expressed in the manner shewn by the following identity,

\[
N = ar^n + br^{n-1} + cr^{n-2} + \ldots + hr + k.
\]
Each of the digits \( a, b, c, \ldots, h, k \) is less than \( r \); and any one or more of them after the first may happen to be zero.

435. To express a given whole number in any proposed scale.

By a given whole number we mean a whole number expressed in words, or else expressed by digits in some assigned scale. If no scale is mentioned the common scale is to be understood.

Let \( N \) be the given whole number, \( r \) the radix of the scale in which it is to be expressed. Suppose \( k, h, \ldots, c, b, a \) the required digits, \( n + 1 \) in number, beginning with that on the right hand: then

\[
N = ar^n + br^{n-1} + cr^{n-2} + \ldots + hr + k.
\]

Divide \( N \) by \( r \), and let \( M \) be the quotient; then it is obvious that \( M = ar^{n-1} + br^{n-2} + \ldots + h \), and that the remainder is \( k \). Hence the first digit is found by this rule: divide the given number by the proposed radix, and the remainder is the first of the required digits.

Again, divide \( M \) by \( r \); then it is obvious that the remainder is \( h \); and thus the second of the required digits is found.

By proceeding in this way we shall find in succession all the required digits.

436. We shall now solve some examples.

Transform 32884 into the scale of which the radix is seven.

\[
\begin{array}{c|c|c|c|c|c}
7 & 32884 \\
7 & 4697 \ldots 5 \\
7 & 671 \ldots 0 \\
7 & 95 \ldots 6 \\
7 & 13 \ldots 4 \\
& 1 \ldots 6
\end{array}
\]

Thus \( 32884 = 1 \cdot 7^5 + 6 \cdot 7^4 + 4 \cdot 7^3 + 6 \cdot 7^2 + 0 \cdot 7^1 + 5 \), so that the number expressed in the scale of which the radix is seven is 164605.
Scales of Notation.

Transform 74194 into the scale of which the radix is twelve.

\[
\begin{array}{c|c}
12 & 74194 \\
12 & 6182 \ldots 10 \\
12 & 515 \ldots 2 \\
12 & 42 \ldots 11 \\
& 3 \ldots 6 \\
\end{array}
\]

Thus \(74194 = 3 \cdot 12^4 + 6 \cdot 12^3 + 11 \cdot 12^2 + 2 \cdot 12 + 10\).

In order to express the number in the scale of which the radix is twelve in the usual manner, we require two new symbols, one for ten, and the other for eleven: we will use \(t\) for the former, and \(e\) for the latter. Thus the number expressed in the scale of which the radix is twelve is \(36e2t\).

Transform 645032, which is expressed in the scale of which the radix is nine, into the scale of which the radix is eight.

\[
\begin{array}{c|c}
8 & 645032 \\
& 72782 \ldots 4 \\
\end{array}
\]

The division by eight is performed thus: First eight is not contained in 6, so we have to find how often eight is contained in 64; here 6 stands for six times nine, that is fifty-four, so that the question is how often is eight contained in fifty-eight, and the answer is seven times with two over. Next we have to find how often eight is contained in 25, that is how often is eight contained in twenty-three, and the answer is twice with seven over. Next we have to find how often eight is contained in 70, that is how often is eight contained in sixty-three, and the answer is seven times with seven over. Next we have to find how often eight is contained in 73, that is how often is eight contained in sixty-six, and the answer is eight times with two over. Next we have to find how often eight is contained in 22, that is how often is eight contained in twenty, and the answer is twice with four over. Thus 4 is the first of the required digits.

We will indicate the remainder of the process; the student should carefully work it for himself, and then com-
pare his result with that which is here obtained.

\[
\begin{array}{c|c}
8 & 72782 \\
8 & 8210 \ldots 2 \\
8 & 1028 \ldots 3 \\
8 & 113 \ldots 6 \\
8 & 12 \ldots 5 \\
1 & \ldots 3.
\end{array}
\]

Thus the number \(= 1.8^6 + 3.8^5 + 5.8^4 + 6.8^3 + 3.8^2 + 2.8 + 4\), so that, expressed in the scale of which the radix is eight, it is 1356324.

437. It is easy to form an unlimited number of self-verifying examples. Thus, take two numbers, expressed in the common scale, and obtain their sum, their difference, and their product, and transform these into any proposed scale; next transform the numbers into the proposed scale, and obtain their sum, their difference, and their product in this scale; the results should of course agree respectively with those already obtained.

Examples. XLIII.

1. Express 34042 in the scale whose radix is five.
2. Express 45792 in the scale whose radix is twelve.
3. Express 1866 in the scale whose radix is two.
4. Express 2745 in the scale whose radix is eleven.
5. Multiply \(e4t\) by \(te\); these being in the scale with radix twelve; transform them to the common scale and multiply them together.
6. Find in what scale the number 4161 becomes 10101.
7. Find in what scale the number 5261 becomes 40205.
8. Express 17161 in the scale whose radix is twelve, and divide it by \(te\) in that scale.
9. Find the radix of the scale in which 13, 22, 33 are in geometrical progression.
10. Extract the square root of \(eel001\), in the scale whose radix is twelve.
XLIV. Interest.

438. The subject of Interest is discussed in treatises on Arithmetic; but by the aid of Algebraical notation the rules can be presented in a form easy to understand and to remember.

439. Interest is money paid for the use of money. The money lent is called the Principal. The Amount at the end of a given time is the sum of the Principal and the Interest at the end of that time.

440. Interest is of two kinds, simple and compound. When interest is charged on the Principal alone it is called simple interest; but if the interest as soon as it becomes due is added to the principal, and interest charged on the whole, it is called compound interest.

441. The rate of interest is the money paid for the use of a certain sum for a certain time. In practice the sum is usually £100, and the time is one year; and when we say that the rate is £4. 5s. per cent. we mean that £4. 5s., that is £4½, is paid for the use of £100 for one year. In theory it is convenient, as we shall see, to use a symbol to denote the interest of one pound for one year.

442. To find the amount of a given sum in any given time at simple interest.

Let $P$ be the number of pounds in the principal, $n$ the number of years, $r$ the interest of one pound for one year, expressed as a fraction of a pound, $M$ the number of pounds in the amount. Since $r$ is the interest of one pound for one year, $Pr$ is the interest of $P$ pounds for one year, and $nPr$ is the interest of $P$ pounds for $n$ years; therefore

$$M = P + Pnr = P(1 + nr).$$

443. From the equation $M = P(1 + nr)$, if any three of the four quantities $M$, $P$, $n$, $r$ are given, the fourth can be found: thus

$$P = \frac{M}{1 + nr}, \quad n = \frac{M - P}{Pr}, \quad r = \frac{M - P}{Pn}.$$
444. To find the amount of a given sum in any given time at compound interest.

Let \( P \) be the number of pounds in the principal, \( n \) the number of years, \( r \) the interest of one pound for one year, expressed as a fraction of a pound, \( M \) the number of pounds in the amount. Let \( R \) denote the amount of one pound in one year; so that \( R = 1 + r \). Then \( PR \) is the amount of \( P \) pounds in one year. The amount of \( PR \) pounds in one year is \( PRR \), or \( PR^2 \); which is therefore the amount of \( P \) pounds in two years. Similarly the amount of \( PR^2 \) pounds in one year is \( PR^2 R \), or \( PR^3 \), which is therefore the amount of \( P \) pounds in three years.

Proceeding in this way we find that the amount of \( P \) pounds in \( n \) years is \( PR^n \); that is

\[
M = PR^n.
\]

The interest gained in \( n \) years is

\[
PR^n - P \text{ or } P(R^n - 1).
\]

445. The Present value of an amount due at the end of a given time is that sum which with its interest for the given time will be equal to the amount. That is, the Principal is the present value of the Amount; see Art. 439.

446. Discount is an allowance made for the payment of a sum of money before it is due.

From the definition of present value it follows that a debt is fairly discharged by paying the present value at once; hence the discount is equal to the amount due diminished by its present value.

447. To find the present value of a sum of money due at the end of a given time, and the discount.

Let \( P \) be the number of pounds in the present value, \( n \) the number of years, \( r \) the interest of one pound for one year expressed as a fraction of a pound, \( M \) the number of pounds in the sum due, \( D \) the discount.

Let \( R = 1 + r \).
At simple interest
\[ M = P(1 + nr), \text{ by Art. 442;} \]
therefore \( P = \frac{M}{1 + nr}; \quad D = M - P = \frac{Mnr}{1 + nr}. \)

At compound interest
\[ M = PR^n, \text{ by Art. 444;} \]
therefore \( P = \frac{M}{R^n}; \quad D = M - P = \frac{M(R^n - 1)}{R^n}. \)

448. In practice it is very common to allow the interest of a sum of money paid before it is due instead of the discount as here defined. Thus at simple interest instead of \( \frac{Mnr}{1 + nr} \) the payer would be allowed \( Mnr \) for immediate payment.

**Examples. XLIV.**

1. At what rate per cent. will £a produce the same interest in one year as £b produces when the rate is £c per cent.?

2. Shew that a sum of money at compound interest becomes greater at a given rate per cent. for a given number of years than it does at twice that rate per cent. for half that number of years.

3. Find in how many years a sum of money will double itself at a given rate of simple interest.

4. Shew, by taking the first three terms of the Binomial series for \((1 + r)^n\), that at five per cent. compound interest a sum of money will be more than doubled in fifteen years.
Miscellaneous Examples.

1. Find the values when \( a = 5 \) and \( b = 4 \) of
   \[ a^3 + 3a^2b + 3ab^2 + b^3, \] of \( a^2 + 10ab + 9b^2, \] of \( (a - b)^3, \)
   and of \( (a + 9b)(a - b). \)

2. Simplify \( 5x - 3 \left[ 2x + 9y - 2 \{ 3x - 4(y - x) \} \right]. \)

3. Square \( 3 - 5x + 2x^2. \)

4. Divide \( 1 \) by \( 1 - x + x^2 \) to four terms: also divide \( 1 - x \) by \( 1 - x^3 \) to four terms.

5. Simplify \( \frac{4x^3 - 17x + 12}{6x^2 - 17x + 12}. \)

6. Find the L.C.M. of \( 4x^2 - 9, \ 6x^2 - 5x - 6, \) and \( 6x^2 + 5x - 6. \)

7. Simplify \( \frac{x + a}{a - x} - 2 \frac{x + a}{x + a} + 2 \frac{x + a}{x + a}. \)

8. Solve \( \frac{x - 2}{3} + \frac{x + 5}{6} = \frac{7x - 6}{9}. \)

9. The first edition of a book had 600 pages and was divided into two parts. In the second edition one quarter of the second part was omitted, and 30 pages were added to the first part; this change made the two parts of the same length. Find the number of pages in each part in the first edition.

10. In paying two bills, one of which exceeded the other by one third of the less, the change out of a £5 note was half the difference of the bills; find the amount of each bill.

11. Add together \( y + \frac{1}{2}z - \frac{1}{3}x, \frac{1}{2}x - \frac{1}{3}y, x + \frac{1}{2}y - \frac{1}{3}z; \)
and from the result subtract \( \frac{1}{6}x - y - \frac{1}{3}z. \)
12. If \( a = 1, \, b = 3, \) and \( c = 5, \) find the value of \[
\frac{2a^3 + b^3 + c^3 + a^2(b - c) + b^2(2a - c) + c^2(2a + b)}{2a^3 - b^3 + c^3 + a^2(b - c) - b^2(2a - c) + c^2(2a + b)}.
\]

13. Simplify \((a + b)^2 - (a + b)(a - b) - \{a(2b - 2) - (b^2 - 2a)\}\).

14. Divide \(2x^5 - x^4y - 4x^3y^2 + 5x^2y^3 - 4y^5\) by \(x^3 - xy^2 + 2y^3\).

15. Reduce to its lowest terms \[
\frac{x^4 - 2x^3 + x^2 - 1}{x^4 + x^2 + 1}.
\]

16. Find the L.C.M. of \(x^2 - 9x - 10, \, x^2 - 7x - 30, \) \((x + 1)(x + 3)(x - 10)\), and \(x^2 + 4x + 3\).

17. Simplify \[
\frac{2}{x^2 - 9x - 10} + \frac{3}{x^2 - 7x - 30} - \frac{5}{x^2 + 4x + 3}.
\]

18. Solve \(x - \frac{x - 2}{3} = \frac{x + 15}{4} - \frac{x}{5}\).

19. Solve \[
\frac{3}{2} (x - 1) - \frac{2}{3} (x + 2) + \frac{1}{4} (x - 3) = 4.
\]

20. Two persons \(A\) and \(B\) own together 175 shares in a railway company. They agree to divide, and \(A\) takes 85 shares, while \(B\) takes 90 shares and pays £100 to \(A\). Find the value of a share.

21. Add together \(a + 2x - y + 24b, \, 3a - 4x - 2y - 81b, \) \(x + y - 2a + 55b\); and subtract the result from \(3a + b + 3x + 2y\).

22. Find the value of \[
\frac{a^2b}{7} + \sqrt[7]{7ab(2c^2 - ab)} - (2a - 3b)^3,
\]
when \(a = 3, \, b = 2\frac{1}{3}, \) and \(c = 2\).

23. Simplify \(\{x(x + a) - a(x - a)\} \{x(x - a) - a(a - x)\}\).

24. Divide \[
\frac{x^3 - x}{6} + \frac{1}{8} - \frac{5x^2}{36} \]
by \(\frac{x}{3} - \frac{1}{2}\); and verify the result by multiplication.

25. Find the g.c.m. of \(x^4 + 3x^2 - 10\) and \(x^4 - 3x^2 + 2\).
26. Simplify \( \frac{2a^2}{b^2-4a^2} - \frac{b}{b+2a} + \frac{a}{2a-b} \).

27. Find the \text{l.c.m.} of \( x^2-4 \), \( 4x^2-7x-2 \), and \( 4x^2+7x-2 \).

28. Solve \( \frac{2x}{3} - \frac{x-1}{15} + \frac{\frac{1}{3}x-1}{6} = 4 \).

29. A man bought a suit of clothes for £4. 7s. 6d. The trousers cost half as much again as the waistcoat, and the coat half as much again as the trousers and waistcoat together. Find the price of each garment.

30. A farmer sells a certain number of bushels of wheat at 7s. 6d. per bushel, and 200 bushels of barley at 4s. 6d. per bushel, and receives altogether as much as if he had sold both wheat and barley at the rate of 5s. 6d. per bushel. How much wheat did he sell?

31. If \( a=1 \), \( b=2 \), \( c=-\frac{1}{2} \), \( d=0 \), find the value of

\[
\frac{a-b+c}{a-b-c} - \frac{ad-bc}{bd+ac} - \sqrt{\left(\frac{b^3-a^3}{a^3-c^3}\right)}.
\]

32. Multiply together \( x-a \), \( x-b \), \( x+a \), and \( x+b \); and divide the result by \( x^2+x(a+b)+ab \).

33. Divide \( 8x^5-x^2y^3+\frac{1}{2}y^5 \) by \( 2x+y \).

34. Find the g.c.m. of \( 4x(x^2+10)-25x-62 \) and \( x^2-7x+10 \).

35. Reduce to its lowest terms \( \frac{12x^2-15xy+3y^2}{6x^3-6x^2y+2xy^2-2y^2} \).

36. Simplify \( \frac{1}{1+\frac{a}{a+b+c}} - \frac{a}{b+c+d} \).

37. Solve \( \frac{x-1}{9} - \frac{2-x}{4} - \frac{2x-1}{14} + \frac{2-3x}{30} = 0 \).
38. Solve \[ \frac{2x-1}{3} - \frac{x+4}{9} = \frac{5x-1}{27} \].

39. \( A \) can do a piece of work in one hour, \( B \) and \( C \) each in two hours: how long would \( A, B, \) and \( C \) take, working together?

40. \( A \) having three times as much money as \( B \) gave two pounds to \( B \), and then he had twice as much as \( B \) had. How much had each at first?

41. Add together \( 2x+3y+4z \), \( x-2y+5z \), and \( 7x-y+z \).

42. Find the sum, the difference, and the product of \( 3x^2-4xy+4y^2 \) and \( 4x^2+2xy-3y^2 \).

43. Simplify
\[ 2a - 3(b-c) + \{a - 2(b-c)\} - 2\{a - 3(b-c)\} \].

44. Find the g.c.m. of \( x^4+67x^2+66 \) and \( x^4+2x^3+2x^2+2x+1 \).

45. Simplify \( \frac{x^4-1}{x^3-1} \times \frac{x+1}{x^4+2x^3+2x^2+2x+1} \).

46. Find the l.c.m. of \( x^2-4 \), \( x^2-5x+6 \), and \( x^2-9 \).

47. Reduce to its lowest terms \( \frac{3x^3-4x^2-x-14}{6x^3-11x^2-10x+7} \).

48. Solve \( 3(x-1)-4(x-2)=2(3-x) \).

49. Solve \( \sqrt{9+4x} = 5-2\sqrt{x} \).

50. How much tea at 3s. 9d. per lb. must be mixed with 45 lbs. at 3s. 4d. per lb. that the mixture may be worth 3s. 6d. per lb.?

51. Multiply \( 3a^2+ab-b^2 \) by \( a^2-2ab-3b^2 \), and divide the product by \( a+b \).

52. Find the g.c.m. of \( 2x(x-3)+3(x-6\frac{2}{3})+15 \) and \( 2x^3-5x^2-6x+15 \).

53. Simplify \( \frac{1}{1-x} + \frac{1}{1+x} \).
54. Simplify \( \frac{(a+b)^2}{a-b} \div \frac{ab+b^2}{a^2-ab} \).

55. Solve \( \frac{1}{y} + \frac{2}{x} = \frac{2x + 3}{xy}, \quad \frac{1 - 2x^2}{x} = \frac{y}{x} - (1 + 2x) \).

56. Solve \( x + \frac{3}{y} = \frac{7}{2}, \quad 3x - \frac{2}{y} = \frac{26}{3} \).

57. Solve \( 2(x-3) - \frac{1}{5}(y-3) = 3, \quad 3(y-5) + \frac{1}{3}(x-2) = 10 \).

58. Solve \( 7yz = 10(y+z), \quad 3zx = 4(z+x), \quad 9xy = 20(x+y) \).

59. Solve \( \frac{a}{x} + \frac{b}{y} = m, \quad \frac{b}{x} - \frac{a}{y} = n \).

60. The denominator of a certain fraction exceeds the numerator by 2; if the numerator be increased by 5 the fraction is increased by unity: find the fraction.

61. Divide \( x^5 - \frac{1}{x^3} \) by \( x - \frac{1}{x} \).

62. Reduce to its lowest terms \( \frac{33x^2 - 49x - 10}{21x^3 - 14x^2 - 29x - 10} \).

63. Simplify \( \left( a - \frac{2a}{x + \frac{1}{x}} \right) \div \left( \frac{x}{2} + \frac{1}{2x} - 1 \right) \).

64. Solve \( 3(x-1) + 2(x-2) = x - 3 \).

65. Solve \( \frac{x-1}{3} = \frac{y+1}{4}, \quad \frac{2x-3}{5} = \frac{13-2y}{7} \).

66. Solve \( 5x + 2 = 3y, \quad 6xy - 10x^3 + \frac{y-2x}{a} = 8 \).

67. Solve \( \frac{x+y}{7} - \frac{2y-x}{3} = 3, \quad \frac{3y+2x}{4} + \frac{9(x-1)}{8} = \frac{x}{2} \).
68. Solve $\sqrt{(x^2 + 40)} = x + 4$.

69. Solve $\frac{x^2 + 3x + 2}{x + 1} - \frac{x^2 - x - 6}{x + 2} = \frac{5x}{2}$.

70. A father's age is double that of his son; 10 years ago the father's age was three times that of his son: find the present age of each.

71. Find the value when $x = 4$ of

$$\sqrt{(2x + 1)} - \left(x + \frac{6}{\sqrt{x}}\right) - \left(3 - \frac{x^2}{4 - 3\sqrt{2x}}\right).$$

72. Reduce $\frac{3x^3 - 16x^2 + 23x - 6}{2x^3 - 11x^2 + 17x - 6}$ to its lowest terms; and find its value when $x = 3$.

73. Resolve into simple factors $x^3 - 3x + 2$, $x^3 - 7x + 10$, and $x^3 - 6x + 5$.

74. Simplify $\frac{1}{x^3 - 3x + 2} + \frac{3}{x^3 - 7x + 10} - \frac{4}{x^3 - 6x + 5}$.

75. Solve $\frac{1}{14} \left(3x + \frac{11}{3}\right) - \frac{1}{7} (4x - 2\frac{2}{3}) = \frac{1}{2} (5x - 1)$.

76. Solve $9x^2 - 63x + 68 = 0$.

77. A man and a boy being paid for certain days' work, the man received 27 shillings and the boy who had been absent 3 days out of the time received 12 shillings: had the man instead of the boy been absent those 3 days they would both have claimed an equal sum. Find the wages of each per day.

78. Extract the square root of $9x^4 - 6x^3 + 7x^2 - 2x + 1$; and shew that the result is true when $x = 10$.

79. If $a : b :: c : d$, shew that

$$a^2c + ac^2 : b^2d + bd^2 :: (a + c)^3 : (b + d)^3.$$ 

80. If $a, b, c, d$ be in geometrical progression, show that $a^2 + d^2$ is greater than $b^2 + c^2$.

81. If $n$ is a whole positive number $7^{2n+1} + 1$ is divisible by 8.
82. Find the least common multiple of \( x^2 - 4y^2 \), \( x^3 + 6x^2y + 12xy^2 + 8y^3 \), and \( x^3 - 6x^2y + 12xy^2 - 8y^3 \).

83. Solve \( \frac{3}{x} + \frac{1}{y} = \frac{1}{2} \), \( \frac{4}{x} - \frac{3}{y} = 2 \frac{5}{6}. \)

84. Solve \( x^2 + 2x + 2 \sqrt{x^2 + 2x + 1} = 47. \)

85. The sum of a certain number consisting of two digits and of the number formed by reversing the digits is 121; and the product of the digits is 28: find the number.

86. Nine gallons are drawn from a cask full of wine, and it is then filled up with water; then nine gallons of the mixture are drawn, and the cask is again filled up with water. If the quantity of wine now in the cask be to the quantity of water in it as 16 is to 9, find how much the cask holds.

87. Extract the square root of

\[ 16x^6 + 25y^6 - 30xy^5 - 24x^4y^2 + 9x^2y^4 + 40x^3y^3. \]

88. In an arithmetical progression the first term is 81, and the fourteenth is 159. In a geometrical progression the second term is 81, and the sixth is 16. Find the harmonic mean between the fourth terms of the two progressions.

89. If \( \sqrt{5} = 2.23606 \), find the value to five places of decimals of \( \frac{6}{\sqrt{5} - 1} \).

90. If \( x \) be greater than 9, show that \( \sqrt{x} \) is greater than \( \sqrt{x + 18} \).

91. Divide \( (x - y)^3 - 2y(x - y)^2 + y^2(x - y) \) by \( (x - 2y)^2 \).

92. Find the g.c.m. and the l.c.m. of

\[ 24(x^3 + x^2y + xy^2 + y^3) \text{ and } 16(x^3 - x^2y + xy^2 - y^3). \]

93. Simplify

\[ \frac{x}{x^3 + x^2y + xy^2 + y^3} + \frac{y}{x^3 - x^2y + xy^2 - y^3} + \frac{1}{x^2 - y^2} - \frac{1}{x^2 + y^2}. \]
94. Solve \( \frac{6x + 7}{13} + \frac{2x + 5}{7} = 3 - \frac{8x + 1}{9} \).

95. Solve
\[ xy + 20(x - y) = 0, \quad yz + 30(y - z) = 0, \quad 3x - 2z = 0. \]

96. Solve \( 3x^2 - 2x + \sqrt{(3x^2 - 4x - 6)} = 18 + 2x. \)

97. A rows at the rate of \( 8 \frac{1}{3} \) miles an hour. He leaves Cambridge at the same time that \( B \) leaves Ely. \( A \) spends 12 minutes in Ely and is back in Cambridge 2 hours and 20 minutes after \( B \) gets there. \( B \) rows at the rate of \( 7 \frac{1}{2} \) miles an hour; and there is no stream. Find the distance from Cambridge to Ely.

98. An apple woman finding that apples have this year become so much cheaper that she could sell 60 more than she used to do for five shillings, lowered her price and sold them one penny per dozen cheaper. Find the price per dozen.

99. Sum to 8 terms and to infinity \( 12 + 4 + 1 \frac{1}{3} + \ldots \).

100. Find three numbers in geometrical progression such that if 1, 3, and 9 be subtracted from them in order they will form an arithmetical progression whose sum is 15.

101. Multiply \( x^7 - x^5 + x^3 - x^2 + x^2 - x + x^3 - 1 \) by \( x^4 + 1 \); and divide \( 1 - x^3 \) by \( 1 - x^4 \).

102. Find the L.C.M. of \( x^3 - a^3, \ x^3 + a^3, \ x^4 + a^2x^2 + a^4, \ x^3 - ax^2 - a^2x + a^3, \) and \( x^3 + ax^2 - a^2x - a^3. \)

103. Simplify \( \frac{a^3 - b^3}{a^2 - b^2 + \frac{2b^3}{1 + \frac{a + b}{a - b}}} \).

104. Solve
\[ \frac{x + 5}{6} + \frac{1}{9} \left( \frac{x}{2} + \frac{2}{5} \right) - \frac{2}{3} (3 + 2x) = \frac{4x - 14}{3} + x + 10. \]

105. Solve \( \frac{6}{x - 1} + \frac{8}{x - 5} - \frac{7}{x + 1} + \frac{18}{x + 5}. \)
106. Solve \( x^2 + y^2 + z^2 = 50, \)
\( yz + xy - zx = 7, \)
\( xy - yz - zx = 47. \)

107. A and B travel 120 miles together by rail. B intending to come back again takes a return ticket for which he pays half as much again as A; and they find that B travels cheaper than A by 4s. 2d. for every 100 miles. Find the price of A's ticket.

108. Find a third proportional to the harmonic mean between 3 and \( \frac{3}{7} \), and the geometric mean between 2 and 18.

109. Extract the square root of
\[ \frac{x}{y} \left( 2 + \frac{x}{y} \right) - \frac{y}{x} \left( 2 - \frac{y}{x} + \frac{x}{y} \right). \]

110. If \( a : b :: b : c \), shew that \( b^4 = \frac{a^2 - b^2 + c^2}{a^{-2} - b^{-2} + c^{-2}}. \)

111. Divide \( x^{\frac{3n}{2}} - x^{-\frac{3n}{2}} \) by \( x^n - x^{-n} \).

112. Reduce \( \frac{x^3 + 3x^2 - 20}{x^4 - x^2 - 12} \) to its lowest terms, and find its value when \( x = 2. \)

113. Solve \( \frac{x - 3}{x - 4} - \frac{13}{3} - \frac{x + 2}{3(6-x)}. \)

114. Find the values of \( m \) for which the equation \( m^2x^2 + (m^2 + m)ax + a^2 = 0 \) will have its roots equal to one another.

115. Solve \( 3xy + x^2 = 10, \quad 5xy - 2x^2 = 2. \)

116. Solve \( \frac{1}{x} + \frac{1}{y} = 5, \quad \frac{x}{y} + \frac{y}{x} = 2\frac{1}{2}. \)

117. Find the fraction such that if you quadruple the numerator and add 3 to the denominator the fraction is doubled; but if you add 2 to the numerator and quadruple the denominator the fraction is halved.
118. Simplify \( \{ -(x^3)^{\frac{1}{3}} \}^{-\frac{1}{2}} \times \{ -(\frac{1}{x})^{-3} \}^{\frac{1}{2}} \).

119. The third term of an arithmetical progression is 18; and the seventh term is 30: find the sum of 17 terms.

120. If \( \frac{a+b}{2}, \frac{b+c}{2} \) be in harmonical progression, shew that \( a, b, c \) are in geometrical progression.

121. Simplify \( a - \frac{1}{b + \frac{1}{b + \frac{ab}{a-b}}} \).

122. Extract the square root of
\[ 37x^2y^2 - 30x^2y + 9x^4 - 20xy^3 + 4y^4. \]

123. Resolve \( 3x^3 - 14x^2 - 24x \) into its simple factors.

124. Solve \( \frac{x + 5}{2x - 1} - \frac{3(5x + 1)}{5x + 4} = \frac{4}{2x - 1} - 2\frac{1}{2}. \)

125. Solve \( x^3 + \frac{1}{x^3} = \frac{65}{8}. \)

126. Solve \( x^2 - y^2 = 9, \quad x + 4 = 3(y - 1). \)

127. Solve \( y + \sqrt{(x^2 - 1)} = 2, \quad \sqrt{(x + 1)} - \sqrt{(x - 1)} = \sqrt{y}. \)

128. If \( a, b, c, d \) are in Geometrical Progression,
\[ a : b + d :: c^2 : c^2d + d^3. \]

129. The common difference in an arithmetical progression is equal to 2, and the number of terms is equal to the second term: find what the first term must be that the sum may be 35.

130. Sum to \( n \) terms the series whose \( m^{th} \) term is \( 2 \times 3^m. \)

131. Simplify \( \frac{1 + \sqrt{(1 - 2x)}}{1 - \sqrt{(1 - 2x)}} + \frac{x - \sqrt{(1 - 2x)}}{x}. \)

132. Find the g.c.m. of \( 30x^4 + 16x^3 - 50x^2 - 24x \) and \( 24x^4 + 14x^3 - 48x^2 - 32x \).
133. Solve \( x^2 - x - 12 = 0 \).

134. Form a quadratic equation whose roots shall be 3 and -2.

135. Solve \( x^4 + \frac{1}{x^4} = a^4 + \frac{1}{a^4} \).

136. Solve \( \frac{x^2}{\sqrt{x^2 + 5}} = 1 + \frac{1}{\sqrt{x^2 + 5}} \).

137. Having given \( \sqrt{3} = 1.73205 \), find the value of \( \frac{6}{\sqrt{3} - 1} \) to five places of decimals.

138. Extract the square root of \( 61 - 28 \sqrt{3} \).

139. Find the mean proportional between \( \frac{x + y}{x - y} \) and \( \frac{x^2 - y^2}{x^2 y^2} \).

140. If \( a, b, c \) be the first, second and last terms of an arithmetical progression, find the number of terms. Also find the sum of the terms.

141. If \( a, c, b, a \) are 2, 3, 4, 5, find the values of
\[
\frac{a+b+c}{a-b+c}, \quad \frac{ab-cd}{ac-bd}, \quad \text{and} \quad \sqrt{\frac{a-1}{b-3}}.
\]

142. In the product of \( 1 + 4x + 7x^2 + 10x^3 + 15x^4 \) by \( 1 + 5x + 9x^2 + 13x^3 + 17x^4 \), find the coefficient of \( x^4 \).

Divide \( 21x^5 - 2x^4 - 70x^3 - 23x^2 + 33x + 27 \) by \( 7x^2 + 4x - 9 \).

143. Simplify \( \frac{a^4 - b^4}{a^2 + b^2 + 2ab} \div \frac{a-b}{a^2 + ab} \), and
\[
\frac{\sqrt{x}}{\sqrt{x} - \sqrt{a}} - \frac{\sqrt{a}}{\sqrt{x} + \sqrt{a}} - \frac{x-a}{x+a}.
\]
144. Solve the following equations:

(1) \[ \frac{60 - x}{14} - \frac{3x - 5}{7} = 6 - \frac{24 - 3x}{4} \]

(2) \[ \frac{x + 4}{x + 3} = \frac{5x + 12}{43x + 9} \]

(3) \[ \frac{3x + 5y}{20} + \frac{5x - 3y}{8} = 3, \quad \frac{x + 1}{y + 2} = \frac{2}{3} \]

145. Solve the following equations:

(1) \[ \frac{20}{8 - x} + \frac{21}{6 - x} = 11 \]

(2) \[ \sqrt{\frac{x}{2}} + \sqrt{3x + 1} = 7 \]

(3) \[ 3x^2 - 4xy = 7, \quad 3xy - 4y^2 = 5 \]

146. A bill of £20 is paid in sovereigns and crowns, and 32 pieces are used: find how many there were of each kind.

147. A herd cost £180, but on 2 oxen being stolen, the rest average £1 a head more than at first: find the number of oxen.

148. Find two numbers when their sum is 40, and the sum of their reciprocals is \( \frac{5}{48} \).

149. Find a mean proportional to \( 2\frac{1}{2} \) and \( 5\frac{5}{8} \); and a third proportional to 100 and 130.

150. If 8 gold coins and 9 silver coins are worth as much as 6 gold coins and 19 silver ones, find the ratio of the value of a gold coin to that of a silver coin.

151. Remove the brackets from

\[ (x - a)(x - b)(x - c) - [bc(x - a) - \{(a + b + c)x - a(b + c)\}x] \]

152. Multiply \( a + 2\sqrt{a^2 b} + 2\sqrt{b} \) by \( a - 2\sqrt{a^2 b} + 2\sqrt{b} \).

153. Find the g.c.m. of \( x^4 - 16x^3 + 93x^2 - 234x + 216 \) and \( 4x^3 - 48x^2 + 186x - 234 \).
154. Solve the following equations:

(1) \[ \frac{13x-1}{4} - \frac{28-5x}{3} = 17 - \frac{3x+1}{8}. \]

(2) \[ \frac{2x+3}{3x+9} = \frac{2x-8}{3x-13}. \]

(3) \[ x - y = 3, \quad 3 \left( \frac{1}{y} + \frac{1}{x} \right) = 11 \left( \frac{1}{y} - \frac{1}{x} \right). \]

155. Solve the following equations:

(1) \[ \sqrt{x+1} + \sqrt{2x} = 7. \]

(2) \[ 7x - 20\sqrt{x} = 3. \]

(3) \[ 7xy - 5x^2 = 36, \quad 4xy - 3y^2 = 105. \]

156. A boy spends his money in oranges; if he had bought 5 more for his money they would have averaged an half-penny less, if 3 fewer an half-penny more: find how much he spent.

157. Potatoes are sold so as to gain 25 per cent. at 6 lbs. for 5d.: find the gain per cent. when they are sold at 5 lbs. for 6d.

158. A horse is sold for £24, and the number expressing the profit per cent. expresses also the cost price of the horse: find the cost.

159. Simplify \( \sqrt{4a^2 + \sqrt{(16a^2x^2 + 8ax^3 + x^4)}} \).

160. If the sum of two fractions is unity, shew that the first together with the square of the second is equal to the second together with the square of the first.

161. Simplify the following expressions:

\[ a - \left[ b - \{ a + (b-a) \} \right], \]

\[ 25a - 19b - \left[ 3b - \{ 4a - (5b - 6c) \} \right] - 8a, \]

\[ \left[ \{ (a^{-m})^{-n} \}^{-p} \right] \div \left[ \{ (a^{2m})^{-3p} \}^{2n} \right]. \]
162. Find the g. c. m. of \(18a^3 - 18a^2x + 6ax^2 - 6x^3\), and 
\(60a^2 - 75ax + 15x^2\).

163. Find the L. c. m. of \(18(x^2 - y^2)\), \(12(x - y)^2\), and 
\(24(x^3 + y^3)\).

164. Solve the following equations:

(1) \(\frac{2x - 4}{7} + \frac{3x - 2}{5} = 7\).

(2) \(\frac{9x + 20}{36} = \frac{4x - 12}{5x - 4} + \frac{x}{4}\).

(3) \(\frac{x + 4}{2} = \frac{2}{x + 1}\).

(4) \(2(x - y) = 3(x - 4y)\), \(14(x + y) = 11(x + 8)\).

165. Solve the following equations:

(1) \(32x - 5x^2 = 12\).

(2) \(\sqrt{2x + 3} = \sqrt{x - 2}\).

(3) \(x^2 + y^2 = 290\), \(xy = 143\).

(4) \(3x^2 - 4y^2 = 8\), \(5x^2 - 6xy = 32\).

166. A and B together complete a work in 3 days which would have occupied A alone 4 days: how long would it employ B alone ?

167. Find two numbers whose product is \(\frac{2}{5}\) of the sum of their squares, and the difference of their squares is 96 times the quotient of the less number divided by the greater.

168. Find a fraction which becomes \(\frac{1}{3}\) on increasing its numerator by 1, and \(\frac{1}{4}\) on similarly increasing its denominator.
MISCELLANEOUS EXAMPLES.

169. If \( a : b :: c : d \), show that
\[
\frac{1}{a} + \frac{1}{b} = \frac{1}{c} + \frac{1}{d}.
\]

170. Find a mean proportional between 169 and 256, and a third proportional to 25 and 100.

171. Remove the brackets from the expression
\[
b - 2 \{b - 3 [a - 4 (a - b)]\}.
\]

172. Simplify the following expressions:
\[
x + \frac{2x^2 + y^2}{xy} - \frac{3xy^2 - 3x^3 - y^3}{x^2y} - \frac{4xy^3 - 2x^2y^2 - y^4}{x^2y^2},
\]
\[
(p - q - m)p - (m + q - p)q + (q + m)m + m(p - m) + q^2,
\]
\[
\left(\frac{r^p + q}{r^q}ight) / \left(\frac{a^q}{r^p - q}\right).
\]

173. Find the g.c.m. of \( x^4 + ax^3 - 9ax^2 + 11a^3x - 4a^4 \) and \( x^4 - ax^3 - 3ax^2 + 5a^3x - 2a^4 \).

174. Solve the following equations:
\[
(1) \quad x - \frac{2x + 1}{3} = \frac{x + 7}{5}.
\]
\[
(2) \quad \frac{10x + 17}{18} - \frac{12x + 2}{13x - 16} = \frac{5x - 4}{9}.
\]
\[
(3) \quad 9x + \frac{8y}{5} = 70, \quad 7y - \frac{13x}{3} = 44.
\]
\[
(4) \quad \frac{6x + 7}{3x + 1} = \frac{2x + 19}{x + 7}.
\]

175. Solve the following equations:
\[
(1) \quad x + 4 - \frac{7x - 8}{x} = 3.
\]
\[
(2) \quad 2x^2 - 3y^2 = 2, \quad xy = 20.
\]
\[
(3) \quad 2y^2 - x^2 = 1, \quad 3x^2 - 4xy = 7.
\]
\[
(4) \quad x + y = 6, \quad x^2 + y^3 = 126.
\]
176. When are the clock-hands at right angles first after 12 o'clock?

177. A number divided by the product of its digits gives as quotient 2, and the digits are inverted by adding 27: find the number.

178. A bill of £26. 15s. was paid with half-guineas and crowns, and the number of half-guineas exceeded the number of crowns by 17: find how many there were of each.

179. Sum to six terms and to infinity $12 + 8 + 5\frac{1}{3} + \ldots$.

180. Extract the square root of $55 - 7\sqrt{24}$.

181. If $x = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$, and $y = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$, find the value of $x^2 + xy + y^2$.

182. Reduce to its lowest terms $\frac{3x^2 - 16x - 12}{x^3 - 8x^2 - 12x + 144}$.

183. If two numbers of two digits be expressed by the same digits in a reversed order, shew that the difference of the numbers can be divided by 9.

184. Solve the following equations:

1. $\frac{3x - 3}{4} - \frac{3x - 4}{3} = \frac{21 - 4x}{9}$.

2. $\frac{2x + 3y}{6} + \frac{x}{3} = 8$, $\frac{7y - 3x}{2} - y = 11$.

3. $4x - \frac{14 - x}{x + 1} = 14$.

185 Solve the following equations:

1. $\sqrt{(x + 3)} \times \sqrt{(3x - 3)} = 24$.

2. $\sqrt{(x + 2)} + \sqrt{(3x + 4)} = 8$.

3. $x^4 - x^2(2x - 3) = 2x + 8$.

186. Find two numbers in the proportion of 9 to 7 such that the square of their sum shall be equal to the cube of their difference.
187. A traveller sets out from A for B, going 3½ miles an hour. Forty minutes afterwards another sets out from B for A, going 4½ miles an hour, and he goes half a mile beyond the middle point between A and B before he meets the first traveller; find the distance between A and B.

188. Two persons A and B play at bowls. A bets B four shillings to three on every game, and after playing a certain number of games A is the winner of eight shillings. The next day A bets two to one, and wins one game more out of the same number, and finds that he has to receive three shillings. Find the number of games.

189. If \( m = x - x^{-1} \) and \( n = y - y^{-1} \), show that \( \frac{mn}{(m^2 + 4)(n^2 + 4)} = 2 \left( xy + \frac{1}{xy} \right) \).

190. Sum to nineteen terms \( \frac{9}{4} + \frac{3}{2} + \frac{3}{4} + \ldots \).

191. Multiply \( \frac{x^2}{2} - \frac{x}{3} + \frac{1}{4} \) by \( \frac{x^2}{4} + \frac{x}{3} - \frac{1}{2} \).

Divide \( \frac{3x^5}{4} - 4x^4 + \frac{77}{8} x^3 - \frac{43}{4} x^2 - \frac{33}{4} x + 27 \) by \( \frac{x^2}{2} - x + 3 \).

192. Reduce to its lowest terms

\[
\frac{4x^3 - 27x^2 + 58x - 39}{x^4 - 9x^3 + 29x^2 - 39x + 18}.
\]

193. Find the L.C.M. of \( x^3 + 2x^2 y + 4xy^2 + 8y^3 \) and \( x^3 - 2x^2 y + 4xy^2 - 8y^3 \).

194. Solve the following equations:

1. \( \frac{1}{4} (x + 6) - \frac{1}{12} (16 - 3x) = 4\frac{1}{3} \).

2. \( \frac{5x - 9}{13} - \frac{23 - 2x}{9} = 3x - 20 \).

3. \( \frac{1}{5} (x + y) = \frac{1}{3} (2x + 4), \frac{1}{3} (x - y) = \frac{1}{2} (x - 24) \).
195. Solve the following equations:

(1) \[ \frac{3}{4} (x^2 - 3) = \frac{1}{8} (x - 3). \]

(2) \[ \sqrt{x + 3} + \sqrt{3x - 3} = 10. \]

(3) \[ x + y = 6, \quad (x^2 + y^2)(x^3 + y^3) = 1440. \]

196. The express train between London and Cambridge, which travels at the rate of 32 miles an hour, performs the journey in \(2 \frac{1}{2}\) hours less than the parliamentary train which travels at the rate of 14 miles an hour: find the distance.

197. Find the number, consisting of two digits, which is equal to three times the product of those digits, and is also such that if it be divided by the sum of the digits the quotient is 4.

198. The number of resident members of a certain college in the Michaelmas Term 1864, exceeded the number in 1863 by 9. If there had been accommodation in 1864 for 13 more students in college rooms, the number in college would have been 18 times the number in lodgings, and the number in lodgings would have been less by 27 than the total number of residents in 1863. Find the number of residents in 1864.

199. Extract the square root of

\[ a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4, \]

and of \( (a + b)^4 - 2(a^2 + b^2)(a + b)^2 + 2(a^4 + b^4). \)

200. Find a geometrical progression of four terms such that the third term is greater by 2 than the sum of the first and second, and the fourth term is greater by 4 than the sum of the second and third.

201. Multiply \( 8 - 3x + \frac{38x - 6x^2 - 58}{7 - 2x} \) by \( 9 - 2x + \frac{7x^2 - 55 + 30x}{6 - 3x} \).

202. Find the g.c.m. of \( x^4 + 4x^2 + 16 \) and \( x^4 - x^3 + 8x - 8. \)
203. Add together \( \frac{1}{2+3x} \), \( \frac{2x-5}{(2+3x)^2} \), \( \frac{x^2-x+6}{(2+3x)^3} \).

Take \( \frac{1}{1+x+x^2} \) from \( \frac{1}{1-x+x^2} \).

204. Solve the following equations:

1. \( \frac{3x+5}{8} - \frac{21+x}{3} = 39 - 5x \).
2. \((a+b)(a-x) = a(b-x)\).
3. \( \frac{2x+3y}{16} + \frac{x}{12} = 2 \frac{x}{4}, \frac{7y-3x}{2} - 2y = 3 \).

205. Solve the following equations:

1. \( 6x + \frac{35-3x}{x} = 44 \).
2. \( 4(x^2+3x) - 2 \sqrt{x^2+3x} = 12 \).
3. \( x^2 + xy = 15, y^2 + xy = 10 \).

206. A person walked out from Cambridge to a village at the rate of 4 miles an hour, and on reaching the railway station had to wait ten minutes for the train which was then \( 4 \frac{1}{2} \) miles off. On arriving at his rooms which were a mile from the Cambridge station he found that he had been out \( 3 \frac{1}{4} \) hours. Find the distance of the village.

207. The tens digit of a number is less by 2 than the units digit, and if the digits are inverted the new number is to the former as 7 is to 4: find the number.

208. A sum of money consists of shillings and crowns, and is such, that the square of the number of crowns is equal to twice the number of shillings; also the sum is worth as many florins as there are pieces of money: find the sum.

209. Extract the square root of

\[ 4a^4 + 8ax^3 + 4ax^2 + 16b^2x^2 + 16ab^2x + 16b^4. \]

210. Find the arithmetical progression of which the first term is 7, and the sum of twelve terms is 348.
211. Divide \( 6x^5 - 25x^4y + 47x^3y^2 - 49x^2y^3 + 62xy^4 - 45y^5 \) by \( 2x^2 - 7xy + 9y^2 \).

212. Multiply \( 3 + 5x - \frac{12 + 41x + 36x^2}{4 + 7x} \) by \( 5 - 2x + \frac{26x - 8x^2 - 14}{3 - 4x} \).

213. Reduce to its lowest terms \( \frac{4x^3 - 45x^2 + 162x - 185}{x^4 - 15x^3 + 81x^2 - 185x + 150} \).

214. Solve the following equations:

1. \( \frac{3x - 2}{5} - \frac{1 - 5x}{11} = 9 \).

2. \( x + \frac{1}{4}y = 17, \ y + \frac{1}{4}x = 8 \).

3. \( \frac{1}{x} + \frac{1}{y} = \frac{1}{2}, \ \frac{1}{x} + \frac{4}{z} = 4, \ \frac{1}{y} + \frac{5}{z} = \frac{18}{5} \).

215. Solve the following equations:

1. \( \frac{1}{x} - \frac{1}{x + 3} = \frac{1}{6} \).

2. \( 10xy - 7x^2 = 7, \ 5y^2 - 3xy = 20 \).

3. \( x + y = 6, \ x^4 + y^4 = 272 \).

216. Divide £34.4s. into two parts such that the number of crowns in the one may be equal to the number of shillings in the other.

217. A number, consisting of three digits whose sum is 9, is equal to 42 times the sum of the middle and left-hand digits; also the right-hand digit is twice the sum of the other two: find the number.

218. A person bought a number of railway shares when they were at a certain price for £2625, and afterwards when the price of each share was doubled, sold them all but five for £4000: find how many shares he bought.
219. Four numbers are in arithmetical progression; their sum is 50, and the product of the second and third is 156: find the numbers.

220. Extract the square root of \(17 + 12 \sqrt{2}\).

221. Divide \(x^9 - 1\) by \(x^3 - 1\); and

\[ m(qx^2 - rx) + p(mx^3 - nx^2) - n(qx - r) \text{ by } mx - n. \]

222. Simplify

\[ \frac{ax^n - bx^{n+1}}{a^2bx - b^3x^3} \text{ and } \frac{a^2 + b^2 + c^2 + 2ab + 2ac + 2bc}{a^2 - b^2 - c^2 - 2bc}. \]

223. Find the L. C. M. of \(7x^3 - 4x^2 - 21x + 12\) and \(21x^2 - 26x + 8\).

224. Solve the following equations:

(1) \[ \frac{2x - 4}{7} - \frac{2 - 3x}{5} = 7. \]

(2) \[ 17x - 13y = 144, \quad 23x + 19y = 890. \]

(3) \[ \frac{1}{x} - \frac{1}{y} = \frac{1}{8}, \quad \frac{1}{x} + \frac{1}{z} = \frac{1}{9}, \quad \frac{1}{x} - \frac{1}{y} = \frac{5}{72}. \]

225. Solve the following equations:

(1) \[ \frac{x}{100} - \frac{21}{25x} = \frac{1}{4}. \]

(2) \[ 0.075x^2 + 2.75x = 150. \]

(3) \[ \sqrt{(x + y)} + \sqrt{(x - y)} = \sqrt{c}, \]

\[ b(x - a) + a(b - y) = 0. \]

226. A person walked out a certain distance at the rate of \(3\frac{1}{2}\) miles an hour, and then ran part of the way back at the rate of 7 miles an hour, walking the remaining distance in 5 minutes. If he was out 25 minutes: how far did he run?

227. A man leaves his property amounting to £7500 to be divided between his wife, his two sons, and his three daughters as follows: a son is to have twice as much as
a daughter, and the widow £500 more than all the five children together: find how much each person obtained.

228. A cistern can be filled by two pipes in 1\(\frac{1}{3}\) hours. The larger pipe by itself will fill the cistern sooner than the smaller by 2 hours. Find what time each will separately take to fill it.

229. The third term of an arithmetical progression is four times the first term; and the sixth term is 17: find the series.

230. Sum to \(n\) terms \(3\frac{1}{3} + 2\frac{1}{2} + 1\frac{2}{3} + \ldots\)

231. Simplify the following expressions:

\[\frac{b}{a+b} - \frac{a+b}{2a} + \frac{a^2+b^2}{2a(a-b)},\]

\[\frac{a^2-ab+b^2}{a^3-3ab(a-b)-b^3} \times \frac{a^2-b^2}{a^2+b^2} + \frac{a^2+b^2}{a^3-3ab(a-b)-b^3} \times \frac{a^2-b^2}{a^2+b^2} + \frac{a^2+b^2}{a^3-3ab(a-b)-b^3} \times \frac{a^2-b^2}{a^2+b^2}.

232. Reduce to its lowest terms \(\frac{x^2+11x+30}{9x^3+53x^2-9x-18}\).

233. Solve the following equations:

\[(1)\ \frac{1}{x} + \frac{1}{2x} - \frac{1}{3x} = 7.
\]

\[(2)\ \frac{3}{1+x} + \frac{3}{1-x} = 8.
\]

\[(3)\ \frac{4x+5y}{40} = x-y, \quad \frac{2x-y}{3} + 2y = \frac{1}{2}.
\]

234. Solve the following equations:

\[(1)\ \frac{48}{x+3} = \frac{165}{x+10} - 5.
\]

\[(2)\ ax^2 + b^2 + c^2 = a^2 + 2bc + 2(b-c)x \sqrt{a}.
\]

\[(3)\ \sqrt{(x+y)} + \sqrt{(x-y)} = 4, \quad x^2 + y^2 = 41.
\]
235. A body of troops retreating before the enemy, from which it is at a certain time 26 miles distant, marches 18 miles a day. The enemy pursues it at the rate of 23 miles a day, but is first a day later in starting, then after two days' march is forced to halt for one day to repair a bridge, and this they have to do again after two days' more marching. After how many days from the beginning of the retreat will the retreating force be overtaken?

236. A man has a sum of money amounting to £23. 15s. consisting only of half-crowns and florins; in all he has 200 pieces of money: how many has he of each sort?

237. Two numbers are in the ratio of 4 to 5; if one is increased, and the other diminished by 10, the ratio of the resulting numbers is inverted: find the numbers.

238. A colonel wished to form a solid square of his men. The first time he had 39 men over; the second time he increased the side of the square by one man, and then he found he wanted 50 men to complete it. Of how many men did the regiment consist?

239. Extract the square root of
\[ a^6 + 2a^5b + 3a^4b^2 + 4a^3b^3 + 3a^2b^4 + 2ab^5 + b^6, \]
and of
\[ a^2 + 4b^2 + 9c^2 + 4ab + 6ac + 12bc. \]

240. Multiply \( x^\frac{3}{2}y^\frac{1}{2} - 2xy + 4x^\frac{3}{2}y^\frac{3}{2} \) by \( x^\frac{1}{2} + 2y^\frac{1}{3} \).

241. Simplify
\[ 40xy - (9x - 8y)(5x + 2y) - (4y - 3x)(15x + 4y), \]
and
\[ \frac{1 + x}{1 - x} + \frac{1 - x}{1 + x} - \frac{1 - x + x^2}{1 + x^2} - \frac{1 + x + x^2}{1 - x^2} + 2. \]

242. Find the g.c.m. of \( x^4 + ax^3 + 2a^2x^2 + 3a^3x + a^4 \), and \( x^4 + ax^3 + 2a^2x^2 + 3a^3x + a^4 + a^2b^2 \).

243. Two shopkeepers went to the cheese fair with the same sum of money. The one spent all his money but 5s. in buying cheese, of which he bought 250 lbs. The other
bought at the same price 350 lbs., but was obliged to borrow 35s. to complete the payment. How much had they at first?

244. The two digits of a number are inverted; the number thus formed is subtracted from the first, and leaves a remainder equal to the sum of the digits; the difference of the digits is unity: find the number.

245. Find three numbers the third of which exceeds the first by 5, such that the product of their sum multiplied by the first is 48, and the product of their sum multiplied by the third is 128.

246. A person lends £1024 at a certain rate of interest; at the end of two years he receives back for his capital and compound interest on it the sum of £1156: find the rate of interest.

247. From a sum of money I take away £50 more than the half, then from the remainder £30 more than the fifth, then from the second remainder £20 more than the fourth part; at last only £10 remains: find the original sum.

248. Find such a fraction that when 2 is added to the numerator its value becomes \( \frac{1}{3} \), and when 1 is taken from the denominator its value becomes \( \frac{1}{4} \).

249. If I divide the smaller of two numbers by the greater, the quotient is 21, and the remainder is 0.04162; if I divide the greater number by the smaller the quotient is 4, and the remainder is 0.742: find the numbers.

250. Shew that \( \frac{(xy^2)^3 - (x^2y)^3 + x}{x + y} = \frac{x^3}{x^4 + y^4} \).

251. Simplify

\[
6a + [4a - \{8b - (2a + 4b) - 22b\} - 7b] - [7b + \{8a - (3b + 4a) + 8b\} + 6a].
\]

252. Multiply \( a - x \) successively by \( a + x \), \( a^2 + x^2 \), \( a^4 + x^4 \), \( a^8 + x^8 \); also multiply \( a^{m-n} b^{n-p} \) by \( a^{n-m} b^{p-n} c \).
253. Find the g.c.m. of $45a^2x + 3a^2x^2 - 9ax^3 + 6x^4$ and $18a^2x - 8x^3$.

254. Solve the following equations:

(1) $\frac{x - x - 2}{3} = \frac{x + 23}{4} - \frac{10 + x}{5}$.

(2) $\frac{x + y}{6} = 26, \quad \frac{x - y}{2} = 46$.

(3) $a - x = \sqrt{(a^2 - x)\sqrt{(4a^2 - 7x^2)}}$.

255. Divide the number 208 into two parts, such that the sum of one quarter of the greater and one third of the less when increased by 4, shall equal four times the difference of the two parts.

256. Two men purchase an estate for £9000. $A$ could pay the whole if $B$ gave him half his capital, while $B$ could pay the whole if $A$ gave him one-third of his capital: find how much money each of them had.

257. A piece of ground whose length exceeds the breadth by 6 yards, has an area of 91 square yards: find its dimensions.

258. A man buys a certain quantity of apples to divide among his children. To the eldest he gives half of the whole, all but 8 apples; to the second he gives half the remainder, all but 8 apples. In the same manner also does he treat the third and fourth child. To the fifth he gives the 20 apples which remain. Find how many he bought.

259. The sum of two numbers is 13, the difference of their squares is 39; find the numbers.

260. A horse-dealer buys a horse, and sells it again for £144, and gains just as many pounds per cent. as the horse had cost him. Find what he gave for the horse.

261. Simplify

$$(a + b)(a - b) - \{a + b - c - (b - a - c) + (b + c - a)\}(a - b - c).$$
262. Multiply \( x^8 + x^6 + x^4 + x^2 + 1 \) by \( x^2 - 1 \); and
\[
\frac{a}{x} - \frac{2x}{a} - 1 \text{ by } \frac{a}{x} - \frac{2a}{x} + 1.
\]

263. What quantity, when multiplied by \( x - \frac{1}{x} \), will give \( x^3 - \frac{1}{x^3} - \left(x - \frac{1}{x}\right)^2 \)?

264. Simplify the following expressions:
\[
\frac{3x^3 - 13x^2 + 23x - 21}{6x^3 + x^2 - 44x + 21},
\]
\[
\left\{ \frac{a + b}{2(a - b)} - \frac{a - b}{2(a + b)} + \frac{2b^2}{a^2 - b^2} \right\} \frac{a - b}{2b}.
\]

265. Solve the following equations:

(1) \[
\frac{5x + 3}{x - 1} + \frac{2x - 3}{2x - 1} = 6.
\]

(2) \[
\sqrt{3 + x} + \sqrt{x} = \frac{6}{\sqrt{3 + x}}.
\]

(3) \[
\frac{5x}{9} + 9y = 91, \quad \frac{5y}{9} + 9x = 167.
\]

266. Solve the following equations:

(1) \( x^2 - x - 6 = 0 \).

(2) \[
\frac{x + 1}{x - 1} + \frac{x + 2}{x - 2} = \frac{2x + 13}{x + 1}.
\]

(3) \( x^2 - xy + y^2 = 7, \quad x + y = 5 \).

267. The ratio of the sum to the difference of two numbers is that of 7 to 3. Shew that if half the less be added to the greater, and half the greater to the less, the ratio of the numbers so formed will be that of 4 to 3.

268. The price of barley per quarter is 15 shillings less than that of wheat, and the value of 50 quarters of barley exceeds that of 30 quarters of wheat by £7. 10s.: find the price per quarter of each.
269. Shew that
\((bcd + cda + dab + abc)^2 - (a + b + c + d)^2abcd\)
\(= (bc - ad)(ca - bd)(ab - cd).\)

270. Extract the square root of
\(x^4 + x^3 - \frac{5x^2}{12} - \frac{x}{3} + \frac{1}{9},\)
and of \(33 - 20\sqrt{2}.\)

271. If \(a = y + z - 2x, b = z + x - 2y,\) and \(c = x + y - 2z,\)
find the value of \(b^2 + c^2 + 2bc - a^2.\)

272. Divide \(x^4 - 21x + 8\) by \(1 - 3x + x^2.\)

273. Add together \(\frac{a + x}{a - x}, \frac{a - x}{a + x},\) and \(\frac{a^2 + x^2}{a^2 - x^2}.
Take \(\frac{3a + x}{3a - x}\) from \(\frac{27a^2 + 3ax + 7x^2}{15a^2 + ax - 2x^2}.\)

274. Multiply \(3x - \frac{12ax - 5x^2}{4a - 3x}\) by \(4x - \frac{20ax - 7x^2}{5a - 2x}.\)
Divide \(1 - \frac{1}{1 + x}\) by \(1 + \frac{x^2}{1 - x^2}.\)

275. Simplify \(\frac{1}{a + \frac{1}{a + \frac{1}{b + \frac{1}{c + d}}}}\) and \(\frac{1}{a^2} - \frac{1}{ax} + \frac{1}{x^2}\).

276. Solve the following equations:

(1) \(\frac{6}{x} - \frac{12}{x} + \frac{20}{x} = 7.\)

(2) \(5y - 3x = 2, \ 8y - 5x = 1.\)

(3) \(\frac{3x - 2y}{4} - \frac{x - y}{2} = 1, \ \frac{x}{3} + \frac{y}{2} = 4.\)
277. Solve the following equations:

1. \[ a^2(x-a)^2 = b^2(x+a)^2. \]
2. \[ \frac{x}{x-2} + \frac{5x+1}{x+3} = 5. \]
3. \[ \sqrt{(13x-1)} - \sqrt{(2x-1)} = 5. \]

278. A person walked to the top of a mountain at the rate of \(2\frac{1}{2}\) miles an hour, and down the same way at the rate of \(3\frac{1}{2}\) miles an hour, and was out \(5\) hours: how far did he walk altogether?

279. Shew that the difference between the square of a number, consisting of two digits, and the square of the number formed by changing the places of the digits is divisible by 99.

280. If \(a:b::c:d\), show that

\[ \sqrt{(a^2+b^2)} : \sqrt{(c^2+d^2)} :: \sqrt[3]{(a^3+b^3)} : \sqrt[3]{(c^3+d^3)}. \]

281. Find the value of \[ \frac{\sqrt{(a-(a-b))}}{\sqrt{(a^2+b^2)}} + \frac{\sqrt[3]{(5a-(a-b))}}{a+b}, \]
when \(a = 3, \ b = 4.\)

282. Subtract \((b-a)(c-d)\) from \((a-b)(c-d)\): what is the value of the result when \(a = 2b\), and \(d = 2c\)?

283. Reduce to their simplest forms:

\[ \frac{x^2 - 2ax - 24a^2}{x^2 - 7ax - 44a^2} \text{ and } \frac{x-y}{x+y} - \frac{x}{x-y} + \frac{y}{y-x}. \]

284. Solve the equations:

1. \[ \frac{4}{3+x} - \frac{1}{x} = \frac{9}{7x}. \]
2. \[ \frac{3x-2y}{5} - \frac{x-y}{2} = 1, \quad \frac{x}{3} + \frac{y}{2} = 4. \]
3. \[ \sqrt{(2x-1)} + \sqrt{(3x+10)} = \sqrt{(11x+9)}. \]
285. Solve the equations:

\[ (1) \quad 10x + \frac{2}{1-x} = 9. \]

\[ (2) \quad \left( \frac{x}{a} - \frac{2a}{x} - 1 \right) \left( 1 + \frac{a}{x} - \frac{2x}{a} \right) = 0. \]

\[ (3) \quad x^2 - xy + y^2 = 7, \quad 5x - 2y = 9. \]

286. In a time race one boat is rowed over the course at an average pace of 4 yards per second; another moves over the first half of the course at the rate of 3\(\frac{1}{2}\) yards per second, and over the last half at 4\(\frac{3}{4}\) yards per second, reaching the winning post 15 seconds later than the first. Find the time taken by each.

287. A rectangular picture is surrounded by a narrow frame, which measures altogether ten linear feet, and costs, at three shillings a foot, five times as many shillings as there are square feet in the area of the picture. Find the length and breadth of the picture.

288. If \(a:b::c:d\), shew that

\[ a+b+c+d : a+b-c-d :: a-b+c-d : a-b-c+d. \]

289. The volume of a pyramid varies jointly as the area of its base and its altitude. A pyramid, the base of which is 9 feet square, and the height of which is 10 feet is found to contain 10 cubic yards. Find the height of a pyramid on a base 3 feet square that it may contain 2 cubic yards.

290. Find the sum of \(n\) terms of the arithmetical progression \[ \frac{1}{1+x}, \quad \frac{1}{1-x^2}, \quad \frac{1}{1-x}, \ldots \]

291. Find the value of \(a^3 - b^3 + c^3 + 3abc\), when \(a = 0.03, b = 1, c = 0.07\).
292. Simplify \( \frac{(ac-bd)^2 + (ad+bc)^2}{c^2 + d^2} - a^2 \), and show that
\[
bc(b^2 - c^2) + ca(c^2 - a^2) + ab(a^2 - b^2) - (a + b + c)(a^2(b - c) + b^2(c - a) + c^2(a - b)) = 0.
\]

293. If \( a + b + c = 0 \), show that \( a^3 + b^3 + c^3 = 3abc \).

294. Reduce to its lowest terms
\[
\frac{x^4 + 2x^3 + 6x - 9}{x^4 + 4x^3 + 4x^2 - 9}.
\]

295. Solve the following equations:

(1) \( \frac{10x + 17}{18} - \frac{12x + 2}{13x - 16} = \frac{5x - 4}{9} \).

(2) \( 6x - 5y = 1, \ y - x = 12 \).

(3) \( \frac{x}{8} + 8y = 66, \ \frac{y}{8} + 8v = 129 \).

296. Solve the following equations:

(1) \( \frac{x + 1}{4} + \frac{3x + 1}{x + 4} = 4 \).

(2) \( \sqrt{2x + 2} \sqrt{4x - 3} = 20 \).

(3) \( \sqrt{3x + 1} - \sqrt{2x - 1} = 1 \).

297. A siphon would empty a cistern in 48 minutes, a cock would fill it in 36 minutes; when it is empty both begin to act: find how soon the cistern will be filled.

298. A waterman rows 30 miles and back in 12 hours, and he finds that he can row 5 miles with the stream in the same time as 3 against it. Find the times of rowing up and down.

299. Insert three Arithmetical means between \( a - b \) and \( a + b \).

300. Find \( x \) if \( 2^x : 2^{2x} :: 8 : 1 \).
ANSWERS.

I.  1. 22.  2. 26.  3. 89.  4. 564.
   5. 274.  6. 10.  7. 6.  8. 6.  9. 34.

II. 1. 55.  2. 81.  3. 94.  4. 8.  5. 27.
   6. 81.  7. 12.  8. 11.  9. 21.  10. 15.

III. 1. 5.  2. 16.  3. 9.  4. 224.  5. 459.
   6. 7.  7. 74.  8. 12.  9. 8.  10. 238.
   11. 420.  12. 144.  13. 43.  14. 15.  15. 9.  16. 2.

IV. 1. 7.  2. 88.  3. 43.  4. 2.  5. 72.
   6. 1.  7. 1.  8. 16.  9. 14.  10. 5.  11. 7.
   12. 5.  13. 11.  14. 7.  15. 4.  16. 2.

V. 1. $15a - 9b$.  2. $3x^2 - 3y^2$.  3. $9a + 9b + 3c$.
   4. $4x + 2y + 4z$.  5. $a - b$.  6. $3x - 3y - 2b$.  7. $2a + 2b$.
   8. $a + b - c$.  9. $-2a + 2b + 2d$.  10. $2x^3 - 2x^2 - 8x + 10$.
   11. $5x^4 + 4x^3 + 3x^2 + 2x - 9$.  12. $4a^3 + 2a^2b - 4ab^2 + b^3 - 7b^2$.
   13. $a^2x + 3a^3$.  14. $6ab - 9a^2x + 7ax^2 + ax^3$.  15. $5x^2$.
   16. $10x^2 + 8y^2 + 12x + 12$.  17. $x^4$.  18. $x^3 + y^3 + z^3 - 3xyz$.

VI. 1. $3a + 4b$.  2. $4a + 2c$.  3. $a + 5b + 4c + d$.
   4. $2x^2 - 2x - 4$.  5. $3x^4 - x^3 - 14x + 18$.
   6. $x^2 - ax + 2a^2$.  7. $-5xy - 5xz + 2y^2 + yz$.
   8. $3x^2 + 13xy - 16xz - y^2 - 13yz$.  9. $2a^3 - 6a^2b + 6ab^2 - 2b^3$.
   10. $3x^3 + 4x + 16$, $x^3 + 8x^2$.

VII. 1. $a$.  2. $2c$.  3. $a + a^3$.  4. $a - 3b$.
    5. $-2b + 2c$.  6. $3x + 3y - z$.  7. $a - b + c + d - e$.
    8. $a - b + 2c - d$.  9. $3c$.  10. $3a - 3b$.  11. $2a - b$.
    12. $5a$.  13. $a$.  14. $4a$.  15. $4a - 16b - 2c$.
    16. $3a - 2c$.  17. $9 + 3x$.  18. $7x + 6$.  19. $a$.
    20. $16 - 12x$.  21. $12x - 15y$.  22. $4c$.  23. $3a - 2c$.
    24. $-8x^3 - 8x$.

T. A.
VIII. 1. $8x^5$.  
2. $12v^9$.  
3. $4a^3b^3$.  
4. $15v^6y^5z^2$.  
5. $49x^4y^4z^4$.  
6. $12x^3b - 9ab^2$.  
7. $24a^4 - 27a^3b$.  
8. $6x^4y - 8x^2y^3 + 10x^2yz^2$.  
9. $x^4y^5z^2 - x^2y^5z^6 + x^4y^2z^6$.  
10. $4x^2y^4z^4 + 6x^3y^5z^2 - 10x^4y^3z^3$.  
11. $2x^2 + 3xy - 2y^2$.  
12. $6x^4 - 96$.  
13. $x^4 - 2x + 1$.  
14. $1 - 2x - 31x^2 + 72x^3 - 30x^4$.  
15. $x^5 - 41x - 120$.  
16. $x^5 + 151x - 234$.  
17. $2x^5 - 18x^4 + 39x^3 - 25x^2 + x + 1$.  
18. $x^6 + 1008x + 720$.  
19. $4x^6 - 5x^5 + 8x^4 - 10x^3 - 8x^2 - 5x - 4$.  
20. $x^8 + 2x^6 + 3x^4 + 2x^2 + 1$.  
21. $x^3 - 9x^2 + 9x - 5$.  
22. $a^4 + 4a^3x + 4a^2x^2 - a^4$.  
23. $-10b^3 - ab^2 + 26ab - 7a^3$.  
24. $a^4 - a^2b^2 + 2ab^3 - b^4$.  
25. $a^4 + 3a^2b^2 + 4b^4$.  
26. $12x^3 - 17x^2y + 3xy^2 + 2y^3$.  
27. $x^6 - x^4y^2 + x^2y^4 - y^6$.  
28. $6x^4 + 17x^3y + 26x^2y^2 + 19xy^3 + 4y^4$.  
29. $x^3 + y^3 + 3xy - 2x - 2y + 1$.  
30. $x^5 - 32y^5$.  
31. $243x^5 - y^5$.  
32. $x^3 + a^2b + ab^3 + b^3 + 2b^2 - (a - b)x^2$.  
33. $a^3 + b^3 + c^3 - 3abc$.  
34. $a^4 + 8b^2x^2(a^2 - 2) + 16b^4x^4$.  
35. $a^4 - 2a^2b^2 + b^4 + 4abc^2 - c^4$.  
36. $x^4 - 2a^2b^2 + 4ab - c$.  
37. $a^4 - 4a^2b^2$.  
38. $x^3 + x^2(2a + b + c) + 2(ab + ac + bc) + abc$.  
39. $x^3 + 3a^4 + a^8$.  
40. $x^4 - 5a^3b - 4a^4$.  

IX. 1. $5x^3$.  
2. $-3a^3$.  
3. $3xy$.  
4. $-8a^2b^2c^2$.  
5. $4a^6b^2y^2$.  
6. $x^2 - 2x + 4$.  
7. $-a^2 + 4a - 5$.  
8. $x^2 - 3xy + 4y^2$.  
9. $5a^2b^2 + ab - 4$.  
10. $15a^2b^2 - 12ab^3 + 9abc^2 - 5c^4$.  
11. $x - 4$.  
12. $x - 8$.  
13. $x^2 + x + 3$.  
14. $3x^2 - 2x + 4$.  
15. $3x^2 + 2x + 1$.  
16. $x^2 - 3x + 7$.  
17. $x^5 + x^4 + x^3 + x^2 + x + 1$.  
18. $a^2 + ab - b^2$.  
19. $x^3 + 3x^2y + 9xy^2 + 27y^3$.  
20. $x^3 - x^2y + xy^2$.  
21. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$.  
22. $a^4 - 2a^3b + 4a^2b^2 - 8ab^3 + 16b^4$.  
23. $2a^3 - 2a^2b + 4a^2b^3 - 8ab^3 + 16b^4$.  
24. $x^2 + xy + y^2$.  
25. $x^2 + 2xy + 3y^2$.  
26. $x^2 - 2x + 2$.  
27. $x^3 - 3x - 1$.  
28. $x^2 - 5x + 6$.  
29. $x^2 - 4x + 8$.  
30. $x^3 + 5x + 6$.  

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31. \(x^3 - x - 19\). 32. \(1 - 3x + 2x^2 - x^3\).
33. \(x^4 + 2x^3 + 3x^2 + 2x + 1\). 34. \(a^2 + 2ab + 3b^2\).
35. \(a^3 + 2ab^2 + 2ab^2 + b^3\). 36. \(x^4 - 3x^2 + 4x + 1\).
37. \(x^4 + 2x^3 + 3x^2 + 2x + 1\). 38. \(x^8 - x^6 + 2x^5 - 2\).
39. \(x - c\). 40. \(ax^2 + bx + c\). 41. \(a^2 - 2xy + y^2\).
42. \(x^2 + x(y + 1) + y^2 - y + 1\). 43. \(7x + 4x\).
44. \(a + b + c\). 45. \(a + 2b + c\).
46. \(a^2 + a(2b - c) + b^2 - bc + c^2\). 47. \(a(b + c) - bc\).
48. \(x^2 - x(a + b) + ab\). 49. \(x + y - z\). 50. \(x + y + z\).

X. 1. \(225x^2 + 420xy + 196y^2\). 2. \(49x^4 - 70x^2y^2 + 25y^4\).
3. \(x^4 + 4x^3 - 8x + 4\). 4. \(x^4 - 10x^3 + 39x^2 - 70x + 49\).
5. \(4x^4 - 12x^3 + 7x^2 + 24x + 16\).
6. \(x^4 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz\). 7. \(x^4 + 2x^3y + x^2y^2 - y^4\).
8. \(x^4 + x^2y^2 + y^4\). 9. \(x^4 - x^3y^2 - 2xy^3\).
10. \(x^4 - x^2y^2 + 2xy^3 - y^4\). 11. \(x^6 + 2x^4 + 5x^2\).
12. \(x^4 - 18x^2 + 81\). 13. \(a^4 - b^4 - 4\).
14. \(16x^4 + 96x^3y + 144x^2y^2 - 81y^4\). 15. \(a^4x^4 - b^4y^4\).
16. \(a^4x^4 - 2a^2b^2x^2y^2 + b^4y^4\).

XI. 1. \(a^2 + b^2 + c^2\). 2. \(a^2 + b^2 + c^2\).
3. \(a^2 + b^2 + c^2 + 2ac + 2bd\). 4. \(6(a + b + c)\).
5. \(2(a + b + c)\). 6. \(2b(x + y)\). 7. \(bx + ay + (a + b)z\).
8. \(x(2a + c) + y(2b + a) + z(2c + b)\).
9. \(2(a + b + c)(x + y + z)\).
10. \(2(a^2 + b^2 + c^2 - ab - bc - ca)\). 11. \(b - 11a\).
12. \(b^2 - d^2\). 13. \(2a + 4by\). 14. \((x + a)^2\). 15. \(a\).
16. \(2a - 5b + 4c\). 17. \(6\). 18. \(x^3 + x^2y + xy^2 + y^3\).
19. \(x^3 + x^2y + xy^2 + y^3\). 20. \(12abc\). 21. \(a + b + c + d\).
22. \(3b\). 23. \(9a^2 - 30ab + 25b^2\).
24. \(-6c^2 + c(9a + 4b) - 6ab\). 25. \((x^2 + xy + y^2)^2\).
26. \((x^2 - xy + y^2)^2\). 27. \(a^2 - 2ab + 3b^2\).
28. \(x^2 - 8xy + 15y^2\). 29. \(a^4 - a^2b^2 + b^4\). 30. \(a^4 - b^4\).
31. \(2a^2 - 3ab + 4b^2\). 32. \(x - 1\). 33. \((x - 1)(x + 4)\).
34. \(a + x\). 35. \(a^3 + b^3\). 36. \(x^3 - ax + a^2\).
37. \((x + 4)(x + 5)\). 38. \((x + 5)(x + 6)\).
39. \((x - 5)(x - 10)\). 40. \((x - 10)^2\). 41. \((x - 11)(x + 12)\).
42. \((x + 4)(x - 11)\). 43. \((x - 3)(x + 3)(x^2 + 9)\).
44. \((x + 5)(x^2 - 5x + 25)\).
45. \((x - 2)(x + 2)(x^2 + 4)(x^4 + 16)\).
46. \((x - 2)(x + 2)(x^2 + 2x + 4)(x^2 - 2x + 4)\).
47. \((a + 4b)(a + 5b)\). 48. \((x - 6y)(x - 7y)\).
49. \((a + b - 5c)(a + b - 6c)\).
50. \((2x + 2y - a - b)(x + y - 3a - 3b)\).

XII. 1. \(3x^2\). 2. \(4a^2b^2\). 3. \(12x^4y^5z^4\).
4. \(7a^3b^3x^3y^3\). 5. \(2(x + 1)\). 6. \(3(x + 1)\).
7. \(4(a^2 + b^2)^2\). 8. \(x^2 - y^2\). 9. \(x + 5\). 10. \(x - 7\).
11. \(x - 10\). 12. \(x - 12\). 13. \(x^2 + 3x + 4\).
14. \(x^2 - 5x + 3\). 15. \(x^2 - 6x + 7\). 16. \(x^2 - 6x + 5\).
17. \(x + 3\). 18. \(x - 4\). 19. \(x^2 - x + 1\).
20. \(x^2 - x + 1\). 21. \(3x + 2\). 22. \(x^2 - x - 1\).
23. \(x^2 - 2\). 24. \(x - 2\). 25. \(x^2 + 1\).
26. \(x^2 + 3x + 5\). 27. \(7x^2 + 8x + 1\).
28. \(x^4 - 2x^3 + 3x^2 - 2x + 1\). 29. \(x^2 - 3x + 1\).
30. \(x + 1\). 31. \(x + 7\). 32. \(x + 3y\). 33. \(x + a\).
34. \(x - 2a\). 35. \(x - y\).

XIII. 1. \(12a^2b^2\). 2. \(36a^2b^2c^2\). 3. \(24a^2b^2x^3y^3\).
4. \((a + b)(a - b)^2\). 5. \(12ab(a^3 + b^3)\). 6. \((a + b)(a^3 - b^3)\).
7. \((x + 1)(x + 3)(x - 4)\). 8. \((x + 2)(x + 4)(x^2 + 3x + 1)\).
9. \(x(2x + 1)(3x - 1)(4x + 3)\).
10. \((x^2 - 5x + 6)(x - 1)(x - 4)\).
11. \((x^2 + 3x + 2)(x - 3)(x + 5)\).
12. \((x^2 + x + 1)(x^2 + 1)(x + 1)(x - 1)\).
13. \((x^3 - x^2 - 4x + 4)(x - 1)(x - 4)\).
14. \((x^2 - ax + a^2)(x^3 + ax + a^2)(x - a)^2\).
15. \(36a^3b^2c^2\). 16. \(120(a + b)^2(a - b)^2\).
17. $\frac{24(a-b)(a^2+b^2)}{a^3}$. 18. $105ab^2(a+b)(a-b)$.
19. $x^8-1$. 20. $x^8-1$. 21. $x^{12}-1$.
22. $(x+1)(x+2)(x+3)$. 23. $(x+1)(x+2)(x^2+2x-3)$.
24. $(x^3-19x-30)(x^3+5x+10)$.

XIV. 1. $3x + \frac{4x}{7}$. 2. $4ac + \frac{4c}{9}$. 3. $2a + \frac{3b}{4a}$.
4. $2x - \frac{5y}{6x}$. 5. $x + \frac{2}{x+3}$. 6. $2x - \frac{1}{x-3}$.
7. $x^2 + 3ax + 3a^2 + \frac{3a^2}{x-2a}$. 8. $x - 1 - \frac{2x-1}{x^2-x+1}$.
9. $x^3 + x^2 + x + 1 + \frac{2}{x-1}$. 10. $x^3 - x^2 + x - 1$. 11. $\frac{4a^2}{3b}$.
12. $\frac{8(a^2+b^2)}{3(a+b)}$. 13. $\frac{3(a-b)}{2(a+b)}$. 14. $\frac{x^2}{(x^2-1)(x+1)}$.
15. $\frac{4x}{3y}$. 16. $\frac{3a+2b}{a+b}$. 17. $\frac{2(a-b)}{3(a+b)}$. 18. $\frac{(x^2-1)(x+1)}{x^2+1}$.

XV. 1. $\frac{2a^2x}{3y}$. 2. $\frac{a+b}{2b}$. 3. $\frac{a+b}{a-b}$. 4. $\frac{2ax}{ax-3y^2}$.
5. $\frac{4(a+b)}{5(a-b)}$. 6. $\frac{a^2-ab+b^2}{a-b}$. 7. $\frac{x+2}{x+5}$. 8. $\frac{x+7}{x-5}$. 9. $\frac{x+3}{x-7}$.
10. $\frac{x+b}{x+c}$. 11. $\frac{x-b}{x+c}$. 12. $\frac{3x-4}{4x-3}$. 13. $\frac{x+a-b-c}{x+b-a-c}$.
14. $\frac{x+3}{x^2-2x+5}$. 15. $\frac{x-3}{x^2+7x+3}$. 16. $\frac{x+5}{x^2+3x+2}$.
17. $\frac{x+7}{x^2-4x+3}$. 18. $\frac{6x-5}{3x^2+x+1}$. 19. $\frac{5x+4}{3x^2+x+2}$.
20. $\frac{x-a}{x^2-ax+a^2}$. 21. $\frac{x-4}{x+4}$. 22. $\frac{x^2+ax-2a^3}{2x^2+3ax+4a^3}$.
23. $\frac{x-3}{x^2-3x+1}$. 24. $\frac{x+a}{x^2+ax+a^2}$. 25. $\frac{x-3}{x^3+1}$.
26. $\frac{3x^2+x+2}{2x^2+x+3}$. 27. $\frac{3x(x^2-5a^2)}{2x^2+3a^2}$. 28. $\frac{x^3+1}{x^3+x^2+1}$.
29. $\frac{1}{x-1}$. 30. $\frac{x^3}{x^3-a^2y}$. 31. $\frac{1}{x^3-a^3}$. 32. $\frac{y^{m-1}}{x^{m+1}}$. 
33. \( \frac{9x^2}{12x^3} \cdots \)
34. \( \frac{4(x-1)}{4(x^2-1)} \cdots \)
35. \( \frac{a(x+a)}{x^2-a^2} \cdots \)
36. \( \frac{a(a+b)(a^2+b^2)}{a^4-b^4} \cdots \)
37. \( \frac{(x-1)(x+1)^2}{(x-1)^2} \cdots \)
38. \( \frac{a(x^2+ax+a^2)}{x^3-a^3} \cdots \)
39. \( \frac{x^2+ax+a^2}{x^3+a^3x^2+a^4} \cdots \)
40. \( \frac{x-c}{(x-a)(x-b)(x-c)} \cdots \)

XVI. 1. \( \frac{6a-6b-c}{4} \)
2. \( \frac{2x}{a^2-b^2} \)
3. \( \frac{a^2+2ab-b^2}{a^2-b^2} \)
4. \( \frac{2cb}{a^2-b^2} \)
5. \( \frac{a+b+c}{abc} \)
6. \( \frac{1}{x-y} \)
7. \( \frac{12x}{1-9x^2} \)
8. \( \frac{a+x}{ax} \)
9. \( \frac{a+b}{2a-2b} \)
10. \( \frac{4a}{a+x} \)
11. \( \frac{2a^2+9c^2}{6ac} \)
12. \( \frac{b}{a-b} \)
13. \( \frac{b(a+b)}{x^2-b^2} \)
14. \( \frac{2x-3}{x(4x^2-1)} \)
15. \( \frac{16}{(x-2)(x+2)^3} \)
16. \( \frac{a}{a^2-b^2} \)
17. \( \frac{a^4+6a^2x^2+x^4}{a^4-x^4} \)
18. \( \frac{2}{(x+1)(x+2)(x+3)} \)
19. \( \frac{5x^2-7x}{(x^2-1)(x-2)} \)
20. \( \frac{4x^3}{y(x^2-y^2)} \)
21. \( \frac{2x^2}{1-x^2} \)
22. \( \frac{2x^2}{x^2-1} \)
23. \( \frac{2a^2}{x(x^2+a^2)} \)
24. \( \frac{2a^4+6a^2b^2}{a^4-b^4} \)
25. \( \frac{3x^2}{x^2-1} \)
26. \( \frac{4a^2(a^2-ax+x^2)}{a^4-x^4} \)
27. \( \frac{4(x+10)}{x^3-16} \)
28. \( \frac{2x^2-9x+44}{x^3+64} \)
29. \( \frac{x^2-4ax-a^2}{(x^2-a^2)^2} \)
30. \( \frac{2a}{x^2-a^2} \)
31. \( \frac{x^2-2x}{x^3+1} \)
32. \( \frac{3}{a^3-a^2} \)
33. \( \frac{2x^2}{x^3+y^3} \)
34. \( \frac{6}{x(x+1)(x+2)} \)
35. \( \frac{1}{(1+x^2)(1+x^3)} \)
36. \( \frac{2a^2}{x^3+y^3} \)
37. \( \frac{2y^2}{x^3-y^3} \)
38. \( \frac{2x^3+2}{x^4+x^2+1} \)
39. \( \frac{4(a^4x^3-b^4y^3)}{a^4x^4-b^4y^4} \)
40. \( \frac{4x^3}{x^3+x^3+1} \)
41. \( \frac{4a^3}{x^4-a^4} \)
42. \( \frac{4b^7}{a^8-b^8} \)
43. \( \frac{24b^4}{a(a^2-b^2)(a^2-4b^2)} \)
44. \( \frac{48a^3}{(x^2-a^2)(x^2-9a^2)} \)
46. \( \frac{c}{(x-a)(x-b)} \)  \hspace{1cm} 47. \( \frac{x}{(x-a)(x-b)} \)  \hspace{1cm} 48. \( \frac{x(a+b)-ab}{(x-a)(x-b)} \)
49. \( \frac{1}{(a-c)(c-b)} \)  \hspace{1cm} 50. \( \frac{c-a-b}{(c-a)(c-b)} \)  \hspace{1cm} 51. 0
52. \( \frac{1}{c(c-a)(c-b)} \)  \hspace{1cm} 53. 1  \hspace{1cm} 54. \( \frac{3x-a-b-c}{(x-a)(x-b)(x-c)} \)
55. \( \frac{3x^2-a^2-b^2-c^2}{(x-a)(x-b)(x-c)} \)  \hspace{1cm} 56. \( \frac{1}{(x-a)(x-b)(x-c)} \)

XVII. 1. \( \frac{4c}{5a} \)  \hspace{1cm} 2. 1  \hspace{1cm} 3. \( \frac{a^3b^3c^3}{x^3y^3z^3} \)  \hspace{1cm} 4. \( \frac{1}{(x-1)(x+2)} \)
5. \( x-a \)  \hspace{1cm} 6. \( \frac{a^4-b^4}{ab} \)  \hspace{1cm} 7. \( \frac{a^2b^2}{a^3-b^2} \)  \hspace{1cm} 8. \( \frac{ax}{a^2-x^2} \)
9. \( \frac{(x+y)^2}{x^2+y^2} \)  \hspace{1cm} 10. \( \frac{x+c}{x+b} \)  \hspace{1cm} 11. \( \frac{x}{x-y} \)  \hspace{1cm} 12. \( \frac{(a-c)^2-b^2}{abc} \)
13. \( \frac{x^6-ax^5+ax^3-x-a^5}{a^3x^3} \)  \hspace{1cm} 14. \( \frac{x^2}{a^2} + \frac{a^2}{x^2} - \frac{y^2}{b^2} - \frac{b^2}{y^2} \)  \hspace{1cm} 15. 1

XVIII. 1. \( \frac{6ay}{b.x} \)  \hspace{1cm} 2. \( \frac{9c^2x^2}{16a^2y^2} \)  \hspace{1cm} 3. \( \frac{1}{x+y} \)  \hspace{1cm} 4. \( \frac{3(a-b)^2}{b(a+b)} \)
5. \( \frac{x(a+2x)}{a^2} \)  \hspace{1cm} 6. \( \frac{2x}{x-y} \)  \hspace{1cm} 7. \( \frac{a+x}{x+y} \)  \hspace{1cm} 8. \( \frac{x-b}{x-a} \)
9. \( \frac{a+b-c}{c+a-b} \)  \hspace{1cm} 10. \( \frac{1}{x^2-y^2} \)  \hspace{1cm} 11. \( \frac{(x-1)^3}{x-3} \)  \hspace{1cm} 12. \( \frac{y^4-x^4}{y^3} \)
13. \( 5x-1 \)  \hspace{1cm} 14. \( \frac{a^4+a^2+1}{a^2} \)  \hspace{1cm} 15. \( \frac{(x^2+a^2)(x^4+a^4)}{x^3a^4} \)
16. \( \frac{x^2-6a^2}{xa} \)  \hspace{1cm} 17. \( \frac{x-y}{y} \)  \hspace{1cm} 18. \( \frac{x^2+ax+a^2}{ax} \)  \hspace{1cm} 19. \( \frac{a^2+x^2}{x+ax} \)
20. \( \frac{x^4-3x^3a+3a^2x+a^4}{a^2x^3} \)  \hspace{1cm} 21. 1  \hspace{1cm} 22. \( \frac{x-4}{x-5} \)
23. \( \frac{1}{x+1} \)  \hspace{1cm} 24. \( \frac{x^2-a^2}{x(a+b+c)-bc} \)  \hspace{1cm} 25. \( \frac{1}{x+1} \)
26. \( \frac{1}{1+x} \)  \hspace{1cm} 27. \( \frac{x+1}{1+x} \)  \hspace{1cm} 28. \( \frac{1+x^2}{1+x} \)  \hspace{1cm} 29. \( \frac{(x^2+y^2)^2}{x^4+y^4} \)
30. \( x \)  \hspace{1cm} 31. 1  \hspace{1cm} 32. \( \frac{(a^2+b^2)^2}{a^4+b^4} \)  \hspace{1cm} 33. \( \frac{a^2}{b^2} \)  \hspace{1cm} 34. \( \frac{b}{a} \)
ANSWERS.

35. 0. 36. \( \frac{4}{9} \). 37. \( 2\frac{1}{7} \). 38. 0. 39. 0. 40. a.

XIX. 1. 6. 2. 9. 3. 7. 4. 11. 5. 21.

6. 2. 7. 4. 8. 7. 9. 8. 10. 5.

11. 18. 12. 6. 13. 2. 14. 27. 15. 15.

16. 63. 17. 60. 18. 36. 19. 64. 20. 96.

21. 45. 22. 24. 23. 120. 24. 72. 25. 12.

26. 6. 27. 5. 28. 1. 29. 6. 30. 2. 31. 2.

32. 3. 33. \( 1\frac{1}{2} \). 34. 7. 35. \( 1\frac{1}{6} \). 36. 11.

37. 5. 38. \( 2\frac{1}{3} \). 39. 3. 40. 7. 41. 11.

42. 12. 43. 4. 44. 3. 45. 7. 46. 3.

47. \( 5\frac{1}{2} \). 48. \( 1\frac{1}{3} \). 49. 10. 50. 6. 51. 10.

52. 7. 53. 1. 54. 12. 55. 5. 56. \( \frac{1}{7} \).

57. 3. 58. 2. 59. 3. 60. 28. 61. 5.

62. 2. 63. 3. 64. 2. 65. 4. 66. 2.

XX. 1. 10. 2. 8. 3. 12. 4. 6.

5. \( -7 \). 6. 16. 7. 5. 8. \( 3\frac{1}{7} \). 9. \( -6 \).

10. 5. 11. 8. 12. \( \frac{7}{4} \). 13. 3. 14. 2.

15. 7. 16. \( 1\frac{1}{4} \). 17. \( \frac{1}{5} \). 18. 1. 19. 17.

20. 2. 21. 5. 22. 2. 23. 6. 24. 7. 25. 2.

26. 2. 27. 2. 28. \( \frac{50}{29} \). 29. 7. 30. 4.

31. \( -1 \). 32. \( \frac{3}{2} \). 33. \( -23 \). 34. 3. 35. \( 5\frac{1}{2} \).

36. \( \frac{4}{13} \). 37. 0. 38. 20. 39. 3. 40. 5.

41. \( a - b \). 42. \( a + b \). 43. \( b - a \). 44. \( \frac{2ab}{a + b} \).

45. \( 2(a + b) \). 46. \( \frac{a^2 + ab + b^2}{a + b} \). 47. \( \frac{ab}{a + b - c} \).

48. \( \frac{ab(a + b - 2c)}{(a + b)c - a^2 - b^2} \). 49. \( \frac{2ab}{a + b} \). 50. \( \frac{a + b}{2} \).

51. \( \frac{a + b + c + d}{m + n} \). 52. c. 53. \( \frac{a^2}{b - a} \).
54. \( \frac{ab-pq}{a+b+p+q} \).
55. \( \frac{1}{2} (a+b+3) \).
56. \( \frac{c^2-ab}{a+b-2c} \).
57. \( \frac{2(a^2+ab+b^2)}{3(a+b)} \).
58. \( \frac{1}{2} (a+b) \).
59. 4.
60. 50. 61. 25. 62. \frac{13}{81}.
63. \( (a-b)^2 \).
64. \( a \).

XXI. 1. 30. 2. 2. 3. 13, 20. 4. 35, 50, 70.
9. 52. 10. 36, 27. 11. 48, 36. 12. 14, 24, 38.
17. 8, 12. 18. 10. 19. 36, 9. 20. 36, 12.
25. 36, 24. 26. 24, 60, 192. 27. 840. 28. 30000.
29. 420. 30. 24. 31. 500. 32. 10, 14, 18, 22, 26, 30.
33. 36, 26, 18, 12. 34. 50, 100, 150, 250. 35. 5, 6.
36. 24, 36, 56. 37. 88, 44. 33. 130, 150, 130, 90.
39. 13, 27. 40. 75, 25. 41. 85, 35. 42. 1000.
43. 18, 3, 3. 44. 24000. 45. 80. 46. 26, 16, 32, 27, 42.
47. £140. 48. 10\( \frac{1}{2} \)d.

XXII. 1. 72. 2. 20, 30. 3. 200 miles from Edinburgh.
4. 12, 16. 5. 8, 16. 6. 32, 16.
7. 48. 8. 30. 9. 9, 16. 10. 30. 11. 18, 22, 10, 40.
12. 6, 24. 13. 10, 15, 3, 60. 14. 10 shillings. 15. 55, 45.
16. At the end of 56 hours. 17. 27, 17. 18. 168, 84, 42.
19. 16, 25, 7, 42. 20. 240, 180, 144 days. 21. 15, 21.
27. 875, 1125. 28. 25. 29. 10, 20. 30. 20, 80.
31. 5\( \frac{5}{7} \). 32. 40, 50. 33. 11, 17. 34. 28.
35. 24. 36. 1024. 37. 450, 270. 38. 2200, 1620, 1100, 1080.
39. 60. 40. 7 + 12 + 32. 41. 30.
42. 60. 43. 240. 44. 3d, 9d, 18.4d. 45. 50d.
46. £133\( \frac{1}{3} \). 47. 24. 48. 60. 49. £120000.
50. 25. 51. 4\( \frac{1}{2} \), 3\( \frac{1}{2} \). 52. 39. 53. 40.
ANSWERS.

54. 200000000.  55. 6s.  56. 48.  57. $49\frac{1}{11}$ minutes past three.  58. $32\frac{8}{11}$ minutes past three.  59. £288.
60. 2 seconds.  61. 40 minutes past eleven.
62. £300 and £200.  63. 14.  64. 640.

XXIII.  1. 10 ; 7.  2. 17 ; 19.  3. 2 ; 13.
4. 4 ; 1.  5. 5 ; 5.  6. 21 ; 12.  7. 20 ; 10.
8. 2 ; −3.  9. 3 ; 2.  10. 3 ; 2.  11. 3\frac{1}{2} ; 4.
12. 10 ; 7.  13. 19 ; 2.  14. $38\frac{1}{2} ; 70$.  15. 6 ; 12.
16. $\frac{3}{4} \frac{8}{9}; \frac{1}{5} \frac{6}{7}$.  17. 10 ; 5.  18. 12 ; 12.  19. 20 ; 20.
20. 13 ; 5.  21. 9 ; 7.  22. 10 ; 4.  23. 4 ; 9.
24. 5 ; 7.  25. $2\frac{1}{2}$ ; 1.  26. '2 ; '2.  27. 10 ; 8.
28. 12 ; 3.  29. 3 ; 2.  30. 63 ; 14.  31. 3 ; 2.
32. 2 ; 3.  33. 4 ; 12.  34. $a ; b$.  35. $a ; b$.
36. $\frac{ab}{a+b} ; \frac{ab}{a+b}$.  37. $b ; a$.  38. $\frac{ab^2c}{a^2+b^2} ; \frac{a^2bc}{a^2+b^2}$.
39. $\frac{ac}{a+b} ; \frac{bc}{a+b}$.  40. $\frac{1}{a+b} ; 0$.  41. $a ; b$.
42. $a+b$ ; $a-b$.

(43. $(a+b)^2 ; (a-b)^2$.
44. $\frac{c}{a+b} ; \frac{c}{a+b}$.

XXIV.  1. 2 ; 1.  3. 2 ; 3 ; 4 ; 6.  3. 2 ; 1 ; 3.
4. 9 ; 11 ; 13.  5. 4 ; 0 ; 5.  6. 5 ; −5 ; 5.
7. 45 ; −21 ; 1.  8. 10 ; 7 ; 3.  9. 51 ; 76 ; 1.
10. $\frac{2}{3} ; \frac{3}{4} ; 5$.
11. $x=\frac{1}{2}(b+c-a)$, &c.
12. $x=\frac{2}{3}(a+b+c)-a$, &c.
13. $x=\frac{1}{2}(b+c)$, &c.
14. $x=y=z=\frac{abc}{ab+bc+ca}$.
15. $x=a$, $y=b$, $z=c$.
16. $v=3$, $x=4$, $y=5$, $z=2$.

XXV.  1. 42 ; 26.  2. 12 ; 16.  3. 116 ; 166.
4. 24 ; 60.  5. 30d. ; 8d.  6. 49 ; 21.  7. $\frac{4}{15}$.
8. 45 ; 63.  9. 72 ; 60.  10. 30d. ; 15d.  11. 5s. ; 3s.
12. 20; 52. 13. 70; 50. 14. $\frac{3}{5}$. 15. $(24 - 1)20$.
20. 1; 2. 21. 59. 22. 100 lbs. 23. 150 yards;
30. 20 yards per minute. 24. 21; 11. 25. 50; 75.
26. 70; 42; 35. 27. 90; 72; 60. 28. 12 miles.
29. 4 miles walking, 3 miles rowing, at first. 30. $33\frac{1}{3}$ miles per hour; $48\frac{1}{2}$ distance.
31. 45; 30 miles per hour. 32. 30; 50 miles per hour.
33. 60 miles; passenger train 30 miles per hour. 34. 150; 120; 90.
35. $3\frac{3}{5}$s.; 38; $2\frac{1}{2}$. 36. 4; 59; 55. 37. 120; 80; 40.
38. 432. 39. 420; 35; 21 shillings. 40. 2; 4; 94.

XXVI. 1. ± 4. 2. ± 25. 3. ± 7. 4. ± 9.
5. ± 9. 6. ± 6. 7. 1, 2. 8. 2; 3. 9. 2, -12.
10. 3, $-\frac{1}{2}$. 11. $4\frac{1}{3}, -3$. 12. 10, 5.
13. 5, $-\frac{5}{2}$.
14. 6, -3. 15. $3, -\frac{1}{2}$. 16. $\frac{9}{2}, \frac{1}{2}$.
17. 5, $\frac{2}{3}$.
18. 3, -9. 19. $2\frac{1}{2}, -\frac{1}{2}$. 20. $1\frac{1}{2}, -1\frac{1}{2}$.
21. 1, 2.
22. 4. 23. 6, $\frac{9}{4}$.
24. 11, 3. 25. 5, $3\frac{1}{2}$.
26. 44, -2. 27. 7, $-\frac{7}{12}$.
28. 10, -10.
29. 3, $-2\frac{1}{3}$. 30. $\frac{1}{2}, -3$. 31. 2.
32. 2, -3.
33. ± 2. 34. 1, -4. 35. 3, $-\frac{2}{3}$.
36. 6, $2\frac{2}{5}$.
37. 6, $\frac{16}{7}$. 38. 7, $\frac{7}{3}$. 39. 8, $2\frac{4}{11}$.
40. 3, $-4\frac{2}{3}$.
41. 3, -5. 42. 3, $-\frac{5}{7}$. 43. 2, -1.
44. 4, -1.
45. 7, $3\frac{14}{13}$. 46. $1\frac{3}{4}, 1$. 47. $4\frac{1}{3}, \frac{1}{7}$.
48. 3, $-\frac{4}{5}$.
49. 3, -9. 50. -10, $9\frac{3}{5}$. 51. 3, $-1\frac{1}{3}$.
52. 3, $-1\frac{3}{3}$.
53. 4, 0.  54. 1\frac{1}{3}, 0.  55. 13, \frac{5}{7}.  56. 6, -3\frac{1}{3}.
57. 5, -1\frac{1}{3}.  58. 5, 1\frac{1}{5}.  59. 5, -1\frac{1}{4}.  60. 2\frac{2}{3}, 0.
61. \frac{a+1}{a}.  62. (a+b)^2.  63. \pm \sqrt{(ab)}.  64. a, -\frac{b(a+b)}{2a+b}.

XXVII. 1. ±2, ±3.  2. 49.  3. 4.  4. ±4.  5. 5, -3.  6. 3, -2.  7. 6, 0.  8. 12, -3.
9. 9, -12.  10. ±3.  11. 2, -15\frac{1}{3}.  12. 4, 11\frac{1}{3}.
13. 1\frac{2}{5}.  14. 16.  15. 1.  16. 3, 4\frac{4}{5}, 5.  17. 4.
18. 4.  19. \frac{4(a+b)(a^2+b^2)}{(a-b)^2}.  20. \frac{a-1}{2}.  21. 3a^2.
22. 0, ±1\frac{1}{\sqrt{5}}.  23. 0, ±5.  24. 0, ±\sqrt{2}.  25. 2, ±1.
26. 0, ±\sqrt{(ab)}.  27. a, -2a, -2a.  28. a, \frac{3a}{2}, -\frac{a}{2}.

XXVIII. 1. 36, 24.  2. 36, 24.  3. 30, 24.
4. 18, 12, 9.  5. 12, 10.  6. 4, 6.  7. 196.
8. 3, 48.  9. 11.  10. 7.  11. 6, 12.  12. 15.
13. 24.  14. 27 lbs.  15. 88. 9d., 78.  16. £20.
17. 126, 96.  18. 8d.  19. 10, 9 miles.  20. 56.
21. 192, 128.  22. 9 gallons.  23. 64.  24. Equal.
25. 4 per cent.

XXIX. 1. 5, -4; 4, -5.  2. 4, -\frac{25}{7}; 1, -\frac{71}{35}.
3. ±8; ±6.  4. 6, 12; 2, -4.  5. 7, -4; 4, -7.
6. 4, -\frac{48}{13}; 3, -\frac{41}{13}.  7. -24, \frac{6}{5}; 12, \frac{4}{5}.  8. 6, -\frac{4}{81}; 5, \frac{13}{81}.
9. 2, -\frac{29}{24}; 4, -\frac{53}{6}.  10. 6, 0; 5, 0.  11. 2, 0; \frac{3}{2}, 0.
12. 3, \frac{2}{3}; 5.  13. 4, \frac{1}{8}; 8, \frac{1}{4}.  14. \frac{a+b}{a}, 0; \frac{a+b}{b}, 0.  15. a, b.
16. \( a, \frac{(3b-a)a}{a+b}; b, \frac{(3a-b)b}{a+b} \).
17. \( a, \frac{2ab^2}{a^2+b^2}; b, \frac{2ba^2}{a^2+b^2} \).

18. \( a, 0; 0, b. \)
19. \( \pm 4, \pm \frac{7}{\sqrt{2}}; \pm 3, \pm \frac{1}{\sqrt{2}}. \)
20. \( \pm 5; \pm 4. \)

21. \( \pm 7; \pm 6. \)
22. \( \pm 15; \pm 7. \)
23. \( \pm 4, \pm 14; \pm 1, \mp 4. \)

24. \( \pm 9; \pm 4. \)
25. \( \pm 3, \pm 36; \pm 5, \mp \frac{23}{2}. \)
26. \( \pm 9; \pm 3. \)

27. \( \pm 8; \pm 6. \)
28. \( \pm 2; \pm 1. \)
29. \( \pm 9, \pm 8\sqrt{2}; \pm 7, \pm \sqrt{2}. \)

30. \( \pm 4; \pm 1. \)
31. \( 0, 1, \frac{15}{22}; 0, 2, \frac{9}{22}. \)

32. \( \pm \frac{(a+1)b}{\sqrt{(2a^2+2)}}; \pm \frac{(a-1)b}{\sqrt{(2a^2+2)}}. \)
33. \( \pm a, \pm \frac{a+b}{\sqrt{2}}; \pm b, \pm \frac{a-b}{\sqrt{2}}. \)

34. \( \pm a, \pm \frac{a+1}{\sqrt{2}}; \pm 1, \pm \frac{a-1}{\sqrt{2}}. \)
35. \( 6, -4; 4, -6. \)

36. \( 5, 4; 4, 5. \)
37. \( 4, 2; 2, 4. \)
38. \( 4, -3; 3, -4. \)

39. \( 1, 2; 2, 1. \)
40. \( \pm 4, \pm 3; \pm 3, \pm 4. \)
41. \( 2, 1; \frac{2}{3}, \frac{1}{3}. \)

42. \( \pm 5; \pm 3. \)
43. \( 2, 1, -1, -2; 1, 2; -2, -1. \)

44. \( \frac{1}{2}, -\frac{2\pm \sqrt{3}}{2}, -\frac{1\mp \sqrt{13}}{4}; 1, -2\mp \sqrt{3}, \frac{-1\mp \sqrt{13}}{2}. \)

45. \( 3, -\frac{1}{3}; 6, -\frac{2}{3}. \)
46. \( 5, -\frac{5}{3}; 2, -\frac{2}{3}. \)

47. \( 2; 1. \)
48. \( 4, \frac{3}{2}; \frac{1}{4}, -\frac{9}{4}; 2, \frac{9}{2}, -\frac{7}{4}, \frac{3}{4}. \)

49. \( a+b+1, -\frac{a+b+1}{a+1}; b, -\frac{b}{a+1}. \)
50. \( \pm \frac{a}{3}; \pm 3b. \)

51. \( \pm \frac{a}{4}; \pm 2b. \)
52. \( 0, a+b, \frac{1}{2}(a-b)\pm \frac{1}{2} \sqrt{((a+3b)(a-b))}. \)

0, \( a+b, \frac{1}{2}(a-b)\pm \frac{1}{2} \sqrt{((a+3b)(a-b))}. \)
53. \( x = a \mp \sqrt{(abc)}; \& c. \)

54. \( (x+y)(y+z)(z+x) = \pm abc; \& c. \)
55. \( \pm 1; \pm 2; \pm 3. \)
56. \( \frac{8}{3}, \frac{3}{3}, \frac{3}{8}; \pm 2. \)
XXX. 1. 11; 7. 2. 6; 18. 3. 8; 24. 4. 8; 16. 5. 10; 15. 6. 10; 12. 7. 7; 5. 8. 18; 8: 6; 16. 9. 5; 3. 10. 4; 2. 11. 2; 2. 12. 4; 6. 13. 7; 4. 14. 12; 8. 15. 7; 0; 15. 16. 30; 40. 17. 60; 10. 18. 64. 19. 160; £2. 20. 24; 48; 38. 21. 756; 36; 27. 22. 4½ walking; 4½ rowing at first. 23. 10; 12 miles per hour. 24. 6 miles.

XXXI. 1. \(8x^5y^6z^{12}\). 2. \(-8x^5y^6z^9\). 3. \(81a^4b^8c^{12}\).

4. \(\frac{4x^4}{9y^3}\). 5. \(-\frac{64a^3}{27y^6}\). 6. \(\frac{x^{12}}{y^3z^9}\).

7. \(a^6 + 7a^6b + 21a^6b^2 + 35a^6b^3 + 35a^6b^4 + 21a^6b^5 + 7ab^6 + b^7\). 8. \(a^5 - 3a^2b^2 + 3a^2b^4 - b^6\). 9. \(1 - 3x + 3x^2 - x^3\).

10. \(8 + 12x + 6x^2 + x^3\). 11. \(27 - 51x + 36x^3 - 8x^4\). 12. \(1 + 4x + 6x^2 + 4x^3 + x^4\). 13. \(x^4 - 8x^3 + 24x^2 - 32x + 16\). 14. \(16x^4 + 96x^3 + 216x^2 + 216x + 81\). 15. \(2a^3x^3 + 6axb^2y^2\). 16. \(2a^4x^4 + 12a^2x^2b^2y^2 + 2b^4y^4\). 17. \(1 - 4x^2 + 6x^4 - 4x^6 + x^8\). 18. \(2(5x + 10x^3 + x^5)\). 19. \(1 + 2x - x^2 - 2x^3 + x^4\). 20. \(1 + 2x + 3x^2 + 2x^3 + x^4\).

21. \(1 + 6x + 13x^2 + 12x^3 + 4x^4\). 22. \(1 - 6x + 15x^2 - 18x^3 + 9x^4\). 23. \(2(4 + 25x^2 + 16x^4)\). 24. \(1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6\). 25. \(1 + 3x - 5x^3 + 3x^5 - x^6\).

26. \(1 + 9x + 33x^2 + 63x^3 + 66x^4 + 36x^5 + 8x^6\). 27. \(1 - 9x + 36x^2 - 81x^3 + 108x^4 - 81x^5 + 27x^6\).

28. \(2(36x + 171x^3 + 144x^5)\). 29. \(2(36x + 171x^3 + 144x^5)\). 30. \(1 + 4x + 10x^2 + 20x^3 + 25x^4 + 24x^5 + 16x^6\).

31. \(4(ab + ad + bc + cd)\). 32. \(2(a^2 + 2ac + c^2 + b^2 + 2bd + d^2)\). 33. \(1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6\).

34. \(1 - 12x + 60x^2 - 160x^3 + 240x^4 - 192x^5 + 64x^6\). 35. \(8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8\). 36. \(1 - 3x^3 + 3x^6 - x^9\). 37. \(1 + 3x^2 + 6x^4 + 7x^6 + 6x^8 + 3x^{10} + x^{12}\). 38. \(1 \frac{5ab}{7c^3}\). 39. \(-\frac{6ab^3}{5c^2}\). 40. \(\frac{3a}{bc}\). 41. \(\frac{a}{2b^2}\).
10. \( \frac{2ab^2}{c^4} \)

11. \( 4a + 5b \)

12. \( 7a^2 - 6b \)

13. \( 6x^3 + 1 \)

14. \( 8a + 3bc \)

15. \( \frac{5a + 2b}{5a + 2c} \)

16. \( \frac{3x^2 - 4}{2x - 3} \)

17. \( x^2 + x + 1 \)

18. \( 1 - x + 2x^2 \)

19. \( x^2 + 3x + 8 \)

20. \( x^2 - 2x - 2 \)

21. \( 1 - 2x + 3x^2 \)

22. \( 2x^4 - x^2 - 2 \)

23. \( x^3 - ax + 2a^2 \)

24. \( x^2 - ax + b^2 \)

25. \( x^3 - 6x^2 + 12x - 8 \)

26. \( x^3 + 2ax^2 - 2a^2x - a^3 \)

27. \( 1 - x + x^2 - x^3 + x^4 \)

28. \( \frac{2x - 4x - 3y}{3y - 5z + 4z} \)

29. \( 1 + x \)

30. \( 2x - 3y \)

31. \( 1 - x + x^2 \)

32. \( x^2 - (a + b)x + ab \)

33. \( x + 1 \)

34. \( x^2 - xy + y^2 \)

35. \( 34 \)

36. \( 3 \)

37. \( 61 \)

38. \( 72 \)

39. \( 87 \)

40. \( 99 \)

41. \( 123 \)

42. \( 321 \)

43. \( 407 \)

44. \( 55.5 \)

45. \( 6.42 \)

46. \( 914 \)

47. \( 1234 \)

48. \( 5420 \)

49. \( 6201 \)

50. \( 70.58 \)

51. \( 8.008 \)

52. \( 4937 \)

53. \( 12007 \)

54. \( 504.06 \)

55. \( 18042 \)

56. \( 21319 \)

57. \( 75416 \)

58. \( 443329 \)

59. \( 94868 \)

60. \( 249198 \)

61. \( 65574 \)

62. \( 00233 \)

63. \( 412310 \)

64. \( 1135781 \)

65. \( 1863488 \)

66. \( 11956331 \)

67. \( 2x + 3y \)

68. \( 12x^2 + 4y^3 \)

69. \( x - a - b \)

70. \( x^2 + x + 1 \)

71. \( x^2 - ax - a^2 \)

72. \( 2x^2 + 4ax - 3a^2 \)

73. \( 1 - 3x + 4a^2 \)

74. \( 1 - x + x^2 - x^3 \)

75. \( 1 + 2x \)

76. \( 3x - 1 \)

77. \( 27 \)

78. \( 35 \)

79. \( 54 \)

80. \( 61 \)

81. \( 88 \)

82. \( 92 \)

83. \( 138 \)

84. \( 148 \)

85. \( 378 \)

86. \( 392 \)

87. \( 576 \)

88. \( 604 \)

89. \( 1111 \)

90. \( 2755 \)

91. \( 45045 \)

92. \( 17479 \)

XXXIII. \( \frac{1}{3} \)

2. \( \frac{1}{5} \)

3. \( \frac{1}{10} \)

4. \( 100 \)

5. \( \frac{1}{27} \)

6. \( a^{-6} \)

7. \( a^6 \)

8. \( a^{-2} \)

9. \( a^{-1} \)

10. \( a^{7} \)

11. \( x^{\frac{3}{2}} - y^{\frac{3}{2}} \)

12. \( a - b \)

13. \( x^2 + 2x^2 + x - 4 \)

14. \( x^4 + 1 + x^{-4} \)

15. \( a^{-1} - 1 \)

16. \( a^2 - 3a^2 + 3a^{-2} - a^{-2} \)

17. \( a^2 + 2a^2 b^1 + a - x^2 y^1 \)

18. \( x^5 + x^3 y - xy^2 - y^3 \)

19. \( x^3 + y^3 + x^3 y + y^3 \)

20. \( a^3 + a^1 b^1 + b^3 \)

21. \( 16x^{\frac{3}{2}} - 12x^{-\frac{1}{2}} y^{\frac{1}{2}} + 9y^{\frac{1}{2}} \).
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22. \( x + y \).
23. \( a^\frac{3}{4} - a^\frac{1}{4} b^\frac{1}{4} + b^\frac{3}{4} \).
24. \( a^\frac{3}{4} + b^\frac{1}{4} - c^\frac{1}{4} \).
25. \( x^\frac{3}{4} + 2x^\frac{3}{4} a^\frac{1}{4} + 3x^\frac{1}{4} a + 2x^\frac{1}{4} a^\frac{3}{4} + a^2 \).
26. \( x^\frac{3}{4} - 2x^\frac{1}{4} y^\frac{1}{4} + y^\frac{1}{4} \).
27. \( x^\frac{3}{4} - 2x^\frac{1}{4} \).
28. \( x - 2 - x^{-\frac{3}{4}} \).
29. \( x^\frac{5}{3} - 2x^\frac{1}{3} + x^\frac{3}{3} \).
30. \( 2x^\frac{3}{3} - 3 + 4x^{-\frac{3}{3}} \).

XXXIV. 1. \( 7 \sqrt{2} \).
2. \( 9 \sqrt{4} \).
3. \( \frac{8}{3} \sqrt{3} \).
4. \( \frac{\sqrt{4}}{4} \).
5. \( \frac{13\sqrt{15}}{10} \).
6. \( \frac{5\sqrt{2}}{2} \).
7. \( 2 + 2 \sqrt{2} - 2 \sqrt{3} \).
8. \( 2 + \frac{5}{6} \sqrt{6} \).
9. \( 4 + \frac{5}{2} \sqrt{2} \).
10. \( 5 + 2 \sqrt{6} \).
11. \( \frac{24 - \sqrt{15}}{33} \).
12. \( \frac{1}{7} \left( 18 + 9 \sqrt{6} + 4 \sqrt{15} + 6 \sqrt{10} \right) \).
13. \( 3 + \sqrt{5} \).
14. \( 3 - \sqrt{7} \).
15. \( \sqrt{6} + \sqrt{2} \).
16. \( \sqrt{\frac{5}{2}} - \sqrt{\frac{3}{2}} \).
17. \( \sqrt{3} - \sqrt{2} \).
18. \( 2 + \sqrt{3} \).
19. \( \sqrt{3} \).
20. \( \sqrt{10} \).

XXXV. 1. \( \frac{2}{9} \).
2. \( \frac{7}{12}, \frac{5}{8}, \frac{2}{3}, \frac{3}{4}, \frac{8}{9} \).
3. \( \frac{5}{27} \).
4. \( 14, 21 \).
5. \( 24, 30 \).
6. \( 20, 32 \).
7. \( 1 \).
8. \( 15, 10 \).
9. \( 6, 8 \).
10. \( 35, 42 \).
11. \( 4 \).
12. \( \frac{ab}{a + b} \).
13. \( 50, 60, 90 \).
14. \( 0, 2 : 5 \).

XXXVI. 1. \( 14 \).
2. \( 18 \).
3. \( 15 \).
4. \( 12 \).
5. \( 4 \).
6. \( 4 \).
7. \( 2, 2 \frac{1}{2} \).
8. \( 5 \).
9. \( 1, -1 \).
13. \( 45, 60, 80 \).
14. \( 4, 6, 9 \).

XXXVII. 1. \( 4 \).
3. \( 5 : 2 \).
4. \( 2 \).
5. \( 4 \).
6. \( 5 \).
7. \( 8 \).
8. \( abc \).
9. \( \frac{ac^2}{\sqrt{2}} \).
10. \( £113 \frac{1}{2} \).
11. \( 15 \).
12. \( £15360 \).
XXXVIII. 1. 936.  2. 77 1/2.  3. 69.  4. 139 1/2.  
5. 37 1/2.  6. -115.  7. 14, 16, 18.  8. 14 1/3, 14 2/3,...  
9. 6 1/2, 5,...  10. -1/3.  11. 10, 4.  12. 82.  
13. 5, 9, 13, 17.  14. 5, 7, 9.  15. 1, 2, 3, 4, 5.  
16. 18, 19.  17. 18.  5.  19. 1, 4, 7.  20. 1, 2.  
XXXIX. 1. 1365.  2. 134 1/5.  3. 40 2/5.  4. 63 (\sqrt{2}+1).  
5. 6 65/4 86.  6. 463 96.  7. 3 4.  8. 4 3.  9. 3.  10. 41/2.  
15. 4, 16, 64.  16. 8, 12, 18, 27.  17. -9, 27, -81, 243.  
18. 3, 12, 48; or 36, -54, 81.  19. 1, 3, 9,...  20. 3, 6, 12.  
XL. 1. 3 6/5.  1. 2 4/5.  8 1.  2. 3 12/5.  
4. 2 1/15.  12. 2 2/33.  5. 6, 12.  6. 36, 64.  7. 1, 9.  8. 3, 9.  
XLI. 1. 131596.  2. 5040.  3. 126.  4. 30240.  
5. 11. 6. 1990.  7. 15504; 3876.  8. 27; 99.  
XLII. 1. \(a^{13} - 13a^{12}x + 78a^{11}x^2 \ldots - 78a^2x^1 + 13ax^{12} - x^{13}\).  
2. 243 - 810x^2 + 1080x^4 - 720x^6 + 240x^8 - 32x^{10}.  
3. 1 - 14y + 84y^2 - 280y^3 + 560y^4 - 672y^5 + 448y^6 - 128y^7.  
4. \(x^n + 2nx^{n-1}y + 2n(n-1)x^{n-2}y^2 + \frac{4n(n-1)(n-2)}{3}x^{n-3}y^3\).  
5. 1 + 4x + 2x^2 - 8x^3 - 5x^4 + 8x^5 + 2x^6 - 4x^7 + x^8.  
6. 1 + \(\frac{x}{15} + 15x^2 + 30x^3 + 45x^4 + 51x^5 + 45x^6 + 30x^7 + 15x^8 + 5x^9 + x^{10}\).  
7. 1 - 8x + 28x^2 - 56x^3 + 70x^4 - 56x^5 + 28x^6 - 8x^7 + x^8.  
8. 5922.  9. 1590.  10. \(x = \frac{2}{1}, y = \frac{3}{1}, n = 5\).  11. \(x = 4, y = \frac{1}{2}, n = 8\).  
12. \(a^4 - \frac{a^3}{2} - \frac{3a^2}{8} - \frac{7a}{16} - \frac{7a^2}{128}\).  
13. \(1 + \frac{x}{2} + \frac{x^2}{3} + \frac{13x^3}{34}\).  14. \(1 + 2x + 4x^2 + 8x^3 + \ldots\)  
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15. \( r + 1. \)

16. \( \frac{3. 7. 11. 15. 19}{4 \sqrt{5}} \) \((x^3 + 3 - 5y^2). \)

17. \( a\frac{1}{3} + 10a\frac{1}{8} b + 65a\frac{1}{3} b^2 + \frac{1}{3} a\frac{1}{5} b^3 + \frac{4940}{3} a^{-\frac{22}{5}} b^4. \)

18. \( \frac{(r + 1)(r + 2)(r + 3)}{1 \cdot 2 \cdot 3}. \)

19. \( 1 + \frac{1}{2} x \cdot \frac{3x^2}{8} - \frac{3x^3}{16}. \)

20. \( 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{7x^3}{16}. \)

XLIII. 1. 2012132. 2. 22600. 3. 11101001010.


8. 9e21; tc. 9. Radix 5. 10. eee.

XLIV. 1. \( \frac{bc}{a}. \)

3. \( n = \frac{1}{r}. \)

Miscellaneous. 1. 729, 369, 1, 41.

2. \( 41x - 51y. \)

3. \( -30x + 37x^2 - 20x^3 + 4x^4. \)

4. \( 1 + x - x^3 - x^4. \)

5. \( \frac{2x^2 + 3x - 4}{3x - 4}. \)

6. \( (4x^2 - 9)(9x^2 - 4). \)

7. \( \frac{2}{a}. \)

8. 3.

9. 240, 360.

10. \( £2, \ £2\frac{2}{3}. \)

11. \( \frac{7x}{6} + \frac{7y}{6} + \frac{7z}{6}, x + \frac{13y}{6} + \frac{3z}{2}. \)

12. 1.

13. \( 3b^3. \)

14. \( 2x^2 - xy - 2y^2. \)

15. \( \frac{x^2 - x - 1}{x^2 + x + 1}. \)

16. \( (x - 10)(x + 1)(x + 3). \)

17. \( (x - 10^5)(x + 1)(x + 3). \)

18. 5.

19. 7.

20. \( £40. \)

21. \( 2a - 2b - x - 2y, a + 3b + 4x + 4y. \)

22. 11.

23. \( x^4 - a^4. \)

24. \( \frac{x^2 + x - 1}{3}. \)

25. \( x^2 - 2. \)

26. \( \frac{ab - b^2}{b^2 - 4a^2}. \)

27. \( (16x^2 - 1)(x^2 - 4). \)

28. 6.

29. \( 14s, 21s, 52\frac{1}{2}s. \)

30. 100.

31. 1.

32. \( (x^2 - a^2)(x^2 - b^2), (x - a)(x - b). \)
33. \(4x^4 - 2x^2y + x^2y^2 - xy^3 + \frac{y^4}{2}\).

34. \(x - 2\).

35. \(\frac{3(4x-y)}{2(3x^2+y^2)}\).

36. 1.  37. 4.  38. 2.  39. 30 minutes.

40. £18, £6.

41. \(10x + 10x\).

42. \(7x^2 - 2xy + y^2, -x^2 - 6xy + 7y^2, 12x^4 - 10x^2y - x^2y^2 + 20xy^3 - 12y^4\).

43. \(a + b - c\).

44. \(x^2 + 1\).

45. \(\frac{1}{x+1}\).

46. \((x^2 - 4)(x^2 - 9)\).

47. \(\frac{x^2 + x + 2}{2x^2 + x - 1}\).

48. 1.

49. \(\frac{16}{25}\).

50. 30 lbs.

51. \(3a^2 - 5a^2b - 12a^2b^2 - ab^3 + 3b^4, 3a^2 - 8a^2b - 4a^2b^2 + 3b^3\).

52. \(2x - 5\).

53. 2.

54. \(\frac{(a + b)b}{a}\).

55. 1; 2.

56. 3; 6.

57. 5; 8.

58. 4; 5; 2.

59. \(a^2 + b^2, \frac{a^2 + b^3}{am + bn}, \frac{bm - an}{am + bn}\).

60. \(\frac{3}{5}\).

61. \(x^4 + x^3 + 1 + \frac{1}{x^2} + \frac{1}{x^4}\).

62. \(\frac{11x + 2}{7x^2 + 7x + 2}\).

63. \(\frac{2ax}{x^2 + 1}\).

64. 1.

65. 4; 3.

66. 2; 4.

67. 3; -3.

68. 3.

69. 2.

70. 20; 40 years.

71. 1.

72. \(\frac{2x - 1}{2x - 1}, \frac{8}{5}\).

73. \((x - 2)(x - 1), (x - 2)(x - 5), (x - 1)(x - 5)\).

74. 0.

75. \(\frac{2}{5}\).

76. \(\frac{17}{3}, \frac{4}{3}\).

77. 3 shillings, 2 shillings.

78. \(3x^2 - x + 1\).

82. \((x^2 - 4y^2)^3\).

83. 3; -2.

84. 5.

85. 47 or 74.

86. 45 gallons.

87. \(4x^3 - 3xy^2 + 5y^3\).

88. \(\frac{52}{5}\).

89. 4\,\text{5409}.

91. \(x - y\).

92. \(8(x^2 + y^2), 4S(x^4 - y^4)\).

93. \(\frac{x^2 + 3y^2}{x^4 - y^4}\).

94. 1.

95. 4, 5, 6.

96. \(\frac{3}{5}, \frac{-5}{3}\).

97. 20 miles.

98. Present price 3 pence per dozen.

99. \(18\left(1 - \frac{1}{3^8}\right)\); 18.

100. 4, 8, 16.
101. \(x^4 - 1, 1 + x^\frac{1}{3} + x^\frac{2}{3}\).
102. \((x^2 - a^2)(x^6 - a^6)\).
103. \(a\).

104. 1.
105. 13, \(\pm \sqrt{\frac{35}{11}}\).
106. \(\pm 3; \pm 4; \pm 5; \pm \frac{35}{11}\).
107. '20 shillings.

108. 48.
109. \(\frac{x}{y} + 1 - \frac{y}{x}\).
110. \(x^n - 1 + x^{-n}\).

112. \(\frac{x^2 + 5x + 10}{x^3 + 2x^2 + 3x + 6}\).
113. \(\frac{38}{7}\).
114. 1 or -3.

115. \(\pm 2; \pm 1\).
116. \(\frac{1}{3}, \frac{1}{2}, \frac{1}{3}\).

117. \(\frac{2}{3}\).
118. \(-x^2\).
119. 612.

121. \(\frac{2a^2 + a^2 - a^2}{2a^2 - b^2 + a - b}\).
122. \(3x^2 - 5xy + 2y^2\).

123. \(x(3x + 4)(x - 6)\).
124. \(\frac{2}{17}\).
125. \(2, \frac{1}{2}\).

126. \(5, -\frac{13}{4}; 4, \frac{5}{4}\).
127. \(1, \frac{5}{3}; 2, \frac{2}{3}\).
129. 3.

130. \(3(3^n - 1)\).
131. \(\frac{1}{x}\).
132. \(2x(3x + 4)\).

133. 4, -3.
134. \(x^2 - x - 6 = 0\).
135. \(x^4 = a^4\) or \(\frac{1}{a^4}\).

136. \(\pm 2\).
137. 8.19615.
138. \(7 - 2\sqrt{3}\).
139. \(\frac{x + y}{x_3}\).

140. \(\frac{c + b - 2a}{b - a}, \frac{(a + c)(c + b - 2a)}{2(b - a)}\).
141. 3, 2, 2.

142. 197, \(3x^3 - 2x^2 - 5x - 3\).
143. \(a(a^2 + b^2), \frac{4xa}{x^2 - a^2}\).

144. (1) 4. (2) 0, 5. (3) 5; 7.
145. (1) 3, \(\frac{80}{11}\). (2) 8.
(3) \(\pm 7; \pm 5\).
146. 16; 16.
147. 20.
148. 16. 24.

149. \(\frac{15}{4}\).
150. As 5 to 1.
151. \(x^3\).
152. \(a^2 + 4b\).

153. \(x - 3\).
154. (1) 5. (2) 3. (3) 7; 4.

155. (1) 8. (2) 9. (3) \(\pm 9; \pm 7\).
156. 30 pence.
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157. 80.  
158. £20.  
159. \( x + 2a \).

161. \( a, 21a - 27b + 6c, \ a^{11mnpr} \).

163. \( 72(x-y)^2(x^3+y^3) \).

164. (1) 9. (2) 8. (3) 12. (4) 20; 2.

165. (1) \( \frac{2}{5} \). (2) 11. (3) \( \pm 11, \pm 13 \);

\( \pm 13, \pm 11 \). (4) \( \pm 2 \); \( \pm 1 \). 166. 12 days.

167. 4, 8. 168. \( \frac{4}{15} \).

170. 208; 400. 171. 23b - 18a. 172. 2, \( p^2 \), \( x^4 \).

173. \( x^3 - 3ax^2 + 3a^2x - a^3 \). 174. (1) 13. (2) 4. (3) 6; 10. (4) 3.

175. (1) 2, 4. (2) \( \pm 5 \); \( \pm 4 \). (3) \( \pm 1, \pm 7 \);

\( \pm 1, \pm 5 \). (4) 1, 5; 5, 1. 176. 164 minutes after 12.

177. 36. 178. 40, 23. 179. 36 \( \left( 1 - \frac{26}{35} \right) \), 36.

180. \( 7 - \sqrt{6} \). 181. 15. 182. \( \frac{3x + 2}{x^2 - 2x - 24} \).

184. (1) 9. (2) 6; 8. (3) 4, \( -\frac{7}{4} \).

185. (1) 13, -15. (2) 7. (3) 2, -1.

186. 288, 224. 187. 20 miles. 188. On the first day A won 8 games and lost 4 games.

190. \( -85\frac{1}{2} \).

191. \( \frac{18x^4 + 12x^3 - 43x^2 + 36x - 18}{144} \), \( \frac{6x^3 - 20x^2 + x + 36}{4} \).

192. \( \frac{4x^2 - 15x + 13}{x^3 - 6x^2 + 11x - 6} \).

193. \( x^4 - 16y^4 \). 194. (1) 8. (2) 7. (3) 40; 16.

195. (1) \( -\frac{3}{3} \), \( \frac{3}{2} \). (2) 13. (3) 2, 4;

4, 2. 196. 56 miles. 197. 24. 198. 23 + 15.

199. \( a^2 - ab + b^2, \ a^2 + b^2 \).

200. 2, 4, 8, 16.

201. \( \frac{x^3 + 9x - 13x^2}{3(7 - 2x)} \).

202. \( x^2 - 2x + 4 \).

203. \( \frac{16x^2}{(2 + 3x)^3} \).

204. (1) 9. (2) \( \frac{a^2}{b} \). (3) 6; 8.
205. (1) 7, $\frac{5}{6}$. (2) 1, -4. (3) ±3; ±2. 206. 10 miles.
207. 24. 208. 6 crown 9 shillings. 209. $2x^2 + 2ax + 4b^3$. 210. '7, 11, 15, ...
211. $3x^3 - 2x^2y + 3xy^2 - 5y^3$. 212. $\frac{x^2}{12 + 5x - 28x^4}$.
213. $\frac{4x^3 - 25x + 37}{x^3 - 10x^2 + 31x - 30}$. 214. (1) 9. (2) 16; 4.
(3) 3; 6; 9. 215. (1) 3, -6. (2) ±7; ±5. (3) 2, 4; 4, 2.
216. 114 of each. 217. 126. 218. 21. 219. 11, 12, 13, 14.
220. $3 + 2\sqrt{2}$. 222. $x^m + x^3 + 1, px^2 + qx - r$.
221. $\frac{a^m-1}{b(a+b+c)} \cdot \frac{c+b+c}{a-b-c}$.
(2) 100, -200. (3) $\frac{ac}{2a + 2\sqrt{(a^2 - b^2)}}; \frac{bc}{2a + 2\sqrt{(a^2 - b^2)}}$.
223. $\frac{7}{12}$ of a mile. 227. 500; 1000; 4000. 228. 2 hours; 4 hours.
229. 2, 5, 8, .... 230. $\frac{5n}{12} (9-n)$.
231. $x^{2a+2b+2c}$, $\frac{b(a^2 + b^2)}{c(a^2 - b^2)}.$ 232. $\frac{x + 5}{9x^2 - x - 3}.$
233. (1) $\frac{1}{2}$. (2) $\pm \frac{1}{2}$. (3) $\frac{1}{4}$; $\frac{1}{5}$.
234. (1) 5, $\frac{27}{5}$. (2) $\frac{b-c + a}{\sqrt{a}}$. (3) 5; ±4. 235. 19.
239. $a^3 + a^2b + ab^2 + b^3, a + 2b + 3c$. 240. $x^2y^3 + 8x^2y^2$.
241. $14xy, \frac{2(1 + x^2 - x^3)}{1 - x^4}$. 242. $x + a$. 243. 105 shillings.
244. 54. 245. 3, 5, 8. 246. 6$\frac{1}{4}$ per cent.
247. 2200. 248. $\frac{5}{21}$. 249. 5$\cdot$678, 1$\cdot$234. 251. $2a - b$. 
252. $a^{16} - x^{16}$, c. 253. $x(3a + 2x)$. 254. (1) 5.

(2) 114; 77. (3) 0, $\frac{a}{2}$. 255. $\frac{a}{112}$; 96. 256. $A$ has £5400, $B$ has £7200. 257. 7; 13. 258. 80.

259. 8; 5. 260. £80. 261. $c^2 + 2bc$.

262. $x^{16} - 1$, $- \frac{1}{x^2 (x^2 + 2x + 2) (x^2 + 3ax^3 - 4a^2x^2 - 3a^3x + 2a^4)}$.

263. $x^2 - x + 1 + \frac{1}{x} + \frac{1}{x^2}$. 264. $\frac{x^2 - 2x + 3}{2x^2 + 5x - 3}$, 1. 265. (1) $\frac{3}{7}$.

(2) 1. (3) 18; 9. 266. (1) 3, -2. (2) $\frac{5}{6}$. (3) 2, 3; 3, 2.

268. 45 shillings, 50 shillings. 270. $x^2 + \frac{x}{2} - \frac{1}{3}$, 5 - $2\sqrt{2}$.

271. 0. 272. $x^2 + 3x + 8$. 273. $\frac{3(a^2 + x^2)}{a^2 - x^2}$.

274. $\frac{4x^4}{(4a - 3x)(5a - 2x)}$, $x(1 - x)$.

275. $\frac{b(c + d) + 1}{ab(c + d) + a + c + d}$, $\frac{a^2 - a(x + x^2)}{a^2 + a(x + x^2)}$. 276. (1) 2.

(2) 11; 7. (3) 4; $\frac{16}{3}$. 277. (1) $\frac{a(a + b)}{a - b}$, $\frac{a(a - b)}{a + b}$. (2) 4, 7.

(3), 5. 278. 7 + 7 miles. 281. $\frac{34}{35}$. 282. 2 $(a - b)(c - d)$, $-2bc$.

283. $\frac{y^2 - 6x}{x - 11a}$, $-\frac{4xy}{x^2 - y^2}$. 284. (1) 4. (2) 6; 4.

(3) $\frac{5}{3}$. 285. (1) $\frac{1}{2}$, $\frac{7}{5}$, (2) $2a$, $-a$, $a$, $-\frac{a}{2}$. (3) $\frac{53}{19}$, $-2$, $\frac{47}{19}$. 286. Second boat 16 minutes.

289. 18 feet. 290. $\frac{n}{2} \left( \frac{\frac{2}{3} + \frac{(n-1)x}{1-x^2}}{1+x} \right)$. 291. 0.

292. $b^2$. 294. $\frac{x^2 + 3}{x^2 + 2x + 3}$. 294. 327.
295. (1) 4. (2) 61; 73. (3) 16; 8.  296. (1) 7, −8.
   (2) 7, −\(\frac{29}{4}\).  (β) 1, 5.  297. 144 minutes.
298. 4½ hours with the stream, 7½ hours against the stream.
299. \(a - \frac{1}{2}b, a, a + \frac{1}{2}b\).  300. 3, −1.

THE END.