A triangle is determined if:

(1) Two sides and the included angle (the angle 'between' the sides) are known.

(2) One side and two angles (similarly placed) are known (if two angles are given, the third may be found by subtracting the sum of the two from 180°).

(3) Three sides are known.

With a right-angled triangle it will suffice to know one side and one other angle. If in case 1, the angle is not 'included', there may be two possible triangles. (The 'ambiguous' case.)

It will be seen from the diagram (or better, by making a 'working' model with strips of wood or Meccano parts) that if the side opposite the angle is smaller than the side which is an arm of the angle (i.e. contains the angle) there will be two cases. It is a matter of common sense that 3 angles do not determine a triangle for they give us no indication of its size. Equiangular triangles are similar and corresponding sides are proportional. (See the section on Proportion.)

We shall meet triangles again when we consider simple trigonometry and surveying.

*Pythagoras' Theorem*

This most important theorem which finds so many uses in mensuration, trigonometry, pure and applied geometry should be approached experimentally by drawing and measuring, and
through its history. Children are interested in the Chinese method of using triangular and square tiles for making patterns for pavements, and they should lose no opportunity of employing the ancient methods of the Egyptian 'rope-stretchers' for setting out right angles on the football pitch. The formal proof as given in Euclid is outside the scope of secondary modern school work.

The 'Chinese' proof (1150 B.C.)

The square on the larger of the two sides containing the right angle is divided into four pieces by finding the centre of the square and through it drawing two lines parallel to two sides respectively of the square on the hypotenuse. The four pieces thus formed and the smaller square are arranged as shown. Children may make models by cutting card or plywood. Many 'area-puzzles' arise from more complex applications of this.

With a top form set of A children it should be possible to give the steps of the general proof. The ordinary text-book proof founded on Euclid may be demonstrated by cutting cardboard, or using coloured 'washes' to fill in the various figures on a single diagram, but it is necessary to know before commencing this proof (1) that the areas of triangles on the same base and between the same two parallels are equal, and (2) the conditions which are necessary for triangles to be identically equal (congruent).

From the secondary modern school point of view the 'result' of
Pythagoras' Theorem is the important thing. It must not be forgotten that the square on one of the smaller sides is equal to the square on the hypotenuse minus the square on the other side. Briefly, Pythagoras' Theorem will find use in:

1. Drawing right angles.
2. Finding square roots by drawing.
3. Many applications to trigonometry, surveying, and navigation.
4. Finding diagonals and other useful distances in figures and solids which contain right angles.

Children will be interested in drawing right-angled triangles to find square roots. From the formula \( n^2 = (n - 1)^2 + 2n - 1 \), it follows that the square root of any odd number may be found by dividing it into two numbers differing by 1, and drawing a right-angled triangle with hypotenuse and one side proportional respectively to these numbers. The other side represents the required square root. Children will be interested to search for sets of numbers such as 3, 4, 5; 13, 12, 5, etc., which will give right-angled triangles. Alternatively, 'subtractive' sets of numbers may be found, e.g. 7 \( 1^2 \cdot 3^2 \). Thus, to find \( \sqrt{7} \) we construct a right-angled triangle with hypotenuse 4 units and another side 3 units.

The Properties of Circles. How a circle is drawn and defined. The mensuration of the circle and the related plane and solid figures is dealt with separately; but a number of properties of lines and angles in relation to circles may be taken together. By reference to the properties of isosceles triangles, it can be demonstrated that 'the angle at the centre of a circle is double the angle at the circumference'.

From this it follows at once that:

(a) Angles in the same segment of a circle are equal.
(b) The angle in a semi-circle is a right angle. This has important applications in trigonometry and geometrical drawing.
(c) Pairs of opposite angles of a quadrilateral drawn in a circle
(a cyclic quadrilateral) are supplementary (i.e. add up to two right angles).

Many 'circle properties' may be demonstrated by using the principle of symmetry about a diameter. Tangents may be illustrated practically by considering a taut driving-belt fitted to a wheel. The fact that 'the angle between tangent and radius is a right angle' finds application in surveying and navigation, and it gives a clue to the method of drawing and calculating the lengths of tangents by using right-angled triangles.

Even where deductive methods are not used, many teachers still adhere to the order of the geometrical topics as they appear in Euclid, or rather in the modern text-books which have been derived from his work. The sound logical development of the system of Euclid (rather than the faulty logic of some of his proofs) has always appealed to teachers. However, for secondary school purposes the 'topic' method of approach may prove to be more satisfactory; and geometry may develop by considering various objects with regard to position, shape, and size. The idea of a locus may be demonstrated by paper-folding, by joined Meccano or wooden strips which can be made to 'draw' the locus under differing conditions, or again by marking out or selecting suitable lines on the surface of the games field, and 'walking' the locus.

Mensuration of Solids

Probably owing to the long influence of Euclid in school geometry, the important practical aspect of the mensuration of solids was frequently overlooked. The 'solid' work in Euclid comes so late in his scheme, and is so 'dry' to the ordinary pupil, that this important topic which is related to the three-dimensional nature of our lives is often forgotten.

The cube may be considered first and built up with layers of unit cubes. There is often some difficulty in proceeding from the measurement of area to that of volume. The idea of a cubic number and cubing will readily follow. The mensuration of solids should proceed with the study of capacities; for not only can a cube of edge 4 inches be built up by using 64 inch cubes, but a
hollow cube of tin-plate may be made and its volume found by filling it with water. From the cube we pass to the cuboid or rectangular solid (often called 'oblong box'). By building one from unit layers the formula \( l \times b \times h \) for its volume may be demonstrated. As the mensuration of this solid is so important in our daily life it is well to consider it from every point of view: its volume as found by filling a hollow model with water; the dimensions and total area of its faces; the numbers of its edges and faces; the diagonals of its faces and its internal diagonal, i.e. greatest length of thin rod which may be placed inside it from one bottom corner to the opposite top corner (the latter when the result of Pythagoras' Theorem is known), etc.

Hollow wooden rectangular solids may be treated by making one from six rectangles of wood, or alternatively by taking a box to pieces. It will be seen, for instance, that top and bottom have 'external' dimensions, the two 'long' sides have internal height and external length, the two ends have internal dimensions and so on. Calculations of the area of wood required may be checked by finding the volume of the wood, by subtracting the internal volume from the external volume, and dividing by its thickness if this is uniform. This, however, is not always practicable.

By sawing up a wooden rectangular block (a cuboid), prisms may be formed. In secondary modern schools it will suffice to deal with right prisms, i.e. prisms where the rectangular faces are vertical, when the triangular sides are horizontal. Hollow models may be made from tin-plate. It is seen from experiment, and by building up the prism from triangular layers, that its Volume \( \approx \) Area of its Triangular Base \( \times \) Vertical Height. Other solids with parallel sides may also be considered. The prism is very important in practice and is found as part of the form of many buildings.

The trapezoid may be regarded as a cuboid plus one or two prisms. A simple example of a trapezoid occurs in a swimming bath which slopes regularly from the shallow to the deep end. Its volume is equal to the area of a vertical section from one end to the other \( \times \) its breadth; and here it is only necessary to multiply length by the average depth to find the area of the section.

A hollow square pyramid may be made from flanged cardboard
or tin-plate, and its volume found by filling with sand or water. Various types of pyramidal models with square, rectangular, and triangular bases may be used for experiment, and the general rule that Volume \( \frac{1}{3} \times \text{Base} \times \text{Vertical Height} \) verified. By carefully made saw-cuts, a prism of wood may be cut into three triangular pyramids, all of which are equal in volume, but this is not always readily seen.

The writer once attended a ‘camp-fire’ where a mathematical scout-master explained the properties of a regular tetrahedron with a simple demonstration. Three equal scout-staffs 6 feet long were fixed together by rope at their tops. The other ends of the staffs respectively touched the ground at the ‘corners’ of an equilateral triangle of 6 feet side drawn on the ground. The camp fire burnt at the centroid of the triangle and over this suspended from the ‘top’ of the tetrahedron was the kettle. The simple properties of the tetrahedron were readily forthcoming! Skeleton figures in 3 dimensions on a large scale are very useful.

A tetrahedron, octahedron, icosahedron, and other more complex solids may be made by paper-cutting, but from the practical point of view it is better to spend the time exploring fully the square and rectangular pyramids, which find considerable application. By Pythagoras’ Theorem the slant heights and the length of the edges may be found. The frustum of a pyramid is sometimes met with in practice, and may be regarded as what remains when a smaller pyramid is taken from the top of a larger similar pyramid. In all cases, experiments with paper, cardboard, wooden or tin-plate models should be performed, and the model should be carefully planned and drawn before it is cut out on a plane surface. Practice in obtaining rough estimates of quantities and volumes may be obtained by visiting an ‘estate’ where building is proceeding, or ships, barges, and other craft in sea and river ports. The use and calibration of a ‘dip-stick’ for finding the capacities of barrels, covered tanks, etc., will also prove interesting.

With A and B, and to a certain extent with C children, geometry may develop by drawing plans and projections which will lead finally to mechanical drawing.
The Mensuration of the Circle, Cylinder, Cone, and Sphere

The approach to \( \pi \).

'And he made a molten sea, ten cubits from the one brim to the other and a line of thirty cubits did compass it round about.' - First Book of Kings, chap. vii, v. 23.

The nature of the ratio between the circumference of a circle and its diameter has been a tantalizing source of experiment and conjecture for thousands of years. It has entered religion, 'magic', architecture, astronomy, mensuration, and science. No single mathematical problem has absorbed so much time as that of endeavouring to evaluate \( \pi \) or trying to find a formula for it. Expressed in the terms of a geometrical construction this may be stated as follows: 'to find the side of a square equal in area to a given circle' and is now known to be an impossible task. This was one of the three great problems of the Greeks, and like the quest for the philosopher's stone in later years, the searches left in their wake many useful discoveries. Some very accurate approximations to \( \pi \) were obtained by Archimedes who used the idea that the area of a circle lies between that of inscribed and circumscribed polygons. By increasing the number of sides the polygons approximated to the circle. Some interesting guesses and approximations to \( \pi \) are given below.

As will be gathered from the quotation referring to the building of Solomon's Temple, the Jews (in common with the Babylonians) thought that \( \pi \) was 3.

\[
\pi \approx \left( \frac{16}{9} \right)^2 = 3.160 - \text{Egyptians}
\]

\( \pi \) lies between 3\( \frac{10}{7} \) and 3\( \frac{10}{7} \) - Archimedes

(found by using polygons)

\( \pi \approx 3.1416 - \text{Ptolemy} \)

\( \pi \approx \frac{49}{16}, \sqrt{10}, \frac{754}{240} - \text{Hindus and Arabs} \)

\( \pi \approx \frac{355}{113} \) (remembered as 113\( \sqrt{4} \), 355) - Metius (Dutch seventeenth century)
When it was felt intuitively that no fraction could be found to
give an exact value for \( \pi \), calculators rivalled one another by
summing various series of numbers in order to obtain its value
correct to considerable numbers of decimal places.

\( \pi \) is not only an irrational but is also a transcendental number,
and a mathematical demonstration of its nature which was first
given in the nineteenth century was sufficient to show that the
circle could never be squared by ruler and compasses.

\( N.B. \) Note that geometrical constructions can be found for
irrationals, e.g. \( \sqrt{2} \), by drawing a right-angled triangle; and it is
the transcendental property, i.e. that \( \pi \) can never be expressed
as the solution of a rational equation, which precludes the squaring
of the circle.)

The importance of \( \pi \) to the mensuration of common objects is
evident. Wherever there are circles, crescents (lunes), spheres, etc.,
\( \pi \) enters. Measurements of angles and trigonometrical functions
are often related to \( \pi \), as elementary trigonometry will show.
Therefore it is important that stress should be laid on sound early
work. Although the approximation \( \frac{22}{7} \) is often used, it is apt to
prove misleading, and it is usually better to express \( \pi \) as a decimal
correct to a given number of places.

Children may commence work by measuring the diameter of
a cycle wheel on the rim of which a chalk mark has been made,
rolling it over a straight line drawn on the floor, and measuring
the length corresponding to a complete revolution or a known
number of exact revolutions. This may be varied by:

1. Using various cylindrical objects such as jam-jars, cocoa
tins, pieces of iron piping, etc.; find the diameter by trapping
between two rectangular blocks and measuring the distance be-
tween them. Then wind strong thread round the cylinder (say)
ten times, unwind the thread, measure its length and divide by
10 and then by the diameter. The fact that the value of \( \pi \) is
independent of the diameter of the cylinder will be noted.

2. A circle is drawn by means of compasses. (Children should
be taught to hold the compasses so that a single rotation in the
same direction will draw the circle.) The circumference may be estimated by (a) an episometer (tracing-wheel), (b) cutting the circle into a number of sectors (each of which is almost an isosceles triangle of small angle), fixing the sectors to paper with paste so that the bases of the triangle are in a 'straight line'. This will then be measured; (c) or what is practically equivalent to the above — using dividers opened to exactly half an inch and marking off this distance along the circumference. This is similar to the polygon method of the Greeks. Many simple practical examples may be founded on the mensuration of the circumference of a circle, e.g. (1) the distance travelled by a revolving wheel and the cycloometer. (2) Edging a circular flower bed. (3) Two driving wheels connected by a belt or a chain. (4) The diameter and circumference of the earth. (5) The distance between the divisions on dials of different diameters, etc.

It should be known thoroughly that the circumference of a circle

\[ = \pi \times \text{diameter} \]
\[ = 2 \pi \times \text{radius} \]

and that \( \pi = 3.142 \) (to three decimal places) and a crude approximation, which is easy of manipulation, is \( \frac{22}{7} \)

*The Area of a Circle*

(1) A circle of 3 inches radius is divided into a convenient number of sectors (24 is a convenient number); these are cut out and pasted on to a piece of paper as shown. If adhesive paper is used the process is facilitated.
A figure is formed approximating to a rectangle of height = radius of circle, and length = \( \frac{1}{2} \) circumference.

\[
\text{Area of circle} = \text{Area of rectangle} = \frac{r \times 2 \pi r}{2} = \pi r^2
\]

(2) A circle is drawn on a piece of squared paper, so that its centre is at a convenient point at the intersection of two lines near the centre of the paper, and its radius is equal to an exact number (say 10) of large divisions of the paper. Here the squares enclosed by the circle (or count a quarter of them and multiply by 4) counting as one, all squares of which more than half is within the circle, and neglecting completely all those of which less than half is within the circle. The total number of large squares is then obtained and compared with 100 (representing the radius squared).

(3) A circle is cut from tin-plate, zinc foil, or cardboard, and weighed. A square of the same material with sides equal to the radius of the circle is also cut out and weighed. The two weights are compared.

The area of a flat ring, e.g. the path round a circular flower bed or the area of metal in the cross-section of a tube, is frequently found in practice. Here the area of the inner circle is subtracted from that of the outer.

If the radii are \( r_1 \) and \( r_2 \)

\[
\text{Area of the ring} = \pi r_1^2 - \pi r_2^2 \\
= \pi (r_1^2 - r_2^2) \\
= \pi (r_1 - r_2)(r_1 + r_2)
\]

(See the Section on Algebra.)

Area of a Sector of a Circle

A sector may be thought of as the 'plan' of a slice of cake, and a segment as the plan of the first piece cut from a round loaf. It will appear from previous work on the determination of the area
of a circle by cutting it into equal sectors, that the area of each sector is the fraction of that of the whole circle given by

\[
\frac{\text{length of arc of sector}}{\text{circumference of circle}}
\]

By experimenting with a rotating arm a child will readily appreciate that the length of arc 'described' by a point on the arm is proportional to the angle through which the arm has turned.

Thus, Area of Sector \( \propto \frac{\text{angle of sector}}{360^\circ} \propto \text{area of the circle.} \)

\textit{The Cylinder}

(1) The surface of a cylinder.

This may be approached practically by considering a cylindrical tin container, closed at both ends, e.g. as used for 'tinned' pears.

The top and bottom may be removed, the open cylinder cut vertically, opened, and flattened to a rectangle. If cardboard or paper models are made and used, the inconvenience of cutting tin-plate will be obviated. We are left with:

- Two circles \( 2 \times (\pi r^2) \) in area
- \( 2\pi r \times h \) in area

(length of rectangle \( \propto \) circumference of tin, breadth \( \propto \) height of tin \( = h \), \( r \) is radius of tin).

Total \( 2\pi r^2 + 2\pi rh \)

\( = 2\pi \left(r + \frac{h}{2}\right) (r + h) \)

It is always necessary to know if the cylinder is closed or open at one or both ends; or if the area of the curved surface alone is required.

(2) The volume of a cylinder.

This may be determined practically by filling cylindrical cans with water. The rule that the volume \( \propto \) area of circular base \( \times \) height is practically intuitive (a child realizes that an inch of peppermint rock contains as much as any other inch of the same stick, but that the lower 'half' of an inverted cone contains less
than the upper half), and it may be demonstrated by building up a cylinder from circular discs cut from wood 1 inch thick.

Volume of a cylinder = \( \pi r^2 h \)

Children should be given the opportunity of measuring the diameters of various common objects, and by simple projects discovering means of investigation in difficult cases:

1. Use of external and internal callipers.
2. Use of sliding callipers or adjustable spanner.
3. Use of rectangular blocks, or two set-squares sliding on the edge of a ruler.
4. The application of the simple theory of similar triangles to make a tube gauge.
5. Indirect method: (a) for finding an external radius. Find the volume of a solid cylinder of known length by displacement of water. Divide volume by length to give area of cross-section \( (\pi r^2) \). Divide by \( \pi \) and find the square root; (b) for finding an internal radius. Fill a hollow cylinder (a piece of piping) of known length with water. Measure the volume of water, and proceed as before. For a fine bore mercury is used and weighed. This is more difficult as the weight has to be divided by density to give its volume, and care is necessary with the units employed.

A frequent source of error is a tendency to confuse radius and diameter when working problems. At first it is probably better to convert all diameters to radii by dividing them by 2 before commencing work. Later, the standard formulae may also be thought about in terms of a diameter, which is sometimes the most convenient way.

There is no need to enumerate examples of the cylinder in our daily lives, for it occurs in all types of engineering, in tubes, wires, rods, dowels, pillars, columns, etc. The convenience of a cylinder compared with a rectangular block for many purposes may be discussed. Of all solids having a constant cross-section throughout their length (we may take a sphere as having no 'length') it has the greatest volume for the smallest surface. A pipe with rubber
walls would always tend to take a cylindrical shape when it contained a fluid under pressure. Our veins and arteries are cylinders. Flat lead water-pipes tend to become cylindrical when the water freezes. Again, cylinders allow an ease of working on their curved surfaces for they contain no corners; and they can be 'machined' accurately by means of lathes and other rotating machinery.

The Cone

Simple experiments with a hollow tin cone and a cylinder open at one end and of the same base and height as the cone, will demonstrate that the cylinder will hold just three times as much water as the cone. This cone may be made by cutting a sector from a piece of tin-plate, rolling it to make a cone, and soldering together the two radii of the sector.

The Volume of a Cone \[= \frac{1}{3} \text{ volume of a cylinder with same base and height} \]

\[= \frac{1}{3} \pi r^2 h \]

\((h \text{ is the perpendicular height}).\)

This may be compared with pyramids and other solids which 'come to a point at the top', and have triangular vertical sections through the apex.

In each case the volume \(= \frac{1}{3} \text{ area of base } \times \text{ perpendicular height}.\)

The Area of the Curved Surface of a Cone

Take a sector of a circle (preferably with an angle greater than a right-angle), bend it until the two radii touch and join them together by means of adhesive paper. It is apparent that:

(a) the slant height of the cone = radius of sector.

(b) the circumference of the base of the cone = length of arc of the sector.
Area of curved surface of cone = area of sector

\[ \text{Length of Arc} = \frac{2\pi r}{2\pi l} \times \pi l^2 = \pi rl. \]

If \( h \) is vertical height of cone \( l = \sqrt{r^2 + h^2} \) (by Pythagoras' Theorem).

\textit{N.B.} If total area of a closed cone is required, area of base must be added.
The Frustum of a Cone

This may be demonstrated by modelling a cone in clay or other suitable material, and 'cutting off the top' by a plane parallel to the base. The lower part is called a frustum.

Frustum = Original cone — smaller cone of similar shape cut off.

If we draw a vertical section of a cone, we get an isosceles triangle; if a straight line is drawn parallel to the base, the original triangle is cut into a smaller isosceles triangle similar to it and a lower trapezium. The similar triangles provide us with a clue to the working of the problem. (The trapezium is the section of the frustum.)

Given vertical height of frustum = \( h \); radii of base and top respectively = \( r_1 \) and \( r_2 \). Let \( h_c \) = vertical height of cone derived by completing the frustum.

Height of small cone = \( h - x \)

By using similar triangles

\[
\frac{r_1}{r_2} = \frac{h}{h - x} \quad \therefore (h - x)r_1 = hr_2
\]

\[
h(r_1 - r_2) = xr_1 \quad \therefore h = \frac{xr_1}{(r_1 - r_2)}
\]

\[
h - x = \frac{xr_2}{(r_1 - r_2)}
\]
volume of frustum = volume of larger cone - volume of smaller cone

\[ \frac{1}{3} \pi \frac{xt_1^3}{t_1 - t_2} - \frac{1}{3} \pi \frac{xt_2^3}{t_1 - t_2} \]

\[ = \frac{1}{3} \pi \left( \frac{t_1^3 - t_2^3}{t_1 - t_2} \right) \]

\[ = \frac{1}{3} \pi x \left( t_1^2 + t_1t_2 + t_2^2 \right) \]

There are many examples of frusta in daily life, such as flower-pots, vellum lamp-shades, buckets, jugs, etc. The theoretical work should always be supported by experiments or projects, e.g. (1) A vellum lamp-shade has a diameter of a foot at the top and 18 in. at the bottom. Its slant height is 1 ft. Draw a pattern to show how it may be cut out in a single piece. (2) Find the capacity of a bucket by calculation and by direct measurement.

The Sphere

The surface area of a sphere is exactly equal to that of a cylinder which will just enclose it. The proof of this is hardly within the scope of early secondary school work, though it may be demonstrated by means of similar triangles.
If a corresponding strip is cut, between the same two parallel horizontal planes, from both the sphere and the cylinder into which it just fits, it will be seen that the strip from the sphere is shorter than that from the cylinder; but as it is broader in proportion, the area is exactly the same. By using similar triangles it can be shown that the radius of the slice of the sphere between A and B is inversely proportional to the length CD (the breadth of the strip). It is essential that the pupil should understand the properties of similar triangles and the 'results' of certain theorems (the angle between tangent and radius is a right-angle).

A knowledge of simple trigonometry will show that the radius (CP) of the strip from the sphere, is proportional to the cosine of the angle COR, but its width CD (if very narrow) is proportional to \( \frac{1}{\cos \angle COR} \) (i.e. secant \( 
abla \overline{OR} \)) if BA is kept constant in length. The product of CP and CD is thus a constant. This principle finds application in map projections.

Thus, area of Surface of Sphere \( = 2\pi r \times 2r = 4\pi r^2 \).

*The Volume of a Sphere*

By means of a plasticine model or a piece of Dutch cheese, it can be demonstrated that we may regard the sphere as being composed of a number of small cones each one having its vertex at the centre of the sphere, and a vertical height equal to its radius. The total area of all the tiny bases of these cones will be the area of the surface of the sphere.

Thus:
Volume of the Sphere \( = \text{Sum of the volumes of all the tiny cones} \)
\[ = \frac{1}{3} \times \text{their common height} \times \text{the sum of the areas of all their bases} \]
\[ = \frac{1}{3} \times 4\pi r^2 \]
\[ = \frac{4}{3} \pi r^3. \]
Various practical experiments may be made to demonstrate this formula, or alternately, to obtain various data by assuming it: e.g. the average radius of a number of small spherical pellets may be determined by finding the volume of a thousand of them by measuring the water they displace when they are dropped into a graduated cylinder or burette, dividing by 1000 and applying the formula.

\[ \text{Let } v = \text{volume of each pellet} \]

\[ \frac{4}{3} \pi r^3 = v \]

\[ \therefore r^3 = \frac{3v}{4\pi} \]

\[ r = \sqrt[3]{\frac{3v}{4\pi}}. \]

Hollow spheres or 'shells' are of frequent occurrence in practice. The volume of metal or other substance in the shell is found by subtracting the internal and external volumes. It usually proves to be more convenient to work in terms of internal and external radii. (The thickness of the material = external radius — internal radius.)

The property of the sphere, that for the smallest surface area it encloses the largest volume, has important applications. Add to this the simple fact of surface tension (or the tension of an enclosing elastic membrane) and we see at once the reason for many of the examples of the spherical shape in nature.

Compare the surfaces of a sphere and cube of equal volumes.

It is interesting to compare the volumes of the following solids, by direct immersion of small models in water in a graduated cylinder:

(a) Cone of height equal to diameter of its base. (b) Sphere of radius equal to that of base of cone. (c) Cylinder with same height and base as cone. (d) Cube of side equal to diameter of sphere.

The volumes are in the proportion 1 : 2 : 3 : 4 (the last is approximate) (i.e. \(\frac{2}{3} \pi r^3; \frac{4}{3} \pi r^3; 2 \pi r^3; 8r^3\)).
Orthographic Projection

Orthographic projections are known to the layman as plans and elevations related together by perpendiculArs and parallels. The matter may be introduced by considering the necessity of making such drawings before commencing the construction of a building. Actual models should be examined and the plan and elevations drawn. Later, it may be possible to borrow, from architects and estate-agents, actual plans and elevations of a building which may be visited and photographed. Many periodicals devoted to architecture, housing, building, and carpentry give both photographs and orthographic projections and the pupil should become accustomed to the visualization of the projections by regarding the photograph, and vice versa. In early work it is sometimes helpful to pin the elevations to a vertical board and the plan to a horizontal board in its correct relative position. If a powerful distant arc or 'point-o-lite', placed at the end of a long room, is available, projections may be demonstrated by means of shadows. Shadow projections may also be obtained by using light from the sun, but this is not so satisfactory as it introduces an awkward tilting of the object and screen. Although orthographic projections are normally more straightforward than radial projections, and are usually dealt with earlier, it is useful at some stage of the work to discuss both types together and also to refer to the nature of stereoscopic vision.

Using pictures, drawings, photographs, and solid objects, it is useful to discuss the difference between perspective and orthographic drawings. A photograph is normally a perspective picture, but a telephotograph taken of a distant façade may approximate to an orthographic 'projection'.

Simple orthographic projections will lead to (a) plans and elevations of furniture and models made in the school workshop, (b) drawings of farm buildings, poultry-houses, sheds, and other structures, which are of particular interest in rural schools, and (c) mechanical drawing for more advanced pupils. Boys are usually interested in engines and machines; and mechanical drawings of simple examples will involve accurate measurements and calculations, and practice in neat and careful drawing.
Radial Projections

A photograph of a large ‘square’ isolated house or shed, taken at an angle of about 45° from one side, suggests a suitable way of introducing the subject. The devices of artists, whose task it is to give the illusion of three dimensions on a plane surface, include the use of perspective. (There are many other methods such as the use of light and shadow, nature and depth of colour, ‘focus’, etc.) This work may with advantage be connected with corresponding topics in art and the study of light, or again, a few optical illusions depending on the principle of perspective may be demonstrated. (A favourite example shows a long street with its horizontal lines converging towards the horizon. Identical figures of men drawn at various ‘distances’ along the pavement give the illusion that the nearer figures are smaller.)

Perspective drawing will probably be dealt with in the art lesson but the mathematics teacher will show that all vertical lines have their direction preserved and that parallel horizontal lines converge to meet in a point on the horizon represented by a horizontal line, on which near its centre is the centre-of-vision (a point opposite the eye). Two other horizontal lines are drawn on the paper: (1) the ground line which passes through the nearest point of the object at ground level, and (2) a line through the station point, i.e. a point on the ground vertically below the eye of the observer. From this point lines parallel to the sides of the object may be drawn to meet the line through the centre-of-vision (usually called the horizontal line). The simple theory of the matter may be developed by using similar triangles. When dealing with perspective, ‘parallax’ methods for finding distances of stars (and terrestrial objects) may also be demonstrated.

A simple model to illustrate radial projections may be made by fixing strong threads to six or seven ‘corners’ of a large solid cube (or to eight corners of a skeleton cube) so that the threads can be joined at a single point some feet away from the cube and they are all kept taut. If now the threads are so arranged that they pierce a piece of gauze or muslin stretched on a frame (held about halfway between the cube and the point where the threads meet) in such a way that they are all straight and taut, a radial projection
of the cube is given by joining the appropriate points where the threads pierce the gauze or muslin. In simple radial projections it is assumed that the object is seen with a single eye or camera-lens.

Architectural scale-drawings of parts of churches and other buildings will prove interesting and useful. Norman, Gothic or Perpendicular windows founded on straight lines and parts of circles when measured, sketched, photographed, drawn accurately and analysed, will reveal the beauty of craftsmanship and link the work to history and architecture. Children are interested to find the simple numerical relations which often exist between the chief measurements of a building or a part of it. Scale drawings of classical structures, and buildings in other more recent styles, will reveal 'harmonic' relationships on measurement. The eye is often quick to detect small departures from these related distances, and we say the building is ugly, disproportionate, top-heavy, and so on. The word 'harmonic' is used by analogy with music, where simple ratios between the frequencies of notes yield concords which sound pleasant to the ear, e.g. a fifth (Doh to Soh, or C to G) has a ratio 2 : 3; a third (Doh to Me or C to E) a ratio 4 : 5; a fourth (Doh to Fah, C to F) a ratio 3 : 4 (speaking ideally, and not in terms of the tempered keyboard scale). Of course the matter is psychological as well as mathematical, and recent investigations have by no means exhausted its potentialities. Similar principles apply widely in Nature both animate and inanimate, as well as in man-made objects. An efficient animal-structure or machine may be beautiful because it has been developed to fulfil certain mechanical principles which may be expressed in formulae. The whole subject may be treated in many fascinating and often difficult ways along lines which combine aesthetic, psychological, mechanical, and mathematical considerations. In this connection teachers will be interested to read Engines of the Human Body (Keith); Growth and Form (D'Arcy Thompson); The Psychology of Beauty (Valentine).

1 See page 153.
CHAPTER X

TRIGONOMETRY; SURVEYING; NAVIGATION; MAP-PROJECTIONS

Trigonometry

Formerly, trigonometry was studied as a separate topic with perhaps occasional references to the geometry of triangles and circles and simple algebraic transformations. Trigonometry 'puts number back into geometry', and besides its great practical utility which will increase its appeal to the ordinary pupil, it offers useful connecting links with geometry, algebra, and graph drawing. In other words, it is a great help when we try to show the essential unity of mathematics. The very simple ideas of trigonometrical ratio prove to be of enormous utility in the sciences both pure and applied. The boy who has already undertaken some 'Boy Scout' surveying will discover that trigonometrical tangents are a great help to him; and the student studying light will find that the law of Willebrod Snell (the 'Law of Refraction') expressed in any other way than 'the sine of the angle of incidence divided by the sine of the angle of refraction is a constant, \( \frac{\sin i}{\sin r} = n \), is exceedingly tedious and clumsy. Besides which, if the student is going to investigate the matter any further, the sine law may give him a clue to form a theory as to what happens when a ray of light is refracted.

The term Sine, and the new way of regarding the terms Tangent and Secant (which have already been used in the geometry of the circle) need some explanation. Early work in trigonometry should aim at showing:

(a) That it grows naturally out of previous work in algebra and geometry, including surveying and drawing.

(b) That it is a tool of very wide application, simplifying or even making possible the work of surveyors, engineers, navigators (and many 'pure' mathematicians).
That the terms used can be justified historically and logically.

In formal trigonometry the approach to the subject has usually been through a consideration of the nature and measurement of angles, right-angled triangles, their properties and the simple trigonometrical functions, the interrelations of sine, cosine, tangent, etc. Although it may be necessary to return to consider radian measure and related topics, in senior schools it will suffice to start trigonometry through its name --- the measurement of triangles. This may be done both practically and historically. Already the children will probably have found the heights of trees, the school flag-pole, and the church spire by measuring the length of a shadow, and at the same time the length of a shadow of a vertical pole (a Scout stave of 6 feet) and comparing them by drawing. Or alternatively, they will find the angle of elevation of the top of the flag-pole by means of a clinometer, measure the distance from the observer to the bottom of the pole, and draw a scale diagram (care being taken to add the distance from the ground to the observer's eye). Children will be interested in some quick method which will enable them to omit the drawing of the right-angled triangle to scale and the measurement of its vertical line.

Although the simple ideas of trigonometry were known at a much earlier date, particularly by the Arabs and Greeks, the word Sine (Latin Sinus) does not appear until the end of the twelfth century, when it was introduced by Gerard of Cremona. The diagram shows how the early trigonometrical functions were not represented by ratios, but rather by lines. As they were all referred to the length of the radius of the circle, i.e. the radius vector, if we were to convert them to ratios these would at least have the one advantage that the length of this line would be the common denominator.

In the diagram $OA = OC = OB$ (the radius) $OA^1$ (a line cutting the circle) was called the Secant, $CA^1$ (a line touching the circle) was called the Tangent.\(^1\)

\(^1\) The term 'sine' preceded the use of 'tangent' and 'secant' in trigonometry by many years.
By the simple theory of similar triangles it is seen that
\[ \frac{OA}{OC} = \frac{OA}{OD} = \text{Secant (as used at present)} \] and
\[ \frac{CA}{OC} = \frac{AD}{OD} = \text{Tangent (as used at present)}. \]

The early writers were not slow to see that a diagram such as the above bore a resemblance to a bow (ACB) with an arrow (OC) stretching the string AOB, before it was released. Indeed, for a long time the line OC was called Sagitta (arrow). The line AD was called the Sinus (bosom or breast line), hence our word Sine.

In our modern style
\[ \frac{AD}{OC} = \frac{AD}{OA} = \text{Sine (as used at present)}. \]

The 'co' prefixes are obvious if one considers such relationships as 'the cosine of an angle is (numerically) equal to the sine of its complement', which can readily be proved by considering the ratios between the sides of a right-angled triangle taken in pairs.

When the desirability of using the ratios of pairs of sides of a right-angled triangle has been considered by the children through work in measuring heights and distances and 'Boy Scout' surveying, they may begin with the relations between the sides of a right-
angled triangle. Simple experiments with similar triangles will show that if triangles are equiangular they also are similar (they have the same shape). As the three angles of a triangle add up to 180°, a knowledge of two of them will fix the shape of any triangle. Thus, if a triangle is right-angled the shape of the triangle is determined, if another angle is known. It follows then, that for every value between 0° and 90° which we may give one of the remaining angles of a right-angled triangle, there will be a perfectly definite shape, and a definite set of ratios between pairs of sides.¹

\[
\begin{align*}
\text{Sine of angle } (\sin) & \quad \text{perpendicular} \quad \frac{P}{H} \\
\text{Cosine of angle } (\cos) & \quad \text{base} \quad \frac{B}{H} \\
\text{Tangent of angle } (\tan) & \quad \text{perpendicular} \quad \frac{P}{B}
\end{align*}
\]

The reciprocals of these: the cosecant, secant, and co-tangent respectively, need not be learnt at this stage.

It should be stressed that if we know the value of any one of these ratios all the others may be determined, either by drawing a right-angled triangle and applying Pythagoras' Theorem, or by a simple manipulation of the formulae by which the trigonometrical functions are connected, viz. —

\[(1) \text{ (A is an angle) tangent } A = \frac{\sin A}{\cos A} \text{ which follows by merely dividing the ratios as set out above.}\]

\[(2) \sin^2 A + \cos^2 A = 1 \text{ which follows from Pythagoras' Theorem.}\]

¹ Recent practice has abandoned the use of Perpendicular in the trigonometrical triangle as it may have any orientation. The letter O (opposite side) is used instead of P. A mnemonic for remembering the three chief ratios reads:

\[
\begin{align*}
\text{O} & \quad \text{H} & \quad \text{B} & \quad \text{H} & \quad \text{O} & \quad \text{B} \\
\text{Oswald has been here obtaining books.} & \quad \text{SIN} & \quad \text{COS} & \quad \text{TAN}
\end{align*}
\]
Tables of sines, cosines, and tangents may be constructed for angles of $30^\circ$, $60^\circ$, $45^\circ$, and later $0^\circ$ and $90^\circ$.

The $30^\circ$, $60^\circ$, $90^\circ$ triangle may be regarded as half an equilateral triangle. (Examine a set-square of this form.) By Pythagoras’ Theorem the sides are in the ratio $1 : \sqrt{3} : 2$.

Thus
\[
\sin 30^\circ = \frac{1}{2},
\cos 30^\circ = \frac{\sqrt{3}}{2},
\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.
\]

In the same way with an angle of $60^\circ$:
\[
\sin 60^\circ = \frac{\sqrt{3}}{2},
\cos 60^\circ = \frac{1}{2},
\tan 60^\circ = \sqrt{3}.
\]

Notice here the complementary relationship with the trigonometrical functions of $30^\circ$. 
In a $45^\circ, 45^\circ, 90^\circ$ triangle two sides are equal and the other side may be found by Pythagoras' Theorem.

The sides are in the ratio $1: \sqrt{2}: 1$.

\[
\sin 45^\circ = \frac{1}{\sqrt{2}} \\
\cos 45^\circ = \frac{1}{\sqrt{2}} \\
\tan 45^\circ = 1
\]

To obtain the trigonometrical functions of 'an angle of $0^\circ$' we must take a right-angled triangle with a very small angle and imagine what happens when this angle approaches $0^\circ$.

The base and the hypotenuse may be considered equal and the perpendicular very small, and it will tend to disappear as the angle approaches $0^\circ$.

\[
\sin 0^\circ = 0 \\
\cos 0^\circ = 1 \\
\tan 0^\circ = 0
\]

A similar method may be used to determine the trigonometrical functions of an angle of $90^\circ$. 
Mathematics should strive for simplicity and elegance. The above results may be set down thus:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sin</td>
<td>$\sqrt{\frac{0}{1}}$</td>
<td>$\sqrt{\frac{1}{4}}$</td>
<td>$\sqrt{\frac{2}{4}}$</td>
<td>$\sqrt{\frac{3}{4}}$</td>
<td>$\sqrt{\frac{4}{4}}$</td>
</tr>
<tr>
<td>Cos</td>
<td>$\sqrt{\frac{4}{4}}$</td>
<td>$\sqrt{\frac{3}{4}}$</td>
<td>$\sqrt{\frac{2}{4}}$</td>
<td>$\sqrt{\frac{1}{4}}$</td>
<td>$\sqrt{\frac{0}{4}}$</td>
</tr>
<tr>
<td>Tan</td>
<td>$\sqrt{\frac{0}{4}}$</td>
<td>$\sqrt{\frac{1}{3}}$</td>
<td>$\sqrt{\frac{2}{2}}$</td>
<td>$\sqrt{\frac{3}{1}}$</td>
<td>$\sqrt{\frac{4}{0}}$</td>
</tr>
</tbody>
</table>

($\pi$ radians = $180^\circ$)

By using generating circles the graphs of sines, cosines, and tangents may be plotted. This is worth doing from several points of view:

(1) It shows that sines, cosines, etc., are continuous functions and ‘repeat’ each $360^\circ$. It gives a quick way of estimating the values of these functions for various angles; and the memory of the shapes of the graphs will give a clue to the calculation of the functions of second, third, and fourth quadrant angles. The pupil will at once grasp that in calculating the functions of angles greater than $90^\circ$, they must be regarded as angles of $180^\circ+$, $180^\circ-$, or $360^\circ-$. Such relations as: ‘the sine of an angle is equal to the sine of its supplement’ can be seen from the graph. A mnemonic to show which functions (and their reciprocals) are positive in any quadrant, is given below (the word CAST is remembered):

$$
\begin{array}{c|c}
(180^\circ - ) \text{ Sine} & 2 \\
(180^\circ + ) \text{ Tan} & 3 \\
\text{CAST} & 10 \\
\text{Cos} (360^\circ - ) & 4
\end{array}
$$

This should be extended by using a radius vector in four positions so that the principle may be continued from acute-angled triangles to those in the second, third, and fourth quadrants.
(2) The sine-curve is of the utmost importance for later work in mathematics and the applied sciences. All periodic functions can be regarded as being formed by adding sine-curves, the frequencies of which are in the ratios of the natural numbers.\textsuperscript{1} Tides and other astronomical occurrences are calculated by compounding sine-curves; and musical tones can be split up into harmonics each of which is represented by such a curve. (There is an ingenious instrument which will generate any musical tone-quality by adding tiny electric currents in the form of sine-waves.) The electrical engineer dealing with alternating-current finds that smaller sine-curves appear and are added to the main sine-curve, representing the alternation of the voltage. The wireless enthusiast who uses the terms 'second-harmonic distortion' and 'percentage modulation' cannot really understand what he is talking about, unless he knows how curves may be combined by adding the ordinates (in the first place with a simple ratio between frequencies, and in the second usually no simple ratio). The curve will demonstrate a principle which may require quite complex mathematical elucidation. Many secondary school boys continue with technical work and various forms of engineering and then some knowledge of simple-harmonic-motion and wave-propagation is useful.

(3) When radian measure has been grasped, the abscissae of the graph will be marked off as $\frac{\pi}{2}$, $\pi$, $\frac{3\pi}{2}$, $2\pi$, instead of $90^\circ$, $180^\circ$, $270^\circ$, $360^\circ$.

If the scales along both axes are the same (the angles being measured in radians), it will easily be seen that the graph goes through the origin at an angle of $45^\circ$ to the $x$ axis, which is equivalent to saying that for a small angle $\theta$ (in radians): $\sin \theta = \theta$.

This is the 'small angle' so often used in physics when dealing with such things as the theory of the pendulum. It is interesting to plot the errors made by using this formula $\sin \theta = \theta$ for different angles ($\theta$ being expressed in radians).

Radian measure offers no difficulties. A radian is the angle of a sector of a circle with an arc equal to its radius. A chord equal to a radius would make an equilateral triangle and therefore

\textsuperscript{1} Fourier's Theorem, 1822.
subtend an angle of $60^\circ$ at the centre. Hence a radian is slightly less than $60^\circ$.

$$2\pi \text{ radians} = 360^\circ$$

$$\therefore 1 \text{ radian} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi} = 57.3^\circ \text{ (approx.)}$$

(4) If cosine and sine-curves, which are identical in everything except position, are drawn on the same graph their complementary relationship may be shown. In later years when tangents to curves and differential coefficients are thought about, the differential relationship between sine and cosine may be demonstrated.

(5) Lastly (and most simple), tangent and sine-curves (particularly the first) may be drawn by the children and used instead of tables for solving problems which arise when surveying is undertaken.

**Application of Trigonometrical Functions to Simple Problems**

(1) Finding the height of a distant spire, the base of which cannot be reached:

Given distance $x$ and angles $\alpha$ and $\beta$,
Find height of steeple $AD$ ($h$).

As the only distances to be considered are horizontal and vertical, sines and cosines and their reciprocals (all of which contain the hypotenuse) are excluded. Tangents or cotangents of the angles will be required. Where there is a height $AD$ common to both triangles it is often more convenient to use this as the denominator of the ratio, i.e. use cotangents.
\[ \cot \alpha = \frac{BD}{h} \]
\[ \cot \beta = \frac{CD}{h} \]
\[ \therefore \ \cot \alpha - \cot \beta = \frac{BD - CD}{h} = \frac{x}{h} \]
\[ \therefore h = \frac{x}{\cot \alpha - \cot \beta} \]

or with tangents it is easier to use the complementary angles.

The angle complementary to \( \alpha \) is \( \angle B\bar{A}D \), call it \( \gamma \).

The angle complementary to \( \beta \) is \( \angle C\bar{A}D \), call it \( \delta \).

\[ \tan \gamma = \frac{BD}{h} \]
\[ \tan \delta = \frac{CD}{h} \]
\[ \therefore \ \tan \gamma - \tan \delta = \frac{BD - CD}{h} = \frac{x}{h} \]
\[ \therefore h = \frac{x}{\tan \gamma - \tan \delta} \]

(If the angles are taken with a clinometer or similar apparatus, do not omit to add the height of the observer.)

Variants of this problem may be made by considering the 'angles of depression', at different times, of a ship sailing to or from an observer on a cliff. If the speed and time are known, the height of the cliff may be calculated. If the height of cliff and interval of time are known, the speed of the ship may be calculated and so on.

The science master will be able to suggest some further applications of the simple trigonometrical functions in mechanics (equilibrium, and the resolution of forces); light (law of refraction); sound (waves and tone quality); electricity and magnetism (resolution of forces, galvanometers, etc.). In surveying, projections of various kinds, navigation, and astronomy we find many more uses for these functions.
No trigonometry of a formal type will normally be undertaken in the secondary modern school but the simple trigonometrical properties of triangles are so important in their applications and present so little difficulty, that it may be well to consider them. (1) *Sine Rule.*

\[
\frac{\text{Sine } A}{a} = \frac{\text{Sine } B}{b} = \frac{\text{Sine } C}{c}
\]

Consider first the acute-angled triangle: ABC. Notice the conventional way of recording the length of the sides, side \(a\) opposite angle \(A\), side \(b\) opposite angle \(B\), etc. Draw the perpendicular \(AD\) of length \(p\).

\[
\frac{\text{Sine } B}{c} = \frac{p}{b}
\]

\[
\frac{\text{Sine } C}{b} = \frac{p}{c}
\]

\[
\frac{\text{Sine } B \times c}{b} = \frac{\text{Sine } C \times b}{c}
\]

As the triangle is a Scalene triangle we have not made any assumptions with regard to its sides, thus without further proof:

\[
\frac{\text{Sine } C}{c} = \frac{\text{Sine } A}{a}
\]

\[
\frac{\text{Sine } A}{a} = \frac{\text{Sine } B}{b} = \frac{\text{Sine } C}{c}
\]

Simple algebra will serve to produce new arrangements of this simple formula. It will be applied to the finding of distances and
angles when surveying by means of triangulation, but to be able to use this formula we must know one angle of the triangle and the length of one side opposite to it and one other measurement.

In the case of a triangle containing an obtuse angle the matter is not rendered much more difficult, for the sine of an angle is also equal to the sine of its supplement, and we consider the triangle containing the acute supplementary angle. The sine-rule applied to the 'triangle of forces' gives Lami's Theorem (when 3 forces acting at a point are in equilibrium each force is proportional to the sine of the angle between the other two).

(2) The Cosine Rule.

\[
\begin{align*}
    a^2 &= b^2 + c^2 - 2bc \cos \Lambda \\
    b^2 &= a^2 + c^2 - 2ac \cos B \\
    c^2 &= a^2 + b^2 - 2ab \cos C
\end{align*}
\]

which when rearranged yield

\[
\begin{align*}
    \cos \Lambda &= \frac{b^2 + c^2 - a^2}{2bc} \\
    \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\
    \cos C &= \frac{a^2 + b^2 - c^2}{2ab}
\end{align*}
\]
There are several ways of approaching the cosine rule. If a triangle is explored algebraically by drawing a perpendicular to form two right-angled triangles, and Pythagoras’ Theorem is applied we obtain the proof in the easiest way:

Using the lengths given in the diagram

From triangle ABD \( c^2 = p^2 + (b - x)^2 \)
\[ = p^2 + b^2 - 2bx + x^2. \quad (1) \]

From triangle BDC \( a^2 = p^2 + x^2. \quad (2) \)

\[ \therefore c^2 = a^2 + b^2 - 2bx \]
\[ c^2 = a \cdot b^2 - 2ab \cos C. \]

\( \cos C = \frac{x}{a} \quad \therefore x = a \cos C \)

\( x \) is called the projection of \( a \) on \( AC \). The length of such projection is found by multiplying the original length ‘\( a \)’ by the cosine of the angle between the original line and the line on to which it is projected.

In the case of an obtuse-angled triangle the cosine rule still applies, but, as will be seen from a diagram, we get (following the same steps as before)

\[ c^2 = a^2 + b^2 + 2ab \cos \hat{A} \hat{C} \hat{D} \]

but \( \hat{B} \hat{C} \hat{D} \) is the supplement of \( C \)

\[ \therefore \cos \hat{B} \hat{C} \hat{D} = - \cos C. \]

Thus again

\[ c^2 = a^2 + b^2 - 2ab \cos C, \]

and similarly for \( a^2 \) and \( b^2 \). The cosine rule is useful in finding the angles of a triangle when only the sides are given, or when two sides and the angle between them (the ‘included-angle’) are known.

Notice that the cosine rule contains Pythagoras’ Theorem as a particular case.
The Area of a Triangle.

The area of a triangle = \( \frac{\text{base} \times \text{perpendicular height}}{2} = \frac{ap}{2} \).

Now \( p = c \sin B \)

Area of a triangle = \( \frac{ac \sin B}{2} \)

and similarly (by thinking of each side in turn as base)

Area of a triangle = \( \frac{ab \sin C}{2} = \frac{bc \sin A}{2} \)

or in words, multiply together the lengths of two sides and the \( \sin \) of the angle between them and divide by 2.

These theorems are very useful because surveying relies very largely on the use of triangles and their properties.

Surveying

Surveying in schools may commence with considerations of making maps, plans, and sketches. The requirements for Boy Scout badges will yield useful problems; and the geography, art, and crafts masters will be able to supply helpful suggestions. The properties of similar figures should be revised, but such trigonometry as is necessary may be evolved as the subject develops. Simple work in scale-drawing will have been undertaken already when making plans and elevations of models, reading maps, etc.

The simple arithmetic involved in using scales is not a matter
of any difficulty, although calculations of area and volume from scale drawings are frequently overlooked (see the section on Proportion). The choice of suitable scales for various purposes of plan- or map-drawing should be discussed from a practical point of view, e.g. the school playing-field is ½-mile long, and not as broad as long; we wish to draw a plan not more than two feet in length, what will be a convenient scale? What is the scale of the large school globe? Stretch a piece of string from one port to another to represent the path of a ship or aeroplane travelling along a great circle. Find the actual distance by reference to the scale. Find the distance travelled by a day’s walk or car ride in the country by taking an ordnance-survey map, marking out the route with thread, measuring this, and converting by means of the scale.

Taking a large-scale map, trace the outline of a lake, a park or the boundaries of a city on squared paper (with inch squares divided into tenths), estimate the number of squares by counting, and find the area in square inches on the map. Find the real area from the scale, remembering that here we are dealing with square measure and if the scale is 1 : x the real area is $x^2$ times that of the area of its representation on the map.

Many of the simple surveying instruments used in schools may be constructed in the workshop.

Measurement of length.

Use of 100-link chain and tape; accuracy in pacing (boys should experiment for themselves); estimation of length by time taken at various speeds (e.g. scout’s pace); estimation of distance or length by inspection; scouts will know that distances seen from a height and over water are sometimes deceptive; use of ‘parallax’ in estimating distances (this may be demonstrated in the classroom, from observations taken from the windows of railway carriages; and the ‘parallax’ of fixed stars due to the motion of the earth may be mentioned). The history of chains, rods, and poles may also be considered.
Chain-Surveying

It is recommended that surveying should, if possible, be undertaken in the country away from the school. It may form part of the work of school and scout journeys or camps. A field-book for recording measurements will be necessary and the entries should start at the bottom of each page and work upwards. Two vertical lines about ½ inch apart should be ruled throughout the length of the page leaving an equal space on both sides. In the column between these lines nothing but the chain measurements (Chain-

![Diagram of a survey setup across a river](image)

age) should be written. From the chain-line, offsets to objects at each side are noted, by taking the distance at right-angles from the line to the object. The corresponding 'chainage' is marked in the central column and a rough plan of the object (with accurate dimensions affixed) is made in a convenient position which need not be proportionally correct. Chain surveys may be extended by simple triangulation, which will demand a certain amount of rough measurement and exploration of the area in order to fix suitable 'stations'. The rough plan drawn by eye observation alone is a useful starting-point.

This may be varied by surveying 'round' obstacles and measuring the distance across rivers. A scout staff is placed on one bank of the river opposite to a tree or other suitable object on the other bank. A convenient distance CB is measured out along the bank
and continued in a straight line to A so that CB = BA. A scout staff is placed upright at B. At right-angles to CA, a distance AD is paced out until a staff placed at D is seen to be in line with B and E. DA is equal to CE, for the triangles BCE and BAD are equal in all respects. Other simple methods should be sought.

The history of mathematics yields many interesting examples of the use of triangles in surveying land. Thales, who lived about 600 B.C., used methods very similar to the above.

An example of finding heights and distances of a more difficult type has already been given, but some simple introductory methods are worth performing.

Finding Heights of ‘Accessible’ Objects

(1) A 45° set-square with a plumb-line attached to one of the angular points, so that one of the shorter sides may be kept vertical, may be used for finding the height of an object the base of which is approachable. The top of the object is sighted so that it is in line with the long side of the set-square. The observer walks forward until one of the shorter sides of the set-square is vertical. The distance of the observer from the base of the object plus the distance between the ground and the observer’s eye is then equal to the height of the object.

(2) The length of the shadow of a 6-feet scout staff is compared with that of the object measured at the same time.

From the properties of similar triangles:

\[
\frac{\text{Height of Object}}{\text{Height of Staff}} = \frac{\text{Length of Shadow of Object}}{\text{Length of Shadow of Staff}}
\]

\[\therefore \text{Height of Object} = \frac{\text{Height of Staff} \times \text{Length of Shadow of Object}}{\text{Length of Shadow of Staff}}\]

The result may also be found by drawing to scale.

(3) A clinometer may be improvised by taking a large protractor and fixing to the centre (of the circle of which it is a part) a plumb-line. It is better, however, to construct the instrument properly by using a half circle of wood graduated with a scale reading from
$0^\circ$ to $90^\circ$ on each side of the ‘horizontal point’. The plummet may be constructed by using black thread and a small weight. The angle of elevation, $\alpha$, the horizontal distance of the observer from the object and the height of the observer’s eye above the ground must be found. The height of the object is found by making a scale drawing, or by using the tangent of the angle $\alpha$. Height of object = Distance ($d$) $\times$ $\tan \alpha$ + height of observer ($h$).

(4) In another method the eye (as near as possible to the ground level) is ‘put into line’ with the top of a vertical staff of known height and the top of the object whose height is to be measured. The distances between the eye and the base of the staff and the eye and the base of the tower are measured. A scale-drawing is made and the simple proportions of similar triangles used in the calculation.

*Triangulation using a Base of known Length*

In chain-surveying using triangles, three sides have to be measured, but fixing a base-line of known length and taking bearings on distant objects from both ends of the base-line not only simplifies the work considerably, but makes easy surveys
which would otherwise be difficult or impossible. It will be remembered that a triangle is determined if one side and two angles are known.

Surveying from a base-line has great importance and pupils will be interested to learn something of the history of the matter. In England, a base-line was marked out on Hounslow Heath and the survey proceeded from this by triangles throughout the British Isles. 'A base of verification' nearly seven miles in length was measured on Salisbury Plain. The degree of accuracy required for such surveys, both in England and in the colonies, gave a great incentive to physical investigations on gravity, pendulums, the expansion of metals, etc. The range-finder with a base of known length and a mirror or telescope at each end, depends on a similar principle of triangulation, and may be mentioned.

The base-line should be at least 2 chains long, and measured out accurately on horizontal ground which commands a view of prominent distant objects, such as a chimney, a church, an isolated tree, the top of a barn, etc.

The bearing of the base-line may be found by observations on the sun or by using a compass (in the latter case making the requisite adjustment for the difference between magnetic and true bearings). Consult Whitaker's Almanack or any sheet of 1" Ordnance Survey map.

For measuring the angles of the bearings of the distant objects
at each end of the base-line we may use a circular scale graduated in degrees (fixed to a horizontal drawing-board placed on a tripod), or a theodolite; or again, a plane-table method may be used, and the plan of the base-line and the distant objects drawn as the work proceeds. When using a tripod, it should be firmly fixed on the ground so that its centre is directly over the end of the base-line, and the instrument should be levelled. In taking angular measurements, care is taken not to touch the instrument except when rotating the telescope or sighting-tube. If a round of angles is taken, a check on the working may be obtained by adding the angles together to see whether their total is 360°. It is possible to make a serviceable theodolite in the workshop, but the survey will be more realistic if a good instrument capable of both horizontal and vertical readings can be borrowed.

A useful level which can be fixed to the top of a tripod can be improvised in the following manner. Two short pieces of glass tubing each about 4 inches long are put through holes bored vertically at each end of a piece of wood about 18 inches × 2 inches × 1 inch. About 2 inches of each piece of glass tubing appears above the surface of the wood and the protruding ends underneath the wood are joined together by suitable rubber tubing. Mercury is poured into one of the tubes so that the rubber connection is quite full and the liquid rises in the tubes to a level somewhat above the top surface of the wood. This simple instrument can be fixed to the top of a post or a tripod and sighting is done by finding the line of the two mercury surfaces. Other liquids will serve the purpose but mercury is far superior.

The plane-table offers the best way of simple and quite accurate plan drawing by triangulation from a measured base. The apparatus consists of a tripod on which is fixed a horizontal board, a good magnetic compass, a sighting ruler, and a spirit-level. The plane-table is used as follows: It is set up and levelled (two positions of the spirit-level at right-angles to each other will suffice) over the position A at the end of the base-line. A piece of drawing-paper is carefully pinned to the board. The direction of the magnetic north is marked on the paper; a point near the centre of it is chosen to mark point A at the end of the base-line.
With this point touching one edge of the sight it is turned round until it is in line with the point B at the other end of the base-line, now marked by a vertical ranging-rod. A line is drawn on a suitable scale to represent the base-line. The ruler is now sighted to a distant object the line of which is immediately drawn from point A. This is repeated with other objects, until series of rays each bearing the name of the distant object have been drawn from point A. The table is then removed from A, the position of which is marked by a ranging rod, and taken to point B. Here a similar set of operations is performed.

This may be varied by simple magnetic-traverses by means of a prismatic compass. Rough work may be performed by using an ordinary compass and pacing the required distances. One of the chief tasks of the surveyor is levelling. This is normally performed by a telescopic level of known height above the ground, which is sighted upon a graduated staff held in a vertical position at a measured distance away. Gentle slopes may be broken into a series of triangles, and the vertical rise for each distance travelled along the slope noted in the field book. Steep gradients may be determined by using the clinometer or theodolite. The levels may be referred to the ordnance bench mark (\(\overline{\text{\small \N}}\)), of which the positions and height above sea-level (the high-water mark at Liverpool) are marked in the 6-inch ordnance survey maps. This will lead to a consideration of gradients and contours. In regard to the former it must be clearly stated whether a gradient of 1 in \(x\) means a rise of 1 foot vertically for every \(x\) feet along the slope or along a horizontal line. With small angles of slope there is little difference.

Contours may be demonstrated by making models of the hill in clay and cutting through in horizontal planes. Although they may be more properly undertaken in geography lessons, some discussions should be held on the history of map-making, modern methods of surveying, the ordnance survey maps, and their conventional signs.

With younger senior school children 'surveying' may take the form of hunting for buried treasure and Boy Scout games of a similar nature. Many adventure stories which are popular with
young people contain references to clues which give distances and
directions. The chart found at the bottom of the seaman’s chest
still carries a thrill with the secret it holds. Children will be
interested in the types of stories which have been evolved since
The Gold Bug was written by Poe. The islands mentioned in
Treasure Island, Robinson Crusoe, Swiss Family Robinson, Coral Island,
Gulliver’s Travels and in many other stories which have appeared
more recently may be drawn; or children may imagine that an
island on the Pacific Ocean contains treasure, they may find a
suitable place by examining a globe, and then write a story giving
the clues.

Many of the historical attempts at surveying and map-making
are of great interest and children will like to see models of the
simple instruments available to the ancients. Eratosthenes was
able to measure the circumference of the earth by finding that
when the sun was directly overhead at Syene (Assouan) on the
tropic of Cancer, it was 7½ degrees from the zenith at Alexandria,
500 miles away. Thus he argued that the circumference of the
earth is $\frac{360}{7\frac{1}{2}} \times 500$ miles or 25,000 miles, which gives a radius of
approximately 4,000 miles. Hipparchus, who made a table of
numbers which we should now call ‘sines’, estimated the distance
of the moon by using angles obtained by observations from two
places and applying the properties of a right-angled triangle; and
Aristarchus (310 B.C. to 230 B.C.), who anticipated to some extent
the ideas of Copernicus and Galileo on the heliocentric theory,
was able to make a rough comparison of the distances of the moon
and sun from the earth.

**Navigation**

This subject presents a very wide field, particularly since air
navigation has yielded its own problems, and wireless signals have
assisted the navigator both in the air and on the sea. The subject
is full of historical interest in connection with our maritime
achievements, and it may be developed from work on time, map-
projections, and graph-drawing.
Longitude and Time

The earth makes a complete revolution (or turns through $360^\circ$) in 24 hours; thus, in an hour it turns through $\frac{360^\circ}{24}$ or $15^\circ$. $15^\circ$ of longitude are therefore equivalent to an hour's difference of time. Now, the sun rises in the east and sets in the west and the earth is revolving 'from west to east'. As we go eastwards from a certain point on the earth's surface we find that the new time is ahead of ours and we have to put our watches on, perhaps to the extent of only an hour, or half an hour, if we are visiting central or western Europe. If we sail west towards America we find that we have to put our watches back, and when we reach the east coast of America time is five or six hours slow compared with English time.

If we are voyaging round the world in this direction, i.e. England to New York, New York to the Pacific coast, across the Pacific Ocean and so on, we are elongating every day and thus at some convenient place before we return we have to add one to the reckoning of the number of days (this is done by 'jumping' from Monday to Wednesday for example), and vice versa. When time can be determined by observations on the sun or stars at any particular place, and this time compared with Greenwich time obtained from an accurate clock or by wireless signals, at once we have a means of determining the longitude of that place, for every hour of difference represents $15^\circ$. It can easily be understood why, during the past centuries, so much skill has been applied to the task of constructing accurate clocks, called chronometers, for the purpose of finding longitude at sea. (It must be understood that there is nothing absolute about time, and that the position of Greenwich, as the place through which passes the line of zero longitude, is quite arbitrary.)

Latitude

Take a large globe suitably mounted and draw a circle to go through its poles and Greenwich. This is a meridian, and like all

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1 A Gaumont British 16 mm. film, with other visual aids, is useful in the teaching of Latitude and Longitude.
circles of longitude it is a great-circle, i.e. its diameter passes through the centre of the earth. Now draw the equator and consider the arc of $90^\circ$ on the circle of longitude between the equator and the pole. Each degree of latitude

$$\frac{\text{length of earth quadrant}}{90} = \frac{\text{length of polar circumference}}{360} = 69 \text{ miles.}$$

(N.B. As the earth is an oblate spheroid the lengths of polar and equatorial circumferences are not quite equal, but along the equator a degree of longitude is practically 69 miles.) A minute of latitude (or longitude at the equator) $\frac{69}{60} = 1.15$ miles. This is called a sea-mile, and it is at once evident that it is a convenient unit for navigators. A knot is a speed of one sea-mile an hour; it derives its name from the knots in the rope which was 'paid out' when the 'log' was thrown overboard. If the knots were placed at equal distances along the rope, the length of rope which left the ship in a given time, measured in terms of these knots, would be proportional to the speed of the ship relative to the water. Modern mechanical or electrical logs give readings, in various parts of the ship, of the speed of the ship and the distance covered by it since the log was set. The passage of the ship through the water rotates vanes which communicate with an instrument not unlike the speedometer of a car. Other patterns depend on the motion and pressure of water through a tube (the pitot tube used in aerial navigation relies on the increase of pressure of air caused by the forward motion of the aeroplane). Rotating-vane types of instrument are also used in aerial navigation.

**How to find Latitude**

The principle of the sextant should be investigated. A simple sextant to demonstrate the action of the instrument may be made in the workshop and used to measure roughly the angles subtended at the eye by pairs of distant objects, and to find the 'altitude' of the sun. As the sun is so far away it is assumed that all its rays are
parallel when they reach the earth. The altitude of the sun at a place is the angle made by the sun’s rays to a tangent plane drawn to the earth at that point. If the sun is overhead at the Equator it will be seen from the diagram that its altitude at noon at any place is equal to \((90^\circ - \text{latitude})\).

Thus, latitude \(= 90^\circ - \text{sun’s altitude at noon}\). However, the sun is only overhead at the Equator twice in the year.

During the winter months the sun’s declination (obtainable for any date by consulting \textit{Whitaker’s Almanack}) must be subtracted from the expression \((90^\circ - \text{sun’s altitude at noon})\) and in the summer months the appropriate angle must be added. Observations on the Pole Star and other stars may also be used. Demonstrations with a globe and a distant source of light will make the matter clear.

\textit{Degrees of Longitude}

Divide the arc of the polar meridian between the Equator and the pole into 90 equal parts of \(1^\circ\) each. Through each of these draw a ‘small circle’ in a plane parallel to that of the Equator. These are parallels of latitude. In an actual demonstration two or three such small circles will suffice.
DB = radius of small circle at latitude $x^\circ$ North

DB = OB $\cos x^\circ$

$\Rightarrow \cos x^\circ \times$ radius of the earth.

Thus the circumference of a 'small circle' ($360^\circ$ of longitude)

$\Rightarrow$ cosine of latitude $\times$ circumference of Equator.

A degree of longitude $= 60 \times \cos x$ sea miles.

Thus when going East or West

Change of longitude in degrees $= \frac{\text{number of sea miles}}{60 \times \cos x}$

'Dead Reckoning'

Before the days of radar, wireless-direction signalling, and under-water signalling the sailor unable by climatic conditions to observe the sun and the stars, had to estimate his position by means of 'dead reckoning'. From his log-book he knew at what time he last fixed his course, he could estimate his speed by means of the log, and hence calculate the number of miles he had travelled in a certain direction.
Suppose his course set at B is $\gamma$ East of true North (obtain by correcting magnetic North) and the distance travelled is $x$ sea miles (by log).

Then 'nortthing' $BA = x \cos \gamma$ sea miles

$$-\frac{x \cos \gamma}{60} \text{ degrees of latitude.}$$

Easting $BC = x \sin \gamma$ sea miles, or expressed in degrees of longitude

$$\frac{x \sin \gamma}{60} \text{ cosine of average angle of latitude between A and B.}$$

This is called 'middle-latitude reckoning' of longitude. These figures would be added to the bearings at point B. (Corrections must be added owing to the curvature of the earth, but these need not concern us here, especially if $x$ is not large; and the above process will be repeated a number of times during the whole course.)
Map-Projections

Work on map-projections should be reserved for older children, but it is of considerable importance for everyone who uses a map, and gives interesting applications of previous work on mensuration and trigonometry. The problem of representing figures, originally on a sphere, on flat surfaces is an old one. The type of projections used depends on the purposes for which the flat map is intended. For descriptive maps showing 'distributions' throughout the world the whole globe must be shown on one projection. At other times, a single large land mass such as Africa or Australia is all that is required; or again, navigators in the air or on the sea, surveyors and explorers in the polar regions need special maps for their specific purposes. It is interesting to think of all the different people who are likely to need maps for their work and to imagine what they will require from them. Some need maps which preserve directions; for others this is not necessary but corresponding areas must be preserved and the possibility of shape distortion is a secondary consideration. All children will know that on the popular Mercator projection, Greenland appears to be larger than Africa, although in actual fact the latter country is at least ten times the size of the former. The danger nowadays is that children see many maps without troubling to refer to the globe. Time spent in examining the globe is never wasted. The school should also possess a large globe of a dull black surface on which the outlines of the land masses are marked in white. Chalk lines such as great-circle sailing-courses can be drawn, and subsequently rubbed out.

An excellent way of illustrating various types of map-projections is to take a clear glass globe, coat it with gelatine, draw on this a rough outline of the continents and fill in the land masses with dilute coloured inks (even a fish globe with flat bottom and open top will serve). The room is darkened, a tiny electric bulb is suspended within the globe, and images of the outlined continents will appear on external sheets of paper. The type of projection varies according to the position of the bulb and the position and nature of the surface of the paper — flat, cylindrical, conical. It is often convenient to use a spherical wire cage instead of a globe,
as this shows the distortions of the graticule (network) in projecting. Not all the standard projections may be demonstrated in this manner, however.

Zenithal Projections

The three cases on page 211 are called respectively Gnomonic ('sundial' projection, light at the centre); Stereographic (light at end of the diameter); Orthographic (the light imagined to be an infinite distance away and rays from it parallel). The paper AB is flat. By varying the position of the light between case (i) and case (iii) other projections have been evolved. In the three standard cases given above, the use of simple trigonometrical functions will suffice to show the relation between the projected distances and those on the globe. A simple conical projection is worth demonstrating and although the theory of an elliptical projection (such as that of Mollweide) is a matter of some difficulty, the general idea is not difficult to grasp. A circle is first drawn to represent the surface area of one hemisphere. An ellipse is drawn with its minor (smaller) axis equal to the diameter of this circle and its major (greater) axis equal to twice this length. The projection of the whole world is drawn in this ellipse and although the shape of a country is distorted its area is preserved.

Mercator’s projection is so largely used for wall maps, atlases, and navigators’ charts that it should be discussed; although here again the complete theory is a matter of some difficulty. It is a ‘cylindrical’ orthomorphic projection, which implies that it is a projection on the inside of a cylinder circumscribing the globe and touching it at the Equator, and that the shape (of a small country) is preserved. The characteristics of this method are that all lines of longitude and latitude are vertical and horizontal respectively on the projection, and the latitude scale at any point is equal to that of the longitude scale. Now a small circle of latitude 60° N. is cosine 60° × the length of the Equator, or just half the length of the Equator. Thus on the Mercator projection the scale at latitude 60° is double that at the Equator. Accordingly an area in latitude 60° is magnified to four times that of a similar area at the Equator. In the general case the linear scale increases as the
reciprocal of the cosine (i.e. secant) of the latitude, and the area as the square of this. At greater latitudes this size magnification increases enormously as we approach the poles, where theoretically a small area on the globe is infinitely large. The Mercator projection is thus useless for the polar regions. It is useful for navigators following the ordinary sailing routes, because all meridians run north and south, all ‘parallels’ are east and west on a Mercator chart, and a straight line crossing them maintains a constant bearing. Nevertheless, such a straight line does not represent the shortest distance between two points on the earth’s surface, which is always a part of a great circle. The great-circle route plotted on a Mercator chart is curved, but it can be followed quite easily by regarding it as a series of straight lines (really chords of the curve), each having a constant bearing. The normal sailing routes do not enter extreme latitudes.

Navigators using the ordinary sea routes use the Mercator chart for most purposes, but for wireless direction-finding, and air-navigation where the great-circle routes can be readily followed, the gnomonic projection is frequently used, for straight lines on this projection actually represent the shortest distances between two points on the globe.

Children will be interested to learn something of radiolocation or radar and new air routes which are constantly being explored, such as that from Moscow to Vancouver which practically goes over the North Pole.

Some further topics which are suitable for lectures for Scouts and Guides or for science and mathematics societies are: surveying by means of beams of light; surveying by aerial photography; taking pictures at intervals depending on the speed and height of the aeroplane; surveying by photographs taken from a moderate height; modern direction-finding instruments; instruments used at sea; supersonic signalling and sounding (depth-finding); wireless direction-finding; radar and instruments used for aerial navigation and blind flying.
CHAPTER XI

GOING AHEAD: TOWARDS THE CALCULUS; STATISTICS AND PROBABILITY; THE GROWTH FUNCTION

When we come to apply our mathematics to many things in daily life, in science, and in engineering, we see that we need some method of dealing with quantities which are constantly changing in some way. For instance, speed is the rate of change of position, acceleration is the rate of change of speed, and curvature is the amount of change from 'straightness'. All the sciences contain thousands of examples of quantities which change with time, and often these changes can be expressed in terms of some simple mathematical law. To draw a graph is a most helpful way of understanding in a general way what is happening when these changes take place. For purposes of calculation it is necessary to find, as far as possible, a mathematical process or formula for what is hidden in a graph. The direction of a curved line is said to change from point to point, but when this statement is analysed and thought about carefully it implies all manner of difficult logical and philosophical conceptions. The problems of the changes in position of a moving body during a certain interval of time gave great trouble to the Greeks. Zeno's famous paradoxes are examples of this. It will suffice to recall one of them. Zeno (495-435 B.C.) argued that it is impossible to reach the end of a race-course because you must cover half of any distance before you traverse the whole, and half of that again and so on ad infinitum. Thus there is an infinite number of points in the given distance and 'you cannot touch an infinite number one by one in a given time'. Thus you will never get to the end. It was a common fact of experience, nevertheless, that the end was reached.¹ In this way Zeno argued against the infinite divisibility

¹ The paradox of Achilles and the tortoise is another good example.
of space and time; and he went on to show that if spaces and times respectively contain only finite numbers of points and instants, deductions can still be made which are readily shown to be contradicted by experience.

With the differential and integral calculus, which became useful tools in the hands of mathematicians in the seventeenth century, some explanation of Zeno's difficulties was forthcoming. Isaac Newton (1642-1727) arrived at his first ideas concerning the calculus at the age of twenty-one. He imagined a curve to be traced by a 'flowing' point. The 'infinitely short path' traced by the point in an 'infinitely' short time was called by him 'the momentum' and when this was divided by the 'infinitely short time' the quotient was called the 'fluxion'. The fluxion of $x$ was denoted by $\dot{x}$. We should now use the notation $\frac{dx}{dt}$.

In his *Principia* (1687) Newton quite clearly shows his appreciation of the logical difficulties which beset his method. The philosopher Berkeley, Bishop of Cloyne, wrote in *The Analyst* (1734) that Newton's 'fluxions' could be attacked from the standpoint of logic and philosophy and inquired whether 'the inferences of modern analysis are more distinctly conceived, or more evidently deduced, than religious mysteries and points of faith'. The strength of Newton's calculus was that it was useful and yielded good results: it was not until the nineteenth century that the logical difficulties were overcome. Mathematicians often found methods which were useful many years or even centuries before rigorous proofs of their theorems and of the validity of their assumptions were forthcoming. In fact, in a number of cases the deductive and philosophical significance of certain mathematical methods is not yet agreed upon.¹

Newton was not the only worker in this field. His great contemporary, Leibniz (1646-1716), the German philosopher, produced a method of differentials which eventually proved to be more useful than the method of 'fluxions'. He introduced the now familiar $\int$ sign for integration and to find the differential of $xy$ he subtracted it from $(x + dx) (y + dy)$ and rejected $dxdy$

¹ The meaning of probability is a case in point.
without real justification. This gave him the result \( d(xy) = vdy + ydx \). The 'tidying up' process which justified the calculus on the lines of rigorous logic still goes on, but it did not really begin until the work of Cauchy in 1821. The acrimony which attended the disputes between Newton's and Leibniz's partisans concerning the priority of their discovery need not trouble the teacher, but it may serve to show that the time was more than ripe for these new methods. In some ways Newton and Leibniz were both anticipated in these matters by J. Wallis (1616-1703) and P. Fermat (1601-1665).

A convenient way of introducing the calculus is by considering the motion of a body falling under gravity, and it should lead directly from a treatment of simple algebraical graphs. The relation between elementary differentiation and integration should be made clear from the first.

The calculus should be introduced earlier than is usual at the present time. It will prove invaluable, particularly to science and engineering pupils in grammar, technical, and even secondary modern schools. It should be preceded by a thorough course in graph-drawing and interpretation, a fair knowledge of simple algebraical manipulation and if possible a little physics and mechanics. There is no virtue in pretending to do the calculus in the lower forms by the rule of thumb methods by which, for instance, an unintelligent use of the formula \( \frac{dx^n}{dx} = nx^{n-1} \) is all that is done. Great care must be taken both by the teacher and the pupil to prevent the use of 'infinity' and 'infinitesimals' as though they were ordinary numbers. \( dy \) does not mean \( d \times y \) and \( \frac{dy}{dx} \) is not a fraction, nor can it be further simplified.

We can pass from the meaning of the slope of a straight-line graph to that of a curve at a particular point. It is impossible at this stage to deal with the matter rigorously, and at later stages it would be honest to approach the calculus again through the theory of functions.

Calculus should never degenerate to entirely mechanical proce-
dures, and where assumptions have to be taken on trust concerning 'vanishing quantities', 'limits', and so called infinitesimals, the teacher should be honest about it. The small increase, delta \( x \), in the value of \( x \) (\( \delta x \)) is not to be confused with \( dx \). The former is a small but finite quantity which can be treated as such by mathematical processes.

This elementary work will lead to the idea of a tangent to a curve and maximum and minimum values. At each stage the work can be derived from and reinforced by good graph-drawing representing practical considerations in volumes, mechanics, and general science. For this elementary work, which is so useful in practice, the simple functions which are differentiated should

<table>
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<tr>
<th>Differentiation</th>
<th>Integration</th>
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<tr>
<td>( \frac{d}{dx} x^{n+1} = (n + 1)x^n )</td>
<td>( \int x^n , dx = \frac{x^{n+1}}{n+1} )</td>
</tr>
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<td>( \frac{d}{dx} \log_e x = \frac{1}{x} )</td>
<td>( \int \frac{1}{x} , dx = \log_e x )</td>
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<td>( \frac{d}{dx} \cos x = \sin x )</td>
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<td>( \frac{d}{dx} \sin x = \cos x )</td>
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<td>( \frac{d}{dx} \tan x = \sec^2 x )</td>
<td>( \int \sec^2 x , dx = \tan x )</td>
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<td>( \frac{d}{dx} \cot x = \cosec^2 x )</td>
<td>( \int \cosec^2 x , dx = -\cot x )</td>
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<td>( \frac{d}{dx} e^x = e^x )</td>
<td>( \int e^x , dx = e^x )</td>
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\( d \) should be regarded as an \( dx \) operator and not a fraction.

The constant has been omitted in each case.
include sines, cosines, and tangents of an angle. It is essential that all angles should be measured in radians, but students of science and engineering will accustom themselves to thinking of two right angles as $\pi$ radians and will remember the subdivisions thereof.

Integration can reasonably be thought of as the reverse of differentiation, but it should be applied to the calculation of distances, areas, and volumes, work, energy, and other physical quantities to show its nature. It is obvious that when a function of $x$ is differentiated the 'constant' term, i.e. that independent of $x$, does not appear in the differential coefficient; thus a constant has to be added when we reverse the process and integrate. What is the nature of this constant and what becomes of it in subsequent processes needs carefully watching by reference to the problem in hand. This will lead to integration between limits or definite integrals.

Statistics and Probability

Often we find that the exact qualities of our simple calculations do not help us very much when we come to consider real happenings. It is impossible to make a steel piston exactly one foot in diameter, for instance. As we refine our methods of machining this engine part we can work within certain limits, and it is often essential to know just what are these limits. Throughout the whole science of measurement something must be allowed for errors of observation. This is necessary, for when the results of an observation are subject to further treatment, or combined with other results also subject to error, it is essential to trace the resultant errors through to the final result. This is an important aspect of all mathematical treatments of measurement.

Again, chance and probability play a large part in our lives. Insurance companies cannot predict the 'length of days' of any particular individual, but by working with large numbers of people and by making certain assumptions which are founded on previous observations, they are able to predict the average expectation of life for ten thousand persons of a certain age and all in 'good health'.
In individual cases it is difficult or even impossible to predict what will happen, but by taking a large number a mathematical generalization may be found. This has wide applications. If we could take a single atom of radium we should be quite unable to say when it would disintegrate, even with the aid of advanced mathematical physics. On the other hand, if we take a small fragment of radium, which contains some millions of atoms, we know that its half-life is 2,000 years, that is, its radioactivity will decay to half the original intensity in this period. We cannot tell whether a penny will fall ‘heads’ or ‘tails’ when it is tossed, but we know that as we increase the number of observations the limit tends to be 50% ‘heads’ and 50% ‘tails’. It is easy to work out at each number of trials what are the chances that the total number of ‘heads’ and ‘tails’ will be different and by what amount. This is important, as when only a limited number of observations of particular phenomena can be made, it is useful to know what reliability can be allowed and what are the chances of unreliability.

Street accidents form a sad though important example of elementary probability. If corrections can be made for the season of the year, the prices and availability of petrol and cars, length of the month, etc., deaths on the roads do not show much fluctuation. Although a particular town may be free from accidents for several months others seem to compensate for this during a similar period. The figure for the whole country does not show a great percentage difference from one month to another ‘equivalent’ month. This difference, which could be predicted and worked out by any student of elementary probability, is the cause of much misunderstanding. A drop of say 30 fatal accidents in a month is hailed as a sign that motorists, cyclists, and pedestrians are being more careful and that in course of time the number of accidents will almost vanish. An increase of about the same number is popularly thought to mean just the opposite. A statistician would see that these fluctuations in the number of accidents are just what the mathematical theory would cause us to expect.

The word ‘statistics’ is used loosely. It may mean the collection and perhaps the tabulation of figures relating to various quantities
(prices, characteristics of human beings, exports, imports, finance, housing, etc.), or it may mean the mathematical treatment by graphs, formulae or correlation of data concerning a large number of measurable quantities.

Graphical methods will help to make these matters clear and will lead to a consideration of correlation when two graphs are drawn on appropriate scales on the same piece of graph paper. A simple way which gives a good general impression is the use of a scatter diagram or scattergram.

Here the quantities with regard to which correlation is sought are plotted as points each representing a pair of values. For instance, we might regard the scores in mathematics as $x$ marks and those in science as $y$ marks. Each pupil would have a point $x, y$. With perfect positive correlation all the points would lie on a straight line. With no correlation at all the points would be distributed fortuitously over the graph paper. With increasing degrees of positive correlation the points tend to bunch together in the region of the line (called by the biologists who specialize in measuring various characteristics of living things, ‘the line of regression’).

We may introduce the subject of statistics by considering the distribution of marks in an examination, measurements of intelligence of large numbers of children of the same age groups, the heights and weights of a large number of children of the same age. If we are to plot the number of children in particular mark groups (say from 65% to 70%) using twenty mark groups from 0% to 100%, in each case making a rectangle whose vertical height represents the number of children in the group and the base the mark group (5% in each case), we shall obtain a histogram with a horizontal reading for marks in 5% steps from 0% to 100%. If the successive midpoints of the upper boundary of each rectangle of the histogram are joined together, we get a frequency polygon. If we can imagine a frequency polygon to be drawn for a very large number of cases with imperceptible changes from one mark or other quantity to another, we shall get a curve. When distribution is normal, as it is when we consider large numbers of quantities which occur in the sciences, we find a symmetrical
exponential curve known as 'the curve of error', or 'the curve of normal distribution'. This curve was first studied in connection with errors in astronomical observations and in gunnery. The mathematics of this curve is beyond our scope, but its properties and the areas of parts of it cut off by particular vertical lines have wide applications. Children will be interested in Pascal's triangle, which is built up by using powers of 11 (care being taken to keep the columns separate and not to carry from column to column after $11^4$ or 14641). This can be related to the problems of head and tail distribution when two, three, four . . . coins respectively are tossed. (The case of two coins, where the probabilities are TT, TH, HT and HH, i.e. in the proportion of 1, 2, 1 for two tails, head and tail together and two heads respectively has an interesting application in the simple Mendel laws of cross-breeding. The study of genetics finds much use for statistics and probability.) Later it may be possible to relate these distributions to the coefficients of the binomial expansion of $(x + 1)^n$.

The curve of normal distribution can be used for elucidating ideas concerning dispersion, probable errors, significance, quartile range, etc. Histograms and frequency polygons show why rules concerning distribution break down when an insufficient number of cases is considered.

Sometimes sets of statistics are presented in a pictorial form which may be misleading. For instance, if areas or diagrams are used to represent volumes, we have to remember that similar shaped areas vary as the square of a side containing the area and volumes as cubes of this linear measurement. A diagram of a cargo ship, which is just twice as long as that of another ship, may represent a vessel which might contain eight times as much freight if other dimensions were proportional. In practice this would not be so, and it would be exceedingly difficult to compare them properly in diagrammatic form. We are on safer ground when we use the 'Isotype' method where each unit is defined pictorially.\(^1\)

Averages may also be a cause of misunderstanding. An analysis of 'cricket averages' will at once show the difficulties. Com- pounded averages are even more misleading as will be known to

\(^1\) See the author's *Visual Methods in Education*, Blackwell, 1951.
any teacher who has struggled with lists of marks in different subjects each carrying different maxima. The ordinary arithmetic mean or average (which may be calculated by the deviation method from an assumed mean) may not be the most suitable quantity for our purpose. Sometimes the median score or the magnitude of the middle item when arranged in rank (or an 'order of merit' list) is more useful. When we are talking about the usual or most frequent value of a character where there is a large number of cases we use the word mode. Mathematically, it is a 'loose', though none the less useful, expression.

There is a wealth of material (which can be treated by the simplest graphical and other statistical methods) to be found in the daily and weekly press, in official publications, in local government reports, etc. Whitaker's Almanack is also a useful source from which children can gather data suitable for treatment. Older children are always interested in methods of tabulating, and if it is possible to show them something of the mechanism and operation of the commercial tabulating, punching, and calculating machines this will give the lessons added value from the standpoint of their practical applications.¹

Probability has an interesting history. Centuries ago men wished to find their chances of winning in various games or how the stakes could be divided when games were left uncompleted. In the same way, merchants wished to know the chances of their ships making successful voyages and to make insurance provision with underwriters.

Games of chance, pool 'competitions', gambling in various forms can be used as illustrations and treated mathematically. The history, methods, and uses of various forms of insurance treated in a similar way will also be of value. In this and other work of a statistical nature it is necessary to consider the proportion or percentage of one quantity to the whole and not to stress absolute values only. The next chapter of this book mentions some sources of useful material from everyday life.

¹ An elementary treatment of distribution, correlation, significance tests and variance is given in Statistics in School by the author (Blackwell).
The Growth Function

Already we have seen something of various ways of growing. A quantity may grow by addition according to some simple rule, or by multiplication. From the practical point of view, as well as from that of the development of ‘self-contained’ mathematical processes, there is another method of growing which is of great importance whenever we consider biological growth, the growth of physical quantities (the building up or decay of an electric current in a wire or of sound in a building, for instance), chemical changes, and many other things in applied science and life in general. In this type of growth, the rate of increase at each instant is proportional to the size of the thing increasing. If cultures of bacteria were not limited by food and excretory problems, they would increase in this way for example. We can think of this way of growing by comparing compound interest with simple. With the lower rates now obtaining, compound interest has lost some of its ‘magic’, but it is still useful to compare the two methods by plotting graphs. If we can consider that a rate of interest decreases as the number of periods, during which it is made up, increases, we find that our original amount $A$ has grown to $A \left(1 + \frac{1}{n}\right)^n$ where $\frac{1}{n}$ is the fraction by which it grows in a unit period of time of which there are $n$.

In the limit, after $n$ has become very great, $\left(1 + \frac{1}{n}\right)^n$ is said to equal $e$ which is slightly greater than 2.7 ($e$ is a transcendental number\(^1\) and it occurs in many investigations of physical and biological laws. It is the base of the natural logarithms, and we may write: $y = \log x$ and $e^y = x$). It can be expressed as an expansion of an ‘infinite’ number of terms:

$$e = 1 + 1 + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{1 \times 2} + \frac{x^3}{1 \times 2 \times 3} + \ldots$$

\(^1\) See page 168.
(when the natural numbers are multiplied together we use the factorial notation, e.g. \(1 \times 2 \times 3 \times 4\) \(\text{[4]}\))

If \(e^x\) is differentiated we are left with

\[
\begin{array}{cccc}
1 & + x & + \frac{x^2}{1 \times 2} & + \frac{x^3}{1 \times 2 \times 3} & \ldots
\end{array}
\]

which is still \(e^x\).

Thus the rate of growth of \(e^x\) with \(x\) is itself \(e^x\), and this is a most important property. We can find examples from bacteriology, disease, changes in population, movements of electricity, chemical changes and the probability 'laws' when we are considering large numbers of chance happenings. The exponential \(e\), like \(\pi\), is one of the fundamental quantities which seem to relate various branches 'of mathematics' and also help to interpret the physical world in mathematical terms. We have said very little about the topics mentioned in this chapter although they are of great importance. A full consideration of them would be outside the scope of secondary school work and of this book. Teachers who are able to continue with these exceedingly interesting matters would do well to master the later chapters of Hogben's *Mathematics for the Million* and then to study Nunn's *Teaching of Algebra*. 
CHAPTER XII

CIVIC AND RURAL ARITHMETIC

The Arithmetic of Citizenship

Many of the topics dealt with in the arithmetic of citizenship are closely related to civics and history; and the mathematics master should endeavour to obtain the assistance of his colleagues who teach these subjects, in order to avoid overlapping. Usually the arithmetical technique required is slight, but the problems to which it is applied are of such importance in the life of the citizen that they cannot be overlooked.

1. Banking. Investing Money and Interest

National Savings Certificates

Their history and rates of interest in the various issues since the first world war. Saving for certificates by means of stamps. How the certificates are issued.

Post Office Savings Bank

Saving by stamps on cards. Nature of a deposit account; pass book; home safes; where the interest comes from.

Banks

Deposit accounts and pass books. Interest; current account; bank charges and how reckoned; how to fill in a cheque; crossing a cheque; clearing houses; overdrafts. How banks use money which is deposited with them. How interest is earned. How bank rates are fixed (a general account).

Compound Interest

The teaching of simple interest in schools cannot be justified from the utilitarian standpoint, but it is sometimes retained to illustrate certain principles in proportion and to provide a
standard of comparison with the more useful (and difficult) compound interest. The ‘magic’ of compound interest is more easily realized when it is compared with the linear growth of a sum of money by simple interest. Here a graph will be useful.

At 4% simple interest a sum of money would double itself in 25 years and treble itself in 50 years. At 4% compound interest money doubles itself in less than 18 years, quadruples itself in 35½ years, and is multiplied by eight in little more than half a century.

Compound interest has applications apart from money problems: if we decrease the length of the intervals of time at which the accounts are ‘made up’, and increase correspondingly the number of intervals of time, we get a clue to the ‘law of natural growth’, which leads to later work on exponentials which have enormous significance in higher mathematics, the growth of living matter, probability, physics and other sciences.

In simple interest, increase of the original sum of money takes the form of adding a fixed sum at regular intervals; in compound interest in may be considered as the repeated multiplication of a sum of money by the same factor at the end of each interval.

Consider the case of £1 invested at 4% compound interest. At the end of the first year it (the amount) has become £1 \times \frac{104}{100}.

At the end of the second year it has become £1 \times \frac{104}{100} \times \frac{104}{100}.

At the end of the third year it has become £1 \times \left(\frac{104}{100}\right)^3.

At the end of the fourth year it has become £1 \times \left(\frac{104}{100}\right)^4.

In the case of a Principal of £P and a rate of r% the amount at the end of the fourth year = £P\left(\frac{100 + r}{100}\right)^4 which is sometimes written £P \,(1+r)^4.

It must not be forgotten that interest is found by subtracting the original principal from the final amount. Such formulae as the above may readily be worked out by means of logarithms:
In the general case: \( A = P (1 + r)^n \) (\( n \) is the number of years).

or \( \log A = \log P + n \log (1 + r) \).

This formula may be manipulated so that if any three of \( A, P, r \) and \( n \) are known the other may be determined.

E.g. find the number of years necessary for a sum of money to double itself at 3\% compound interest.

(Consider a Principal of £1 and an Amount of £2.)

\[
2 = (1 + 0.03)^n \\
\log 2 = n \log 1.03 \\
\frac{\log 2}{\log 1.03} = 24 \text{ (nearly)}
\]

(N.B. Divide the logarithms.)

Answer: 24 years.

This is an example of the method of dealing with simple equations where the unknown is an index.

As the banks 'make up their books' at frequent and regular intervals it is not convenient for them to use logarithmic methods for reckoning the interest, especially as they often have various other complications such as the deductions of bank charges, withdrawals, deposits, etc. A few examples should be worked to calculate interest over a short period such as 2\frac{1}{2} years, the interest being calculated and added directly every half-year. More realistic examples will result by adding deposits and subtracting withdrawals. If the interest is 3\% (per year) it will be taken as 1\frac{1}{2}\% per half-year, and so on. The long addition method of performing compound interest, though less interesting and more cumbersome than the logarithmic method, is nevertheless more realistic. The logarithmic method is useful in such instances as the following: compare the rates per cent compound interest offered by recent issues of National Savings Certificates:

- 15s. becoming £1 os. 3d. in 10 years (9th issue)
- 15s. ,, £1 in 10 years (6th issue)
- 16s. ,, £1 in 8 ,, (5th issue)
- 16s. ,, £1 in 7 ,, (4th issue).
2. Stocks

There are many approaches to the subject of stocks, shares, capital, and money in industry. For instance we may treat the matter historically, discussing the uncritical manner in which people formerly invested their money in many spurious companies, how money was raised for various great undertakings in the past, the trading companies, the South Sea ‘bubble’, etc.

The financial pages of the newspapers and the prices quoted in reports of stock exchange activities will yield much material. The matter can best be understood by a ‘play-way’ in which a member of the class assisted by two or three others starts a ‘Company’ with the financial support of the other members who are ‘shareholders’; other rival companies spring up, stockbrokers buy and sell shares for their clients; and thus various technical and popular terms such as ‘a corner’, ‘bull’ and ‘bear’, ‘at par’, should be explained by citing instances. Children will see how circumstances sometimes arise which give the dishonest a means of gaining money illegally; and what is perhaps more important, how ‘sharp practice’ which is not in the best interests of society in general has sometimes appeared. Whether we like it or not, financial matters are most potent forces controlling national and international policy, security, trade, the prices of commodities, and public well-being; and the cause of democracy is served in a better manner by investigating simple cases of everyday finance, than by mechanically working old-fashioned problems on stocks and shares which have no relation to actual practice.

The procedure at the Stock Exchange in England, at Wall Street in New York, and at stockbrokers’ offices will prove interesting and may be illustrated by lantern slides and cinema films. A popular children’s game, ‘Bulls and Bears’, may also prove useful. In some schools children pretend that they have £1,000 to invest, select various stocks, ‘buy’ them at current prices if they are obtainable, sell them in six months’ time and find their total gains or losses.

The arithmetical basis of stocks and shares is not difficult. The price of a stock is given as the number of pounds cash which will buy a hundred nominal pound shares. Often the price is quoted
for blocks of shares other than one hundred, or even for single shares; e.g. stock at 75 means that £75 cash will buy 100 nominal pound shares; a 15s. National Savings Certificate for the purpose of discussion may be regarded as a special government stock at 75.

Income is normally reckoned on stock, not on cash; e.g. 3% stock at 95 means that the 100 pound shares bought for £95 cash will yield £3 a year income. Generally speaking, as may be expected, the cheaper the stock the smaller the interest it yields. Gilt-edged or government stocks do not yield high rates of interest, but they are secure, as they are ‘backed by the financial stability of the country’.

When money has been invested in a ‘business’ which has prospered, owing to public demand for certain commodities and to wise investments, etc., the income yielded is large and the stocks are not only high in price, but very difficult or even impossible to obtain.

Brokerage is reckoned by making the price of the stock dearer to the buyer and quoting it at a lower price for the seller. Examples including brokerage may usually be omitted from school work.

**Example.** What is my total income obtained by investing £400 in 3½% stock at 80 and £600 in 7% stock at 120?

Stock obtained by investing 
\[
\text{£400 at 80} \quad \frac{400 \times 100}{80} \quad \text{pound ‘shares’}
\]

Income from above at 3½% 
\[
\frac{400 \times 100 \times 3\frac{1}{2}}{80 \times 100} = £17 \ 10 \ 0.
\]

Stock obtained by investing 
\[
\text{£600 at 120} \quad \frac{600 \times 100}{120} \quad \text{pound ‘shares’}
\]

Income from above at 7% 
\[
\frac{600 \times 100 \times 7}{120 \times 100} = £35 \ 0 \ 0
\]

Total 
\[
= £52 \ 10 \ 0
\]
Other Topics for Older Pupils

How a Company is formed; Directors and their duties; Shareholders, Prospectuses and Balance Sheets; 'Limited Liability Companies'; Income Tax on Profits and how it is reckoned; Various types of Shares: Preference Shares, Ordinary Shares; Debentures, How Banks lend money to industry; Bank balance sheets. (In each case actual examples may be taken and discussed.) The finances of Co-operative Societies.

Insurances of Various Kinds

Begin by discussing the history of insurance and why it is necessary.

(a) Fire, 'All-In' and Car Insurances

These may be considered 'historically', and the necessity of such insurances mentioned. The arithmetic is quite simple and examples of the calculation of premiums by simple proportion will be worked. If time permits, special insurances, e.g. Lloyds' 'underwriters' and examples of the insurance which they will undertake.

(b) Life Assurance

The origin of life assurance. Expectation of life. Graphs to illustrate how this has increased during the last half-century. A general outline of the methods of estimating expectation of life. Various types of assurance.

(1) Whole Life Assurance.

(2) Endowment Assurance.

(3) Deferred Annuities.

(4) Special 'family' and other assurances:

Information on these matters may be found in the booklets issued by the various assurance companies.

Calculations may be made to find which are the best types of assurance under particular circumstances and admitting certain risks.
3. National Health and Pensions

Its history and the general benefits effected by it. Cards and stamps. Health, Unemployment, etc., Widows, Old Age Pensions. The recent Government schemes of National Assurance.

Details of the schemes are issued officially and may be obtained at Post Offices, Libraries, Ministry of Labour Offices, and the more comprehensive literature from H.M. Stationery Office.

Other schemes for pensions or superannuation may also be discussed.

Rates and Local Finance

A rate demand notice gives details of the allocation of the money collected, by showing how the rate on every pound of rateable value is divided up. This may be used as the basis of a lesson on rates.

The ‘rateable’ value of the property in any district is calculated and a ‘penny rate’ will produce a sum equal to the total rateable value of the district divided by 240. The rateable value of a small house is usually less than its rentable value, as the rents of such houses have increased so much during the last few years. The district or parish rates are divided into two parts, one of which goes to general county expenses and the other for purely local purposes.

Many interesting problems may be made by considering the finances of local government.

A new road or drain may be necessary. The Ministry of Health (from the central funds of the country) will pay a certain percentage (say up to 40%), and the total expense of the undertaking is £30,000 spread over a number of years. By how much (in pence and decimals of a penny) will the rate in the £ be raised? More complex problems will arise when the ‘council’ borrows money for certain projects, and pays it back gradually together with the interest on the sum still outstanding.

Police, schools, highways, parish relief may be the subjects of interesting problems, and the work should be linked with a ‘qualitative’ discussion of rates in history and civics classes.
National Finance

This is a good subject for topical consideration when the daily newspapers contain accounts of budget speeches and proposals. A graph showing the rates of income-tax during the last fifty years or so, and another showing the increase of national expenditure during the same period will prove interesting and give a basis for discussion. Some particular points in the story of national finance during the last few centuries are always acceptable. The national balance sheet may be considered in two aspects:

(a) The Income or Revenue of the Nation

Excise: Stamp duties, duties on alcohol, amusements, tobacco, purchase tax, patent medicines, etc.

Customs: Duties on imports: alcohol, tobacco, perfumes, silks, and other fabrics, ad valorem duties on many important commodities (a) for increasing revenue, (b) for the protection of home producers.

Estate Duties: Duties charged as a percentage of the value of an estate when the owner dies or its ownership is transferred. A graph will show how the percentage duty increases with the value of the estate.

Income-tax: Income-tax varies from year to year and the budget proposals should be discussed and simple problems may be founded on them. An income-tax assessment-form and a demand-note may be examined and the income-tax payable under various circumstances of salaries, profits from a business, allowances for wife and children, depreciation, life assurance, etc., calculated. The treatment must be topical. Mention how PAYE works.

(b) National Expenditure

Graphs will show how expenditure on the Army, Navy, and Air Force, and on the social services, education, etc., have varied since 1920. The national debt and sinking funds may be treated historically and graphically.

Financial matters in general may be treated in a concentric manner. The child starts by playing at shops and gains ideas of dividing his small sums of money between present spending and
saving for a holiday, a toy, or for a more distant future. It is as unwise to preach a gospel of ‘save all’ as to recommend ‘spend all’. The wise man strikes the mean between spending and saving. The slogans heard so often to-day: ‘Spend wisely’, ‘Save as you spend’ are certainly very true, but are not always easy to put into practice. From early experience with the use made of his weekly pence the child may pass, at the end of primary school life, to a discussion of housekeeping accounts, the allocation of wages earned for various purposes, etc. Many of the periodicals devoted to the home and housecraft give examples of the household budget, which may be considered critically. From this we pass to such topics as the following:

(1) How to act as the treasurer of a school club or society. The balance sheet.
(2) The finances of a small shop or business.
(3) How to make a Poultry Farm pay.
(4) The cost of a Sunday School Treat or Christmas Party.
(5) The cost of a holiday at home or abroad, or of a fortnight’s camp.
(6) The financial aspect of running a car — cost, tax, insurance, petrol, oil, garage and ‘parking’, repairs, depreciation, etc.
(7) Building Societies. Borrowing money to buy a house.
(8) Hire-Purchase — advantages and disadvantages, furnishing a house.
(9) Profit and loss in the school concert or play.

A formal treatment of the problems of costing which demands great experience and skill, is outside the scope of school work. Nevertheless, a child may readily glean some ideas about the necessity of costing and how it is done. Graphs will prove to be very helpful. The boy laboriously engaged at the handicraft centre in the construction of a stool and the girl making a garment should have some idea of the cost of the raw materials used. Add to this ‘so much an hour’ for time, and the basic cost of the article is known. On a larger scale other factors would enter into the matter:
(1) Rent; rates and taxes; heating; lighting; cleaning; repairing of the building. This is part of the 'Over-head' cost.

(2) If materials were bought in larger quantities or, if it were worth while, made at or near the factory, they would be cheaper.

(3) If many articles are made to the same pattern and a machine is devised to make or help to make the article, time is saved, fewer people are necessary in the factory, and more articles are produced in less time.

(4) Advertisements, the fees of agents, salesmen, and others who do not actively make the article have to be considered.

Older classes may consider various large manufacturing organizations; and statistics (preferably shown graphically) will serve to show the relation between production and selling prices, the effect of the introduction of machinery, the combining of interests, etc. From a practical point of view some idea of simple costing will be very useful to the pupil who later will work as a smallholder or in other rural occupations.

**Mathematics in Rural Districts**

Although early secondary school work precludes any narrow vocational training, an agricultural environment with its long tradition, to say the least, will influence considerably the content and treatment of the mathematics course in rural districts. It is obvious that revolutionary changes have been made in farming by the application of scientific discoveries, but what is just as important, successful farms are now run on an equally sound mathematical basis. Farming had seen such bad times between the wars that many farmers only kept themselves financially solvent by adopting careful means of estimating quantities and values, book-keeping and costing. Many have suffered bankruptcy for failing to do this. A most important feature of the course taken by any student at an agricultural or similar college is the work on 'costing' and related subjects, which should subsequently help him to run his farm or business on an economic basis. It is only by applied arithmetic that a farmer can tell whether he
is making the best use of his land from the point of view of financial return. As the farmer pays various taxes, some of which are subject to relief, he often employs an expert to do this work for him. No formal treatment of the intricacies of farm costing will enter the school course, but many simple and useful mathematical problems can be evolved by considering farming and rural life in general.

Some Suggested Topics

Mensuration
The surveying of highways and fields; levelling, determining gradients; measuring perimeters, drawing maps and plans. How these help when extensions and improvements are suggested. Earthenware and other drain pipes. Calculation of areas. (How to find the approximate area of a field with four straight sides by multiplying the averages of opposite sides.) Total lengths of furrows; steam ploughing; question of times and costs. (See also the section on Surveying.)

Application of Simple Statistics to Areas
(1) Number of cattle grazing in a certain area.
(2) Yield of various crops under certain conditions.
(3) Time taken for ploughing, sowing, etc.
(4) Weight and cost of fertilizers employed.

Design of Farm Premises
Plans and sections of outbuildings. Design of byres and other buildings, with a view to requirements of light, cleanliness, and convenience. Capacities of tanks, sumps, churns, pits, containers, and dams. Length of wire for electrical wiring of buildings. Design and capacity of a Dutch barn. Volumes of stacks, mounds, and loads and how to estimate them roughly. Convenient ways of levelling ground so as to move the minimum amount of soil. How to find the heights of trees. Estimation of number of cubic feet of timber in a tree-trunk.
The size and weight of a cheese. Grading eggs. Estimations of
weight from volumes. Hay stacks, manure heaps, piles of stones, mounds of earth; applications of circular mensuration; densities, volumes, and weights.

Arithmetic

Proportion: Yield of crops; quantities of seed, fertilizer, etc., required for certain fields; hours of work and number of men employed.

Proportional Division: Mixing fertilizers, insecticides, and remedies, etc.; allocating money, etc.

Percentages: Percentage germination of seeds; percentage composition of butter, milk, soil, manures, etc. Profit and loss. Farm assurance of various kinds.

Money Problems: Carriage of goods; simple balance sheets; subsidies and concessions; the work of various official departments, e.g. land utilization, War Agricultural Executive Committees, Ministry of Agriculture and Fisheries. Calculation of cost of electricity for power and lighting. (A graph will show what are the most economical terms according to the quantity consumed.) Calculation of quantity and cost of various fodders, etc., consumed by livestock.

Older pupils may suggest projects which may be discussed from an arithmetical point of view, such as:

1. The finances of a poultry farm; a fruit farm; a pig farm; bee-keeping, etc.

2. Is it more profitable to graze cattle, or to farm various root and cereal crops under various sets of circumstances?

3. A simple treatment of the arithmetic of Mendelism and breeding.

Graphs

Owing to the case with which they can be read, graphs are particularly helpful for expressing the statistics of the farm. Graphs may be drawn showing rainfall, yield of milk, of eggs, of other crops, fluctuations in prices, cost of electricity, or other ‘power’ which is used, etc.
Mechanics

Levers; pulleys (Class II and Weston); mechanical advantage; special types of levers (lifting truck), and pulleys adapted for farm work.

Moments, stresses and strains, roof trusses. Various types of weighing machines and how they work. Mechanical principles of gates, fences, derricks. General idea of a force. Work as force × distance.

\[ \text{Power as force} \times \text{speed or} \quad \frac{\text{force} \times \text{distance}}{\text{time}} \]

Horse-power, how it is measured; engines.

Minimum horse-power required for certain tasks, e.g. H.P. given by or required for traction-engine, a tractor, a threshing-machine, a chopper, etc.; water turbines and wheels; 'head' of water; how to estimate power of water-wheels. The inclined plane and screw.

Centrifugal force: the milk separator; threshing-machine, etc. The action of standard farm machinery: the plough, rake, drilling and mowing-machine, 'self-binder', stacking-machine, milking-machine, cheese-press, etc. This may be combined with the rural science course, and it should endeavour to show how simple science used quantitatively will help to give efficient working.

Densitics: Home-made hydrometer for milk testing.

The publications of the Ministry of Agriculture and Fisheries, etc., contain data which may be converted into excellent practical problems in simple mathematics. Periodicals such as The Farmer and Stockbreeder and The Farmer's Weekly and those dealing with the keeping of poultry and other livestock are often concerned with the financial and quantitative aspects of the matters with which they deal, and many of the examples they give may be turned into useful exercises or suggest interesting projects.
Chapter XIII

Some Byways of Mathematics: Properties of Numbers; Calculating Machines; History of Mathematics; Mathematical Societies

Properties of Numbers and Puzzles

Many puzzles, and some of the simpler properties of numbers, have been for centuries an interesting relaxation from strenuous forms of mathematical work. Indeed, for nearly five thousand years the 'magic' properties of numbers have been a source of wonder, and have often borne great mystical and religious significance.

The first and last books of the Bible contain references to numbers which were imagined to hold important secrets. 'Researches' to find the 'meaning' of the number 666 — the 'mark of the beast', have occupied whole lifetimes, and the results would fill many books. Just as the quest for the Elixir of Life or the Philosopher's Stone brought many useful discoveries in science, so the pursuit of number relations for their own sake has not been without its effect on orthodox mathematics. Many interesting topics in the history of mathematics can be found concerning the mystical and magical 'properties' of number and formulae. Even so stable-minded a physicist as Heinrich Hertz (the discoverer of 'wireless waves') said less than century ago: 'One cannot escape the feeling that mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them.' He was, of course, referring to the immense use to which formulae can be put in the sciences and their applications, and not to any so-called 'magical' properties.

It would be difficult to justify the spending of much time on the 'theory of number' or mathematical puzzles, from the purely utilitarian point of view; though it is surprising that seemingly 'useless' sets of numbers have recently been applied to branches of
science so far removed as genetics and wave mechanics. From the child’s point of view an investigation of the properties of number assists in the process of ‘making friends with numbers’, and it can form the basis of a fascinating pastime. No work with numbers should be despised, for ability to factorize, decompose, square or find a square root quickly, is of great assistance to many of the processes of calculating.

Children are always interested in magic squares and cubes, number cross-word puzzles and number codes.

Magic Squares

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<td>4</td>
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<td>8</td>
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Find some other simpler examples

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</tbody>
</table>

(The numbers in each column, row and diagonal add up to 34.)

A Magic Cube (distorted in shape to show the numbers)

(Find how many sets of 3 numbers in a straight line add up to 42.)
SOME BYWAYS OF MATHEMATICS

It will be interesting to hunt for some further examples.

The construction of series of numbers is also acceptable to secondary school children in the later forms. With very little help children will deduce some of the properties and possible uses of the series.

As is well known, the Greeks excelled in geometry but not in arithmetic and algebra. Their sets of numbers were founded on geometrical patterns, e.g.

\[
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Triangular Numbers

Othersets may be made from squares, pentagons, 'stars', hexagons, and more complex patterns. (Hogben's chapter, 'The Beginnings of Arithmetic', contains many examples of this type of number.)

The sieve of Eratosthenes and various ways of testing divisibility and finding factors are always fascinating; tests of divisibility by 2, 4, 3, 5, 9, and in special cases of 11, 37 are useful and interesting.\(^1\)

The powers of 11: 11, 1331, 14,641, etc., anticipate the binomial theorem; indeed, attempts have been made to explain the well-known exponential 'curve of error' through these numbers. Arranged in various forms they give Pascal's and other related triangles.

Some of the simple number theorems are easy to prove, whilst others are of the utmost difficulty, and in some cases a general proof has not yet been found:

(1) Any odd number can be expressed as the difference of two squares.

\[
\begin{align*}
3 &= 2^2 - 1^2 \\
5 &= 3^2 - 2^2 \\
7 &= 4^2 - 3^2 \\
9 &= 5^2 - 4^2 \\
\end{align*}
\]

The law is easily seen — divide the odd numbers as nearly as possible into two halves, e.g. 37 divides into 18 and 19. 37 = 19\(^2\) -- 18\(^2\).

\(^1\) If the difference between the sum of the even digits and the sum of the odd digits of a number is 0 or is divisible by 11, the number is divisible by 11.

Numbers which consist of the same three digits, e.g. 888, are divisible by 37.
(Square numbers and square roots can be found by applying Pythagoras' Theorem.) Simple algebraic formulae will serve to prove the general case, e.g. \((n + 1)^2 - (n)^2 = 2n + 1\) (which is always odd if 'n' is an integer).

(2) Every square number ends in 0, 1, 4, 5, 6, or 9.
(3) All cubic numbers can be expressed as the differences of two squares.

\[
\begin{align*}
1^3 &= 1^2 - 0^2 \\
2^3 &= 3^2 - 1^2 \\
3^3 &= 6^2 - 3^2 \\
4^3 &= 10^2 - 6^2 \\
5^3 &= 15^2 - 10^2
\end{align*}
\]

The series of numbers in the vertical columns on the right are interesting and will give a clue to the general formula:

\[
\begin{array}{ccccccc}
0 & 1 & 3 & 6 & 10 & 15 & 21 \\
1 & 2 & 3 & 4 & 5 & 6
\end{array}
\]

Expressed algebraically: \(\left\{\frac{n^2}{2} - \frac{n}{2}\right\}^2 - \left\{\frac{n^2}{2} - n\right\}^2 = n^3\).

(4) Waring's Theorem
That every integer is either a square or the sum of 2, 3, or 4 squares. Perform experiments with simple cases, e.g.

\[46 = 5^2 + 4^2 + 2^2 + 1^2\]

(The general theorem for any higher power, e.g. using 9 cubes, 19 4th powers, etc., was only proved a few years ago.)

(5) Goldbach's Theorem
'Every even number is the sum of two primes.' The general theorem has not yet been proved. Children will write down any even number up to 200 and later to 1000, and then hunt for the primes. Children will be interested in the 'perfect' and 'amicable' numbers of the ancients. A perfect number is equal to the sum of all its factors including 1, e.g. 6, 28, 496.

\[6 = 3 + 2 + 1 \quad 28 = 14 + 7 + 4 + 2 + 1\]