APPENDIX A

GENERAL NOTATION

For all materials except reinforced concrete:

- $f$ = unit fiber stress.
- $v$ = unit shearing stress (horizontal or vertical).
- $V$ = total shear.
- $e$ = distance from neutral axis to extreme fiber.
- $b$ = breadth of rectangular section.
- $d$ = depth of section.
- $A$ = area of section.
- $I$ = moment of inertia.
- $r$ = radius of gyration.
- $S$ = section modulus.
- $M$ = bending moment or resisting moment.
- $l$ = span or length.
- $L$ = span or length.

$P$ or $P'$ = concentrated load or total stress in a member.

$w$ = uniformly distributed load per unit of length.

$W$ = total uniformly distributed load.

$R$ = reactions at supports or resultant of forces.

$E$ = modulus of elasticity.

$y$ = deflection at any point in a beam.

$\Delta$ = total deflection or deflection at any point in a beam.

$\delta$ = unit deformation.

$e$ = eccentricity.

For reinforced concrete:

(a) Rectangular Beams and Slabs

- $f_s$ = tensile unit stress in steel.
- $f_r$ = compressive unit stress in extreme fiber of concrete.
- $f'_c$ = ultimate compressive strength of concrete at age of 28 days, based on tests of 6- × 12-in. or 8- × 16-in. cylinders made in accordance with A.S.T.M. specifications.
- $E_s$ = modulus of elasticity of steel.
- $E_c$ = modulus of elasticity of concrete.
- $n = \frac{E_s}{E_c}$ For values of $f_c$ in flexure not over 900 lb. per sq. in., $n$ is commonly taken as 15.

$M$ = moment of resistance, or bending moment in general.

$A_s$ = steel area.

$b$ = breadth of beam (generally taken as 12 in. in case of slabs).

$d$ = depth of beam to center of steel.

$k$ = ratio of depth of neutral axis to depth, $d$.

$z$ = depth from compressive face to resultant of compressive stresses.

$j = \text{ratio of lever arm of resisting couple to depth, } d.$

$jd = d - z = \text{arm of resisting couple.}$

$p = \text{steel ratio } = \frac{A_s}{bd}$

$l$ or $L = \text{span length of beam or slab.}$

(b) T-beams

$b$ = width of flange.

$b'$ = width of stem.

$t = \text{thickness of flange.}$

$^1$ Notation not found in this appendix appears in text where used.
(c) Beams Reinforced for Compression

- \( A' \) = area of compressive steel.
- \( p' \) = steel ratio for compressive steel.
- \( f_t' \) = compressive unit stress in steel.
- \( C' \) = total compressive stress in concrete.
- \( C'' \) = total compressive stress in steel.
- \( d' \) = depth of center of compressive steel.
- \( z \) = depth to resultant of \( C' \) and \( C'' \).

(d) Shear, Bond and Web Reinforcement

- \( V \) = total shear at any section.
- \( V' \) = total shear at any section carried by the web reinforcement.
- \( s \) = maximum shearing unit stress at any section.
- \( u \) = bond stress per unit area of bar.
- \( o \) = circumference or perimeter of bar.
- \( \Sigma o \) = sum of the perimeters of all tension bars at any section.
- \( s \) = spacing of web members measured at the neutral axis and in the direction of the longitudinal axis of the beam.
- \( a \) = spacing of web reinforcement bars measured perpendicular to their direction.
- \( A_w \) = total cross-sectional area of web reinforcement within a distance of \( a' \) or total area of all bars bent up in any one plane.
- \( f_t \) = tensile unit stress in web reinforcement.
- \( \alpha \) = angle between web bars and longitudinal bars.
- \( v_c \) = allowable shearing stress on plain concrete.
- \( N_s \) = number of stirrups at one end of member.

(e) Flat Slabs

- \( b \) = side of square drop.
- \( c \) = base diameter of the largest right cone or pyramid which lies entirely within the column and the column capital, whose vertex angle is 90°, and whose base is 1\( \frac{1}{8} \) in. below the bottom of the slab or the bottom of the drop, if a drop is present.
- \( L \) = side of square panel.
- \( l_2 \) = that side of any panel, which is at right angles to the section for which moments are desired.
- \( l_1 \) = that side of any panel which is parallel to the section for which moments are desired.
- \( M_e \) = total moment in one direction on critical sections of one column strip and one middle strip.
- \( +M_c \) = positive moment at center of column strip.
- \( -M_c \) = negative moment across panel and capital edge on column strip.
- \( -M_m \) = negative moment across panel edge on middle strip.
- \( +M_m \) = positive moment at center of middle strip.
- \( q \) = distance from center of column to center of gravity of semi-periphery of column capital, divided by \( c \). For round column capitals \( q = \frac{3}{8} \); for square capitals \( q = \frac{5}{8} \); for octagonal capitals \( q = \frac{7}{8} \).
- \( t_c \) = thickness of slab.
- \( t_s \) = thickness of slab and drop combined.
- \( t_e \) = thickness of slab at center, with panelled ceiling.
- \( w \) = load per square foot of panel including weight of slab.
- \( W \) = total load in one panel including weight of slab.

(f) Columns

- \( A \) = cross-sectional area of member, exclusive of any portion used solely for protective cover.
- \( A_t \) = area of longitudinal steel.
- \( P \) = \( A_t \).
- \( A_s = A(1 - p) \) = net area of concrete.
- \( P \) = total safe axial load (including weight of column).
- \( h \) = unsupported height of column.
APPENDIX B

GENERAL PROPERTIES OF SECTIONS

1. Area of a Section.—The area of standard rolled sections may be determined from the rolling mill handbooks. For the section shown in Fig. 1a, the area is readily determined by dividing the figure into simple figures, in this case the rectangles A, B and C and the four triangles D. In the case of the built-up section of Fig. 1b, the total area is readily found by summing up the areas of the plates and angles. These areas may be taken from any rolling mill handbook.

2. Statical Moment of an Area.—Let Fig. 2 represent any area. The statical moment of this area about any axis, as \( OX \), is the moment of each element of this area about the given axis. Assume the area to be divided into strips parallel to the given axis. Such a strip is represented by 1–2 of Fig. 2. Let \( b \) = length of this strip, \( dy \) = width of strip perpendicular to the given axis, \( y \) = perpendicular distance from center of strip to the given axis, and \( Q \) = statical moment of entire area about the given axis.

The area of the strip 1–2 is \( bdy \) and its statical moment about the axis \( OX \) is \( bydy \). For the entire area, \( Q = \text{sum of all such values as } bydy \), that is

\[ -\int_{y_1}^{y_2} bydy \]

(1)

To apply eq. (1) to a given area, the width of section must be expressed as a function of \( y \) and the resulting equation integrated between the given limits.

![Fig. 1](a) ![Fig. 2](b) ![Fig. 3](c) ![Fig. 4](d)

Consider the rectangle of Fig. 3. Required the statical moment of the figure about an axis \( OX \) at a distance \( a \) below the bottom of the figure. From eq. (1), noting that the width of section is constant, we may write

\[ Q = b \int_{a}^{a+d} ydy = \frac{b}{2} (d^2 + 2ad) = bd \left( a + \frac{d}{2} \right) \]

But \( bd = A \) = area of section. Hence

\[ Q = A \left( a + \frac{d}{2} \right) \]

(2)

That is, the statical moment about the given axis is equal to the area of the section multiplied by the distance from the axis to a point half way across the section.

For the triangle of Fig. 4,

\[ Q = \int_{0}^{h} bydy \]

when \( by \) = width of section at any point \( = \frac{b}{k} (h - y) \)

Hence

\[ Q = \int_{0}^{h} \frac{b}{k} (h - y)dy = \frac{bh^2}{6} \]

But \( \frac{bh}{2} \) = area of triangle. Hence

\[ Q = \text{Area times } \frac{1}{3} \text{ of height of triangle above axis} \]

(3)

The statical moment of an area is sometimes called the \textit{first moment} of the area.

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3. Center of Gravity of an Area.—The center of gravity of an area is the point at which the entire area must be concentrated in order that the product of the area times the distance from this point to a given axis may be equal to the statical moment of the area about the given axis.

Let \( A \) represent the area of the section shown in Fig. 5, and let \( y \) represent the distance from an axis \( OX \) to the center of gravity of the section, assumed as located at the point \( c.g. \). From the above definition

\[
Ay = \int_{B}^{A} bdy
\]

Since \( A = \int_{B}^{A} bdy \), we have in general

\[
\int_{B}^{A} bdy = \int_{B}^{A} bdy
\]

\[
\int_{B}^{A} bdy
\]

Equations (4) and (5) give the coordinates of the center of gravity of the section of Fig. 5 with respect to an origin at \( O \).

In these equations the denominators each represent the area of the section. The limits of integration indicate the extreme values of \( x \) and \( y \) for the section.

From eqns. (4) or (5), it can be seen that the statical moment of any area about an axis through its center of gravity is equal to zero, for under the assumed conditions \( x \) or \( y \) must be zero. Since the denominators of eqns. (4) and (5) represent the area of the section, which cannot be zero for a real area, \( y \) or \( x \) can be zero only when the numerators of these equations are equal to zero. But these numerators represent the statical moment about the axis in question. Therefore, the statical moment of an area is zero for an axis through its center of gravity. This relation is of value in the work to follow.

To apply eqns. (4) and (5) to any given figure, the dimensions of the section must be expressed as functions of \( x \) and \( y \) and the integrations performed, as indicated. For the rectangle of Fig. 6, assume a set of axes \( OX \) and \( OY \) through the sides \( OC \) and \( OA \). Let \( c.g. \) represent the required center of gravity. The distance from the \( OX \) axis to \( c.g. \) as given by eq. (4), is

\[
y = \frac{1}{2} \int_{0}^{d} bdy = \frac{bd^2}{2}
\]

In the same way, the distance from the \( OY \) axis to \( c.g. \) as given by eq. (5) is

\[
x = \frac{1}{2} \int_{0}^{b} dx = \frac{b}{2}
\]

The point represented by these coordinates is the center of the section. Hence, the center of gravity of a rectangle is at the center of the section.

For the right angle triangle of Fig. 7a, the axes are taken along the right angle sides. From eq. (4), the distance from the \( OX \) axis to \( c.g. \) is

\[
y = \frac{1}{3} \int_{0}^{d} y dy = \frac{d}{3}
\]

The distance from the \( OY \) axis to \( c.g. \) is found to be \( \frac{b}{3} \). Therefore for a right triangle,
the center of gravity is located at a distance from the bases equal to \( \frac{b}{3} \) of the altitude of the triangle. Note that if the bases of the triangle are bisected and lines are drawn to the opposite vertices of the triangle, the intersection of these lines coincides with the c.g. of the triangle. Figure 7b shows the coordinates of the c.g. for oblique triangles.

When the section is very complicated or the outline very irregular, the above method cannot readily be applied. In such cases approximate methods of integration may be used to advantage. Thus in Fig. 8, the irregular area may be divided into small strips representing rectangles, triangles or other simple areas. At the center of gravity of each of these small areas apply a force which is proportional to the area of the strip. Scale or calculate the distances from these centers of gravity to any axes, as \( OX \) and \( OY \). If \( A \) represent the area of any strip and \( x_A \) and \( y_A \) the distance from the center of gravity of the area \( A \) to the given axis, then

\[
\begin{align*}
x_s &= \frac{\sum x_A A}{\sum A} \\
y_s &= \frac{\sum y_A A}{\sum A}
\end{align*}
\]
Figure 9 shows the location of the centers of gravity of a few simple figures. A convenient graphical method for locating the center of gravity of a trapezoid is shown on Fig. 9a. The construction is as follows: Produce the top of the figure $AB$ to a point $C$ such that $BC = DE = \text{width of base}$. In the opposite direction produce the base $DE$ to a point $F$ such that $EF = AB = \text{width of top}$. Connect $C$ and $F$. Bisect the upper and lower faces, locating points $G$ and $H$. Connect $G$ and $H$. The intersection of lines $CF$ and $GH$ coincides with the center of gravity of the section.

4. Moment of Inertia.—The moment of inertia of an area with respect to any axis is the sum of the products formed by multiplying each element of the area by the square of its distance from the given axis. Let Fig. 10 represent any area, and let it be required to determine a general expression for the moment of inertia of this area with respect to any axis, as $OX$. Divide the area into strips, 1–2, parallel to the axis $OX$. If $dy = \text{width of each strip}$, the area of a strip is $b \cdot dy$. Let $I$ represent the moment of inertia of the given area. Then, from the above definition, the moment of the entire area is

$$I = \int_{y_1}^{y_2} b \cdot y^3 \cdot dy$$

(6)

To apply eq. (6) to a given area, the width $b$ must be expressed as a function of $y$ and the integration performed as indicated.

![Fig. 10.](image)

![Fig. 11.](image)

The moment of inertia of an area as given by eq. (6), is a quantity of the dimensions distance to the fourth power, for $b \cdot dy$ represents an area which has the dimensions distance to the second power, and $y^3$ is also distance to the second power.

5. Moments of Inertia for Parallel Axes (Parallel Axes Theorem).—A very useful and simple relation may be obtained between the moment of inertia of an area for an axis through its center of gravity and any other parallel axis at a distance $a$ from the gravity axis. In Fig. 11 let $OX$ represent an axis through the center of gravity of the section (sometimes called a gravity axis) and let $OA$ be any axis parallel to $OX$ and at a distance $a$ from $OX$. Let $y$, the distance from any element of area 1–2, to an axis, be referred to $OX$. Then from the above definition, the moment of inertia about axis $OA$, which will be denoted by $I_A$ is

$$I_A = \int_{v_1}^{v_2} b(a + y)^3 \cdot dy$$

Expanding this expression, noting that $a$ is a constant, we have

$$I_A = a^2 \int_{v_1}^{v_2} bdy + 2a \int_{v_1}^{v_2} bydy + \int_{v_1}^{v_2} by^2dy$$

In this equation, $\int_{v_1}^{v_2} bdy = A = \text{area of section}$; $\int_{v_1}^{v_2} bydy = Q = \text{statistical moment of area about an axis through its center of gravity, which is equal to zero}$; and $\int_{v_1}^{v_2} by^2dy = I_z = \text{moment of inertia of area about OX, the gravity axis of the section}$. Therefore, the above equation may be written

$$I_A = I_z + Aa^2$$

(7)

Equation (7) is very useful when an area may be divided into smaller areas for which the properties are known. Also, eq. (7) shows that the moment of inertia of a section for an axis through its center of gravity is less than the value for any other axis, for moving the axis away from the center of gravity increases the moment of inertia, as indicated by the positive value for $Aa^2$. 
6. Moments of Inertia for Inclined Axes (Inclined Axes Theorem).—In Fig. 12, let \( OX \) and \( OY \) be any pair of rectangular axes through the point \( O \), and let \( OU \) and \( OV \) be another pair of rectangular axes through \( O \) but at an angle \( \alpha \) from the first axis. Let \( dA \) represent any element of area, whose coordinates with respect to the \( OX, OY \) axes are \( x \) and \( y \), and \( u \) and \( v \) with respect to the \( OU, OV \) axes.

The moment of inertia of the area about the \( OU \) axis, which will be denoted by \( I_u \) is

\[
I_u = \int u^2 dA
\]

and about the \( OV \) axis for which \( I_v \) denotes the moment of inertia, we have

\[
I_v = \int v^2 dA
\]

In these equations the limits of integration must cover the entire section.

From Fig. 12 it can be seen that in terms of \( x \) and \( y \), the values of \( u \) and \( v \) are

\[
\begin{align*}
u &= x \cos \alpha + y \sin \alpha \\
v &= y \cos \alpha - x \sin \alpha
\end{align*}
\]

Substituting these values of \( u \) and \( v \) in the above equations, and expanding the terms, we have

\[
I_u = \int \cos^2 \alpha y^2 dA - 2 \int \sin \alpha \cos \alpha xy dA + \int \sin^2 \alpha x^2 dA
\]

and

\[
I_v = \int \cos^2 \alpha x^2 dA + 2 \int \sin \alpha \cos \alpha xy dA + \int \sin^2 \alpha y^2 dA
\]

In eqs. (9) and (10), \( \int x^2 dA \) and \( \int y^2 dA \) represent respectively, \( I_u \) and \( I_v \), the moments of inertia of the section about the \( OY \) and \( OX \) axes. The term \( \int xy dA \) is known as the product of inertia of the section. It is the sum of all the products obtained by multiplying each element of area by the product of its distances from the \( OX \) and \( OY \) axes. The product of inertia will be denoted by \( J_{xy} \), the subscripts indicating the axes for which the product of inertia is taken. Note that \( J_{xy} \) is also a term whose dimensions are distance to the fourth power.

Equations (9) and (10) may then be written

\[
I_u = I_x \cos^2 \alpha - 2J_{xy} \sin \alpha \cos \alpha + I_y \sin^2 \alpha
\]

and

\[
I_v = I_y \cos^2 \alpha + 2J_{xy} \sin \alpha \cos \alpha + I_x \sin^2 \alpha
\]

By means of eqs. (11) and (12) it is possible to find the moments of inertia for axes \( OU \) and \( OV \) when the moments of inertia and product of inertia for the axes \( OX \) and \( OY \) are known.

A useful relation between the moments of inertia for the two pairs of axes may be obtained by adding eqs. (11) and (12). Noting that \( \cos^2 \alpha + \sin^2 \alpha = 1 \), we have

\[
I_u + I_v = I_x + I_y
\]

The term \( J_{xy} \), the product of inertia, which appears in eqs. (11) and (12), may have positive, negative or zero values, depending upon the location of the coordinate axes. In this respect it differs from the moment of inertia, which has only a positive value due to the fact that the distance to each area is squared.

Figure 13 shows the effect of the position of the coordinate axes on the sign of \( J_{xy} \). Figure 13a shows a rectangle with the \( X \) axis along the base and the \( Y \) axis through the center of the figure. For every area \( dA \) on the right of the \( Y \) axis there is a corresponding area \( dA \) on the left. If positive directions with respect to an origin at \( O \) are taken as upward and to the right, it is evident that the product of inertia, \( J_{xy} = \int xy dA = 0 \). In Fig. 13b, both \( x \) and \( y \) are positive, and \( J_{xy} = \) a positive quantity. For the conditions shown in Fig. 13c, the \( x \)-distances are positive while the \( y \)-distances are negative, and
\( J_{x^2} \) is a negative quantity. In general, if one of a pair of axes is an axis of symmetry for a section, the product of inertia is zero for that pair of axes.

7. Products of Inertia for Parallel Axes.—Let it be required to find the relation between the products of inertia for the figure of Fig. 14 with respect to the pairs of axes \(OX, OY\) and \(OU, OV\). For the conditions shown, the product of inertia with respect to the \(OU, OV\) axes is

\[
J_{uv} = \int (b + x)(a + y)dA = \int abdb + \int adydA + \int xdxA + \int yydA
\]

In this expression, \( \int dA = A \) = area of section, \( \int yydA = J_{xy} \) = product of inertia with respect to the \(OX, OY\) axes, and \( \int yydA\) and \(\int xdxA\) are respectively the statical moments of the area for the \(X\) and \(Y\) axes. If the \(OX, OY\) axes are assumed as passing through the center of gravity of the figure, the above statical moments are zero. The above equation then becomes

\[
J_{uv} = J_{xy} + Aab
\]

That is, the product of inertia for any pair of axes \(OU, OV\), with respect to a pair of parallel axes through the center of gravity of the figure is equal to product of inertia for the gravity axes plus the area of the figure times the product of the coordinates of the center of gravity of the figure with respect to the \(OU, OV\) axes. Due attention must be paid to signs in calculating the several quantities.

8. Principal Axes and Principal Moments of Inertia.—From eq. (11) or (12), it can be seen that the moment of inertia of a section varies with the angle \(\alpha\). To determine the maximum value of the moment of inertia, as given by eq. (11), place \(\frac{dI_u}{d\alpha}\) equal to zero and solve for the value of \(\alpha\). If \(\alpha_u\) is the value of this angle, we have

\[
\tan 2\alpha_u = \frac{2J_{xy}}{I_y - I_x}
\]

There are two angles which answer the conditions imposed by this equation, one in the first and the other in the second quadrant, and furthermore, these angles differ in value by 90 deg. On substituting values of \(\alpha_u\) as given by eq. (15) in eq. (11), two values of \(I_u\) will be derived. By the methods of the calculus, it can be shown that one of these is the maximum value of \(I_u\) and the other is the minimum value. These moments of inertia are known as the principal moments of inertia for the section and the axes for which they occur are known as the principal axes of the section. A similar operation performed on eq. (12) will give results in which the values are the reverse of the above. Note that the maximum and minimum values of moment of inertia occur for axes which are 90 deg. apart.

The product of inertia of an area for a principal axis can readily be shown to be equal to zero. Thus for the axis \(OU\) of Fig. 12, we have

\[
J_{uv} = \int uvdA
\]

On substituting values of \(u\) and \(v\) as given by eq. (8), we have

\[
J_{uv} = J_{xy} \cos 2\alpha + \frac{1}{2} \sin 2\alpha (I_x - I_y)
\]

When \(OU\) is a principal axis, the angle \(\alpha\) has the value given by eq. (15). Substituting values of this angle in the above equation it will be found that \(J_{uv}\) is equal to zero as stated above.

Let \(OX\) and \(OY\) of Fig. 15 be the principal axes of a section, and let \(I_x\) and \(I_y\) be the principal moments of inertia. Let \(OA\) represent any other axis at an angle \(\alpha\) from \(OX\). Remembering that the product of inertia for principal axes is zero, we may write from eq. (11)

\[
I_A = I_x \cos^2 \alpha + I_y \sin^2 \alpha
\]

Equation (16) is a general equation for moment of inertia about any axis in terms of the moments of inertia for the principal axes.

9. Radius of Gyration.—The radius of gyration of an area is the distance from a given axis to a point at which the entire area of the section must be applied in order that the product of the area times the square of this distance to the given axis may be equal to the moment of inertia of the section about the given axis. If \(r\) = radius of gyration of
the section as defined above, \( A = \) area of section, and \( I = \) moment of inertia about the
given axis, we have
\[
I = Ar^2
\]
or
\[
= \sqrt{\frac{I}{A}}
\]
(17)
For a rectangle of width \( b \) and depth \( d \), \( I = \frac{bd^3}{12} \) and \( A = bd \). From eq. (17)
\[
= \sqrt[3]{\frac{bd^3}{12bd}} = d \sqrt{\frac{1}{12}} = 0.289d
\]
If \( r_A, r_x \) and \( r_y \) represent the radii of gyration for the corresponding axes of Fig. 15,
eq (16) may be written in the form
\[
r_A^2 = r_x^2 \cos^2 \alpha + r_y^2 \sin^2 \alpha
\]
(18)
By analytical geometry, it can be shown that eq. (18) represents an
ellipse with semi-major and minor axes of \( r_y \) and \( r_x \) respectively, as
shown in Fig. 16. The ellipse of Fig. 16 is known as the inertia ellipse
for the area of Fig. 15.
If \( n-n \) is a tangent to the ellipse parallel to any axis \( OA \) through
the center of the ellipse, it can be shown that the perpendicular distance
from the axis \( OA \) to the tangent \( n-n \) is equal to the radius of gyration
of the section of Fig. 15. The construction shown in Fig. 16 offers a
convenient method for determination of moment of inertia.

10. Section Modulus.—In the general formula for resisting moment of beams appears
the term \( S = \frac{I}{c} = \) moment of inertia of section about the neutral axis divided by the dis-
tance from the neutral axis to an extreme fiber of the section. This quantity is known as
the section modulus for the beam cross-section. It is a quantity of dimensions distance
to the third power.
For a rectangle of width \( b \) and depth \( d \), we have
\[
S = \frac{\frac{bd^3}{12}}{d} = \frac{bd^2}{6}
\]
Values of section modulus for rolled shapes are given in rolling mill handbooks.
APPENDIX C

DEFLECTION OF BEAMS

ELASTIC CURVE METHOD

1. General Equation of Elastic Curve for Bending.—Let OABL of Fig. 1a show (greatly exaggerated) the bent position of the neutral axis of a beam which, before the application of the loads, was represented by the straight line OL. This deformation of the beam is due to positive bending moment.

If points A and B of Fig. 1a represent two adjacent points on the neutral axis of the bent beam, the normals at these sections will meet in a point F, which is the center of curvature for these points. To locate points on the beam with respect to a given point, let O, a point at one end of the beam, be taken as an origin. Any convenient point will do as well. Let x and y respectively be the horizontal and vertical distances from the origin O to the center of any element AB, and assume that x and y are positive when measured to the right and downward respectively, as shown by the arrows in Fig. 1. It is evident that the angle between the tangents to the curve at A and B is equal to the angle between the radii at these points, as shown in Fig. 1a.

In Fig. 1b let AB and BC show two adjacent elements of the neutral axis, and assume that their projections on a horizontal axis are each equal to dx. Let the vertical projections of these elements be represented by dy_AB and dy_BC. From Fig. 1b, it is evident that the difference between dy_AB and dy_BC is a measure of the change in deflection across the two elements AB and BC. Let d_y^2 denote the change in vertical projection of the two elements. If the deflection is small compared to the dimensions of the beam section, which is the usual case, we may consider AB = BC = dx and ∠BCD = 90 deg. Hence, since dφ is a very small angle,

\[ d_y^2 = -dx \cdot d\phi \]

The minus sign is used in this equation because, as shown in Fig. 1b the angle between element BC and a horizontal axis is less than the corresponding angle for element AB. To conform to the direction notation given above, increasing values of x and y result in increasing angles between successive elements. Since a decreasing angle exists for the conditions shown in Fig. 1b, a minus sign must be used in the above equation.

From Figs. 1a or b, dφ = dx/R, where R = radius of curve for any small element of the neutral axis.

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Hence
\[ dy = -dx^3/R \]
or
\[ \frac{d^2y}{dx^2} = - \frac{1}{R} \] \hfill (1)

The expression \( d^2y/dx^2 \) of eq. (1) is known as the second differential coefficient of \( y \), the deflection of the beam, with respect to \( x \), the distance from the origin to the point at which the deflection is desired. It is a measure of the rate of change of the slope of the neutral axis.

In Fig. 1c, an element of the beam is enlarged to show the effect of bending deformation. The deformed element is shown by 3-4-5-6. In the undeformed element, the faces 3-B-4 and 5-A-6 are parallel lines. If a line 1-A-2 be drawn parallel to 3-B-4, the deformation of the element is represented by 1-5. If the deformation is small, \( 1 - 3 = AB = dx \), and we may write \( \delta = \frac{f}{E} \, dx \), where \( f = \) fiber stress due to a moment \( M \). From eq. (6), p. 23, \( f = \frac{Mc}{I} \) and hence
\[ \delta = \frac{Mc}{EI} \, dx \]

When the deformations are small, \( FAR \) and \( A-1-5 \) may be considered as similar triangles, and we have \( \frac{\delta}{c} = \frac{dx}{R} \). Solving for \( \delta \) and equating the resulting expression to the value of \( \delta \) given above, we have finally \( \frac{1}{R} = M/EI \). Substituting this value of \( \frac{1}{R} \) in eq. (1), we derive
\[ \frac{d^2y}{dx^2} = - \frac{M}{EI} \] \hfill (2)

Equation (2) is the general expression for the differential equation of the elastic curve of a beam subjected to bending due to a clockwise or positive moment. To determine the equation of the elastic curve by means of eq. (2), the moment \( M \) must be expressed as a function of \( x \), and the resulting expression integrated twice.

2. Application of General Equation of Elastic Curve to Solution of Problems in Deflection of Beams.—In the articles which follow, the equation of the elastic curve will be derived and the maximum deflection will be determined for a few typical cases.

Simple Beam with Uniform Load.—For the conditions shown in Fig. 2, the general expression for moment at any point on the beam is \( M = \frac{w}{2} \times (l - x) \). Substituting this value of \( M \) in eq. (2) we have
\[ \frac{d^2y}{dx^2} = - \frac{wx}{2EI} (l - x) \] \hfill (a)

Integrating,
\[ \frac{dy}{dx} = - \frac{w}{2EI} \left( \frac{x^2}{2} - \frac{x^3}{3} \right) + C_1 \] \hfill (b)

In this equation, \( dy/dx \) is the slope of the elastic curve at any point, and \( C_1 \) is a constant of integration which depends for its value upon the conditions of the problem. To determine \( C_1 \), note that the load on the beam is symmetrical with respect to the beam center. It is therefore evident that the elastic curve will be symmetrical about the beam center, and that the tangent to the elastic curve at the beam center is horizontal. Since slope of a horizontal line is zero, we have as a condition for the determination of \( C_1 \) in eq. (b), that \( dy/dx = 0 \) when \( x = \frac{l}{2} \). Substituting these values in eq. (b) and solving for \( C_1 \), we have
\[ C_1 = \frac{wl}{24EI} \]

Eq. (b) then becomes
\[ \frac{dy}{dx} = + \frac{w}{24EI} \left( 4x^3 - 6lx^2 + l^3 \right) \] \hfill (3)
Integrating eq. (3) we have

\[ y = \frac{w}{2AEI} \left( x^4 - 2lx^3 + p^2 \right) + C \]

To determine \( C \), the constant of integration, note from Fig. 2b that \( y = 0 \) when \( x = 0 \). Hence from the above equation \( C = 0 \) and we have finally

\[ y = \frac{wx}{2AEI} \left( x^3 - 2lx^2 + p^2 \right) \tag{4} \]

which is the general equation of the elastic curve for a simple beam carrying a uniform load.

The maximum deflection of the beam of Fig. 2 evidently occurs at the point where the tangent to the elastic curve is horizontal. As stated above, the tangent is horizontal when \( x = \frac{l}{2} \). Substituting this value of \( x \) in eq. (4) we have

\[ y_{\text{max}} = \frac{5wl^4}{384EI} \tag{4a} \]

**Illustrative Problem.**—A simple beam 16 ft. long supports a uniform load of 600 lb. per ft. Determine the maximum deflection of this beam in inches. Assume that the moment of inertia of the beam is 100 in.\(^4\), and that the material is steel for which \( E = 30,000,000 \) lb. per sq. in.

The maximum deflection is given by eq. (4a). In substituting in eq. (4a) attention must be paid to the units in which the several terms are expressed. Since the deflection in inches is desired, all values must be expressed in inch units. Thus \( w = 600 \times \frac{1}{16} \) lb. per in., and \( l = 16 \times 12 \) ft. Values of \( E \) and \( I \) are given directly in inch units. Substituting these values in eq. (4a) we have

\[ y_{\text{max}} = \frac{(5)(600)(16)/(12)^3}{384(30,000,000)} = \frac{884,736,000}{384EI} \text{ in.} \]

Compare this result with the problem of p. 52. Substituting values of \( E \) and \( I \),

\[ y_{\text{max}} = \frac{884,736,000}{(30,000,000)(100)} = 0.295 \text{ in.} \]

**Illustrative Problem.**—Determine the deflection of the above beam in feet, using foot units.

Here \( w = 600 \), \( l = 16 \), \( E = (30,000,000)(144) \) lb. per sq. ft. and \( l = (12)^4 \) ft.\(^4\).

Hence

\[ y_{\text{max}} = \frac{(5)(600)(144)/(12)^4}{384(144)(30,000,000)(100)} = 0.0245 \text{ ft.} \]

**Illustrative Problem.**—Determine the deflection at a point 5 ft. from the left end of the beam of the above problem. Use inch units.

Here \( w = 600 \times \frac{1}{16} = 50 \) lb. per in.; \( x = 5 \) ft. = 60 in.; \( l = 16 \) ft. = 192 in. Values of \( E \) and \( I \) are as given above. Substituting in eq. (4) we have

\[ y = \frac{(50)(60)}{(24)(30,000,000)(100)} \left( \frac{60^2}{2} \right) = 2 \left( \frac{192}{60} \right)^2 + (192)^3 \]

\[ y = 0.246 \text{ in.} \]

**Simple Beam with a Single Concentrated Load.**—For the conditions shown in Fig. 3, the law of variation of moments on section \( AC \) differs from that on section \( CB \). Hence two substitutions must be made in eq. (2). After integrating each equation twice, four constants of integration will appear which must be determined subject to conditions shown in Fig. 3. The detail work is as follows:

For the portion of the beam from \( A \) to \( C \), \( M_x = R_1 x = W(1 - k)x \). Then from eq. (2)

\[ \frac{d^2y}{dx^2} = -\frac{W}{EI} (1 - k)x \]

Integrating twice, we have

\[ \frac{dy}{dx} = -\frac{W}{EI} (1 - k)x^2 + C_1 \]

\[ y = -\frac{W}{EI} (1 - k)x^3 + C_1 x + C_2 \]

For the portion of the beam from \( C \) to \( B \), \( M_x = R_2 (l - x) = Wk(l - x) \). Then from eq. (2)

\[ \frac{dy}{dx} = -\frac{W}{EI} k(l - x) \]

\[ y = -\frac{W}{EI} k(l - x) \]
Integrating twice, we have
\[ \frac{dy}{dx} = -\frac{Wk}{EI} \left( \frac{l^2}{2} - \frac{x^3}{2} \right) + C_3 \quad (c) \]
and
\[ y = -\frac{Wk}{EI} \left( \frac{l^3}{2} - \frac{x^3}{6} \right) + C_4 x + C_4 \quad (d) \]

Equations (b) and (d) are general expressions for the equations of the elastic curves for the portions of the beam on either side of the load \( W \). However, these equations cannot be used until the values of the constants of integration \( C_3, C_4, C_5, \) and \( C_4 \) are known. Since there are four unknown terms, an equal number of independent equations must be set up before the unknowns can be determined. The required independent equations may be derived from the necessary relations which must exist between the elastic curves on the two sides of the load \( W \) in order that the two elastic curves may be joined to form a single continuous curve.

The four independent equations from which the values of the constants of integration may be determined are derived from conditions shown in Fig. 3b. From this figure it is evident that \( y \) in eq. (b) is zero only when \( x = 0 \). Also, \( y \) in eq. (d) is zero only when \( x = l \). At point \( C \), where the elastic curves to the right and left of the applied load are joined, it is evident that values of the slope given by eqs. (a) and (c) must be equal, and also that values of \( y \) given by eqs. (b) and (d) must be equal. Hence we have the four conditions that

1. \( y = 0 \) when \( x = 0 \) in eq. (b).
2. \( y = 0 \) when \( x = l \) in eq. (d).
3. \( \frac{dy}{dx} \) from eq. (a) = \( \frac{dy}{dx} \) from eq. (c) when \( x = kl \) in these equations.
4. \( y \) from eq. (b) = \( y \) from eq. (d) when \( x = kl \) in these equations.

Performing the operations indicated and reducing the resulting expressions to their simplest form, we derive the following condition equations:

1. \( 0 = C_2 \)
2. \( 0 = -\frac{Wk}{3EI} + C_4 + C_4 \)
3. \( C_1 - C_3 = \frac{Wk}{2EI} \)
4. \( (C_1 - C_3) k l = \frac{Wk}{3EI} + C_4 \)

Solving these four equations simultaneously, the values of the constants of integration are found to be

\[ C_1 = \frac{Wl^2}{6EI} \left( 2k + k^3 - 3k^2 \right) \quad C_2 = 0 \]
\[ C_3 = \frac{Wk}{6EI} \left( 2 + k^2 \right) \quad C_4 = -\frac{Wk}{6EI} \]

Substituting these constants in eqs. (b) and (d), the equation of the elastic line is found to be as follows:

From \( A \) to \( C \)
\[ y = \frac{Wx}{6EI} \left( 1 - k \right) \left( 2 - k \right) k l^2 - x^3 \]  
\[ (5) \]

From \( C \) to \( B \)
\[ y = \frac{W}{6EI} k l - x(2l - x) - k l^2 \]  
\[ (6) \]

The general equations for slope of the tangent to the elastic curve at any point, as given by eqs. (a) and (c) with values of \( C_1 \) and \( C_3 \) substituted, are as follows:

From \( A \) to \( C \)
\[ \frac{dy}{dx} = \frac{W}{6EI} (1 - k) (2 - k) k l^2 - 3x^2 \]  
\[ (5a) \]

From \( C \) to \( B \)
\[ \frac{dy}{dx} = \frac{Wk}{6EI} \left[ 3(l - x)^2 - (1 - k^2) l^2 \right] \]  
\[ (6a) \]
On substituting values of \( x \) in the proper equation, the deflection at any point may be determined. At point \( C \), where the load is located, the deflection may be determined from eqs. (5) or (6) by substituting \( x = kl \), and we have

\[
y = \frac{Wl}{3EI}(1-k)^3
\]  
(7)

The maximum deflection for the beam under consideration can be seen from Fig. 4 to be at the point where the curve becomes horizontal—that is, where \( dy/dx = 0 \). To locate this point, note from Fig. 3 that the tangent is horizontal on the portion of the curve between points \( A \) and \( C \). From eq. (5a) we have

\[
\frac{dy}{dx} = 0 = \frac{W}{6EI} (1-k)(2-k)kl^2 - 3x^2
\]

Solving this expression for \( x \), we find that the deflection is a maximum when

\[
x = \left(\frac{k}{3}(2-k)\right)^{\frac{3}{2}}
\]  
(8)

Substituting this value of \( x \) in eq. (5), the maximum deflection is found to be

\[
y_{\text{max}} = \frac{W}{3EI} \left(\frac{k}{3}(2-k)\right)^{\frac{3}{2}}
\]  
(9)

Equations (8) and (9) give the position and the amount of the maximum deflection when the load \( W \) is located at a distance \( kl \) from the left end of the beam.

From eq. (9) it can be seen that the maximum deflection depends upon the position of the load \( W \). Evidently there is some position of the load for which the deflection will be greater than for any other point in the beam. To determine the position of the load for greatest deflection, place \( \frac{dy_{\text{max}}}{dk} \) from eq. (9) equal to zero and solve for \( k \), from which it will be found that \( k = \frac{3}{2} \), or the load should be placed at the beam center. Substituting \( k = \frac{3}{2} \) in eq. (9) and denoting the resulting deflection by \( \Delta \), we have

\[
\Delta = \frac{1}{48} \frac{W}{EI}
\]  
(10)

Equation (10) gives the deflection at the beam center for a load \( W \) placed at that point. This is the greatest deflection for the beam under consideration.

**Illustrative Problem.**—A 2- × 1-in. piece of wood laid flatwise spans a 24-in. opening. The beam carries a 60-lb. load at a distance of 18 in. from the left end of the beam. Determine the deflection under the load and the maximum deflection of the beam. Assume \( E = 1,500,000 \) lb. per sq. in.

The deflection under the load is given by eq. (7) with \( W = 60 \) lb., \( l = 24 \) in., \( k = \frac{3}{2} \), \( E = 1,500,000 \) lb. per sq. in., and \( I = \frac{3}{12} \) in.\(^4 \), or

\[
I = \frac{3}{12}(2)(1)^3 = \frac{3}{4} \text{ in.}^4
\]

Thus

\[
y = \frac{(60)(24)^4(0.75)^3(1 - 0.75)^2}{(3)(1,500,000)(\frac{3}{4})} = 0.0389 \text{ in.}
\]

The maximum deflection for the given loading is found from eq. (9) with values as above, from which

\[
y_{\text{max}} = \frac{(60)(24)^4(1 - 0.75)(0.75)^3(2 - 0.75)^2}{(3)(1,500,000)(\frac{3}{4})} = 0.0484 \text{ in.}
\]

The point at which this deflection occurs is found from eq. (8) to be

\[
x = \left(\frac{3}{2}(2-0.75)\right)^{\frac{3}{2}} = (0.558)(24) = 13.42 \text{ in.}
\]

from the left end of the beam.

Compare these results with those given on p. 49.

**Illustrative Problem.**—Determine the angle between the horizontal and the tangent to the elastic curve at the left end of the beam and at the load point in the preceding problem.

The slope of the tangent at the left end of the beam is given by eq. (5a) with \( x = 0 \) from which

\[
\text{Slope} = \frac{W}{6EI}(1-k)(2-k)kl^2
\]

For the values given in the above problem,

\[
\text{Slope} = \frac{60(1 - 0.75)(2 - 0.75)(0.75)(24)^2}{(3)(1,500,000)(\frac{3}{4})} = 0.0054 \text{ radians}
\]
In circular measure, a radian is \(57^\circ18'\). Hence the required slope is \((0.0054)(57.3) = 0.3090 = 18.55'\).
This angle is measured in a clockwise direction about point 0 of Fig. 1 with \(OL\) as a horizontal axis.

The slope of the tangent at the load point is given by eq. (5a) and eq. (6a) with \(x = kl\) = 18 in. With values of the several terms as above, we have from eq. (6a).

\[
\text{Slope} = \frac{60}{(6)(1,500,000)(\frac{1}{3})} \left[3(24 - 18)^2 - (1 - 0.75^4)(24)^4(0.75)\right] = 0.00432 \text{ radians} = -0.242 \text{ deg.} = -14.52 \text{ min.}
\]

The slope of this tangent is in the direction shown for similar conditions in Fig. 3.

\textbf{Cantilever Beams.}—Assuming the origin of coordinates to be located at point \(O\) of the cantilever beam of Fig. 4a, the moment at any point distance \(x\) from the origin is \(M_x = -\frac{wx^3}{2}\). Substituting this value of \(M\) in eq. (2), we have

\[
\frac{dy}{dx} = +\frac{wx^2}{2EI}
\]

Integrating

\[
\int \frac{dy}{dx} = \int +\frac{wx^2}{2EI} + C_1
\]

To determine \(C_1\), note from Fig. 4a that the tangent is horizontal when \(x = l\). Hence substituting \(dx = 0\) when \(x = l\) in the above equation, we find \(C_1 = -\frac{wl^3}{6EI}\) and

\[
\frac{dy}{dx} = +\frac{w}{6EI}(x^3 - l^3)
\]

Integrating again

\[
y = +\frac{w}{6EI} \left(\frac{x^4}{4} - l^4x\right) + C_2
\]

To determine \(C_2\), note from Fig. 4a that \(y = 0\) when \(x = l\). Therefore \(\frac{wl^4}{24} - \frac{wl^4}{6EI} + C_2 = 0\), from which \(C_2 = +\frac{1}{6}wl^4\), and we have

\[
y = +\frac{w}{24EI} \left(x^4 - 4l^3x + 3l^4\right)
\]

which is the general equation of the elastic curve for the beam of Fig. 4a. The positive value indicates that \(y\) is measured upward from 0. From Fig. 4a it can be seen that the maximum value of \(y\) occurs when \(x = 0\). Placing \(x = 0\) in eq. (12), we have

\[
y_{max} = +\frac{wl^4}{8EI}
\]

Figure 4b shows a cantilever beam with a single concentrated load at a distance \(a\) from the free end. Since the law of variation of moments differs for the two portions of the beam shown by \(AC\) and \(CH\) of Fig. 4b two substitutions must be made in eq. (2). For an origin at the deflected position of the free end of the beam, the moment at any point in the beam is

From \(A\) to \(C\): \(M = 0\)

From \(C\) to \(B\): \(M = -W(x - a)\)

Substituting in eq. (2), the integrations are as follows:

\[
\begin{align*}
\text{From A to C} & \quad \text{From C to B} \\
EI \frac{d^2y}{dx^2} &= 0 & EI \frac{d^2y}{dx^2} &= +W(x - a) \\
EI \frac{dy}{dx} &= C_1 \ldots \ldots \ldots \ldots \quad (a) & EI \frac{dy}{dx} &= W \left(\frac{x^2}{2} - ax\right) + C_3 \ldots \ldots \quad (c) \\
EI y &= C_1x + C_1 \ldots \ldots \ldots \ldots \quad (b) & EI y &= W \left(\frac{x^2}{6} - ax\right) + C_2x + C_4 \ldots \ldots \quad (d)
\end{align*}
\]

The constants of integration may be determined from the following conditions: \(y = 0\) when \(x = 0\) in eq. \((b)\); \(\frac{dy}{dx}\) from eq. \((a)\) = \(\frac{dy}{dx}\) from eq. \((c)\) when \(x = a\); \(y\) from eq. \((b)\) = \(y\) from eq. \((d)\) when \(x = a\); and \(\frac{dy}{dx} = 0\) in eq. \((a)\) when \(x = l\). These conditions are evident
from a study of Fig. 4b. Performing the operations indicated, the values of the constants of integration are found to be

\[\begin{align*}
C_1 &= -\frac{W}{2} (l - a)^2 \\
C_2 &= 0 \\
C_3 &= -\frac{W}{2} (l - 2a) \\
C_4 &= -\frac{W a^3}{6}
\end{align*}\]

Substituting these values in eqs. (b) and (d), the equation of the elastic curve is found to be

From A to C,

\[y = -\frac{W x}{2 E I} (l - a)^2 \tag{14}\]

From C to B,

\[y = \frac{W}{6 E I} \left[x^3 - 3ax^2 - 3x(l - 2a) - a^3\right] \tag{15}\]

The slope of the tangent to the elastic curve, as given by eqs. (a) and (c), is

From A to C,

\[\frac{dy}{dx} = -\frac{W}{2 E I} (l - a)^2 \tag{16}\]

From C to B,

\[\frac{dy}{dx} = \frac{W}{2 E I} \left[x^2 - 2ax - l(l - 2a)\right] \tag{17}\]

From eq. (14) it can be seen that the portion OC of the elastic curve is a straight line.

**Illustrative Problem.** A 4- × 8-in. wooden member, placed with the 8-in. side vertical, forms a cantilever beam 6 ft. long. At a point 2 ft. from the free end, a 1,000-lb. load is placed. Determine the deflection in inches at the free end of the beam, assuming \(E = 1,500,000\) in.-lb.

The formula for deflection at the free end is given by eq. (15) with \(x = l\), from which

\[-\frac{W}{6 E I} (l - a)^3 (2l + a) \tag{18}\]

The minus sign in this equation indicates that the deflection is upward with respect to the free end of the beam. The value of \(I\) for the given beam is \(I = \frac{bd^3}{12} = \frac{(4)(8)^3}{12} = 170\frac{3}{4}\) in.\(^4\).

Substituting values as given above, using distances in feet and multiplying by 1,728 to reduce to inches, we have

\[y_{max} = -\frac{1,000)(1,728)}{(6)(1,500,000)(170\frac{3}{4})} (6 - 2)^3 (12 + 2) = -0.252\text{ in.}\]

**Beam with an Overhanging End.**—Figure 5a shows a beam supported at points B and C and with an overhanging end AB. Between supports the beam carries a uniform load of \(w\) lb. per ft. and at the free end of the overhanging arm the beam supports a single concentrated load of \(W\) lb. The complete equation of the elastic curve will be determined for the given conditions.

The reactions and moments are as follows:

\[R_1 = \frac{w l}{2} + \frac{W}{l} \left(\frac{l + a}{l}\right) \quad R_2 = \frac{w l}{2} - \frac{W a}{l}\]

\[M_{AB} = -W(a - x) \quad M_{BC} = \left(\frac{wx}{2} - \frac{W a}{l}\right)(l - x)\]

where \(M_{AB}\) and \(M_{BC}\) denote respectively, the moment at any point on \(AB\) or \(BC\). The value of \(x\) is positive when measured as shown in Fig. 5a.

Substituting in eq. (2) we have

From A to B

\[EI \frac{dy}{dx} = +W(a - x) \tag{a}\]

\[EI y = +W \left(\frac{ax^2}{2} - \frac{x^3}{2}\right) + C_1x + C_2 \tag{b}\]
From $B$ to $C$

\[ EI \frac{dy}{dx} = \frac{Wa}{l} \left( l - x \right) - \frac{w}{2} \left( lx - x^2 \right) \]

\[ EI \frac{dy}{dx} = \frac{Wa}{l} \left( \frac{lx - x^2}{2} \right) - \frac{w}{2} \left( \frac{lx^2}{2} - \frac{x^3}{3} \right) + C_3 \tag{c} \]

\[ EI y = \frac{Wa}{l} \left( \frac{lx^2 - x^3}{2} \right) - \frac{w}{2} \left( \frac{lx^3}{6} - \frac{x^4}{12} \right) + C_1 x + C_4 \tag{d} \]

The constants of integration are to be determined subject to the following conditions (see Fig. 5b):

- $y = 0$ in (h) when $x = 0$; $y = 0$ in (d) when $x = 0$;
- $y = 0$ in (d) when $x = l$; and $\frac{dy}{dx}$ in (c) = $-\frac{dy}{dx}$ in (a) when $x = 0$.

The minus sign in this last condition is necessary because of the change in positive directions at point $B$. Subject to the above conditions, the constants of integration are found to have the following values:

\[ C_1 = \frac{Wal}{3} - \frac{wL^3}{24} \quad C_3 = -\frac{Wal}{3} + \frac{wL^3}{24} \]

\[ C_1 = 0 \quad C_4 = 0 \]

Substituting these values in the above equations, the general equations for the elastic curve and slope of the tangent at any point are found to be:

**From A to B**

\[ y = \frac{1}{EI} \left[ \frac{Wx}{6} \left( 2nl + 3ax - x^2 \right) - \frac{wL^3}{24} \right] \tag{18} \]

\[ \frac{dy}{dx} = \frac{1}{EI} \left[ \frac{W}{6} \left( 2nl + 6ax - 3x^2 \right) - \frac{wL^3}{24} \right] \tag{19} \]

**From B to C**

\[ y = \frac{1}{EI} \left[ - \frac{Wx}{6l} \left( 2l - x \right) + \frac{wx}{24} \left( l^2 + lx - x^2 \right) \right] (l - x) \tag{20} \]

\[ \frac{dy}{dx} = \frac{1}{EI} \left[ - \frac{W}{6l} \left( 2l^2 - 6lx + 3x^2 \right) + \frac{w}{24} \left( l^2 - 6lx^2 + 4x^3 \right) \right] \tag{21} \]

Figure 5b shows the form of the elastic curve plotted from these equations.

The maximum deflection in the cantilever arm occurs at the free end, point $A$ of Fig. 5.

Placing $x = a$ in eq. (18), we have

\[ y_A = \frac{a}{EI} \left[ \frac{Wa}{3} \left( l + a \right) - \frac{wL^3}{24} \right] \tag{22} \]

The maximum deflection in the span $BC$ occurs where the tangent to the elastic curve is horizontal. This point may be located by placing $\frac{dy}{dx}$ from eq. (21) equal to zero and solving for $x$. Let $k/l$ be this value of $x$, where $k$ is the fractional part of the span between the left support and the point of maximum deflection. Performing the operation indicated, the value of $k$ is given by the cubic equation

\[ k^3 - \left( \frac{3}{2} + \frac{3W}{w} \right) k^2 + \frac{6Wa}{wL^3} k + \left( \frac{1}{4} - \frac{2Wa}{wL^3} \right) = 0 \tag{23} \]

To determine the maximum deflection in any given beam, solve eq. (23) for $k$ by the methods given in Art. 170, p. 537, and calculate the corresponding value of $x$. Substitute this value of $x$ in eq. (20). This procedure is advisable in this case because a general expression for maximum deflection is too complicated and cumbersome.

**Illustrative Problem.** — A 10-in. 254-lb. steel I-beam is used to form a beam of the type shown in Fig. 5. Calculate the deflection at the free end of the cantilever arm and the maximum deflection in the span $BC$. Let $W = 6,000$ lb., $w = 1,300$ lb. per ft., $a = 5$ ft., and $l = 15$ ft. The moment of inertia of the given I-beam section is 122.1 in.4, and $E = 30,000,000$ lb. per sq. in.

The deflection at the free end of the cantilever arm is given by eq. (22). Substituting values given above in this equation, we have, using inch units

\[ y_A = \frac{60}{(30,000,000)(122.1)} \left[ \frac{1}{3} \left( 1,300 \right)(60)(180 + 60) - \frac{(1,200)(180)}{(12)(24)} \right] \]

from which

\[ y_A = 0.0738 \text{ in.} \]
The point in span BC at which the maximum deflection occurs, is found from eq. (23). Substituting the given values in this equation we derive the following cubic equation for \( k \),

\[ 4k^3 - 1.833k^2 + 0.006k + 0.0278 = 0 \]

Solving this equation by the method given in Art. 70b, p. 537, we find \( k = 0.565 \). Hence the distance from point B, Fig. 5, to the point of maximum deflection is

\[ x = kl = (0.565)(15) = 8.47 \text{ ft.} = 102 \text{ in.} \]

Substituting \( x = 102 \) in. and other values as given above, in eq. (20), the maximum deflection is found to be

\[ y_{max} = \frac{1}{(30,000)(0.00)(122.1)} \left\{ \frac{(6,000)(60)(102)}{(6)(180)} \right. \]

\[ - \frac{(1,200)(102)}{(12)(24)} \left[ (180)^2 + (180)(102) - (102)^2 \right] \left. \right\} (180 - 102) \]

from which

\[ y_{max} = 0.183 \text{ in.} \]

### Unit Load Method

3. **Derivation of General Formula.**—The deflection of a beam due to any given loading may be determined by placing the external work done by this loading during the deflection of the beam equal to the internal work done on the fibers of the beam, for it is evident that a body can be at rest and in a state of elastic equilibrium only when the work of applied loads is balanced by work done within the body. In Art. 37, p. 9, it has been shown that the internal work, or elastic resilience of a body, is given by the expression

\[ \text{Elastic resilience} = K = \frac{1}{2} \frac{f^2}{E} \text{ (Volume of body)} \]

where \( f \) = fiber stress due to applied loading and \( E \) = modulus of elasticity of the material.

Let the simple beam of Fig. 6a be acted upon by a 1 lb. or unit load placed at point C, a distance \( a \) from the left end of the beam. At any cross-section of the beam at a distance \( x \) from any convenient origin, let the moment due to the unit load be \( m \) and let the stress on any fiber at a distance \( c \) from the neutral axis, Fig. 6b, be denoted by \( f \). If the length of any fiber parallel to the beam axis is \( dx \), and the area of that element is \( dA \), as shown in Fig. 6b, the volume of that element is \( dA(dx) \), and we have

\[ K = \frac{1}{2} \frac{f^2}{E} dA \ dx \]

From eq. 6, p. 23, \( f = \frac{me}{I} \) and for the entire cross-section of Fig. 6b.

\[ K = \frac{1}{2E} \int_{c_1}^{c_2} m^2 c^2 \ dA \ dx \]

But \( \int_{c_1}^{c_2} c^2 dA = I = \text{moment of inertia of the section} \). Therefore

\[ K = \frac{1}{2E} \frac{m^2}{I} \ dx \]

For all such sections over the entire beam from \( A \) to \( B \),

\[ K = \text{Total average internal work} = \frac{1}{2} \int_{B}^{A} \frac{m^2}{EI} \ dx \quad (1) \]

Let the deflection of the beam at point \( C \) where the unit load is applied be denoted by \( d \) and assume that the load is gradually applied to the beam so that the deflection varies from zero to its maximum value. The average external work done by the unit load is one-half the deflection times the load causing that deflection, or

\[ \text{Average external work} = \frac{d}{2} \quad (2) \]

From eqs. (1) and (2) we derive

\[ \frac{1}{2} \int_{0}^{A} \frac{m^2}{EI} \ dx \quad (3) \]
Equation (3) gives the deflection of point C, Fig. 6a, due to a 1-lb. load placed at that point. The deflection will be in the direction of the 1-lb. load.

It is sometimes desired to determine the angular rotation of any plane of a beam—as, for example, the plane n-n of Fig. 6c. This may be done by applying a unit couple at the plane in question and calculating the external work due to the rotation. On equating this expression to the internal work, as given by eq. (1), the angular rotation may be determined.

It can be shown that the work done by a couple is equal to the moment of the couple times the angular rotation. Let $\alpha_1$ denote the angular rotation of plane n-n of Fig. 6a. The average work done by a unit couple during a rotation $\alpha_1$ is then

$$\text{Average external work} = \frac{1}{2} \alpha_1$$

Hence from eqs. (1) and (4)

$$\int_B^A \frac{Mm}{EI} \, dx$$

Equation (5) gives the angular rotation in radians of any plane due to a unit couple applied to that plane.

The deflection of any point C, Fig. 6d, or the angular rotation of a plane at that point due to any set of applied loads, such as those shown in Fig. 6d, may be determined by proportion from the corresponding values given by eqs. (3) or (5). In Art. 67, p. 82, it is shown that the deflection is directly proportional to the fiber stress. Since fiber stresses are proportional to moment, it is evident that the deflection, or rotation, due to any set of applied loads is to the corresponding value due to unit loading as the moment due to the applied loads is to the moment due to the unit loading. If $y$ = deflection due to applied loads, and $M$ = moment due to applied loads, we have the proportion

$$y : d :: M : m$$

or

$$y = \frac{M}{m} d$$

Substituting the value of $d$ from eq. (3) we have

$$y = \int_B^A \frac{Mm}{EI} \, dx$$

If $\alpha$ = angular rotation due to applied loads, a similar proportion gives

$$\alpha = \int_B^A \frac{Mm}{EI} \, dx$$

In these equations, $y$ = deflection of any desired point; $\alpha$ = angular rotation in radians of a plane at any desired point; $M$ = moment due to applied loads; $E$ = modulus of elasticity of material composing the beam; $I$ = moment of inertia of beam section; and $m$ = a quantity of linear dimensions which is equal to the moment at any section due to a unit load, or unit couple, applied at the point whose deflection, or angular rotation, is desired and in the direction of the desired deflection, or rotation.

In solving problems in deflection and angular rotation by the method given above, it is not necessary that the direction of the deflection or rotation be known beforehand. Proper attention paid to the algebraic sign of the product $Mm$ will show whether the correct direction has been assumed for the unit loading. If $M$ and $m$ are alike in character (both positive or negative moments) it is evident from the above discussion that they cause deflections or rotations in the same direction, while if they are unlike in character they cause deflections or rotations in opposite directions. Denoting positive moments by plus and negative moments by minus, the product of like moments carries a plus sign and the product of unlike moments carries a minus sign. Therefore, assume any convenient direction for the unit load or couple and pay careful attention to the sign of the product $Mm$.

If the final result is positive, the deflection or rotation is in the direction assumed for the unit loading. If the final result has a negative sign, the deflection or rotation is in a direction opposite to that assumed for the unit loading.

4. Application of Unit Load Method to Problems in Deflection and Angle of Rotation—
Beam with Uniform Load (Moment of Inertia Constant).—To find the general equation for the vertical deflection of point C at a distance $x$ from the left end of the beam of Fig. 7a
due to a uniform load, apply a 1-lb. load at point C. Evidently the deflection is downward so the 1-lb. load is to be applied as shown in Fig. 7c.

The moment diagram for the uniform loading is shown in Fig. 6b and the moment diagram for the 1-lb. load is shown in Fig. 6c. Moment equations, expressed in terms of z, the distance from any point to the left end of the beam, are given on the diagrams. From these diagrams it can be seen that values of M for the entire beam are given by a single equation while values of m are given by two equations, for the law of variation of m changes at the point of application of the unit load. Therefore in substituting in eq. (6), the general expression must be made up for the portions of the beam where the law of variation of moments changes.

Substituting in eq. (6), we have

\[ y = \int_{x}^{l} \frac{1}{E} \left[ \frac{wz}{2} (l - z) \right] \left[ \frac{l}{2} \frac{l - z}{l} \right] dz + \int_{x}^{l} \frac{1}{E} \left[ \frac{wz}{2} (l - z) \right] \left[ \frac{l - z}{l} \right] dz \]

The first integral is for the portion of the beam from A to C and the second integral is for the portion from C to B. Performing the operations indicated, noting that \( x \) is a constant and that \( z \) is variable, we have

\[ y = \frac{wz}{2AEI} (l^3 - 2lx^2 + x^3) \]  
(8)

Note that eq. (8) is exactly the same as eq. (4), p. 584. Hence eq. (8) gives the equation of the elastic curve for the given beam.

The angular rotation of a vertical plane through point C will now be determined, using eq. (7). Since the tangent to the elastic curve at any point is perpendicular to any normal section of a beam, substitution in eq. (7) will give the slope of the tangent to the elastic curve at any point. Applying a unit couple at point C, assuming a positive or clockwise rotation, the resulting moment diagram is shown in Fig. 7d. Substituting in eq. (7), we have

\[ \alpha = \frac{dy}{dx} = \int_{x}^{l} \frac{1}{E} \left[ \frac{wz}{2} (l - z) \right] \left[ \frac{l}{2} \frac{l - z}{l} \right] dz + \int_{x}^{l} \frac{1}{E} \left[ \frac{wz}{2} (l - z) \right] \left[ \frac{l - z}{l} \right] dz \]

Integrating and reducing, we have finally,

\[ \alpha = \frac{dy}{dx} = \frac{wz}{2AEI} (l^3 - 6lx^2 + 4x^3) \]  
(9)

This expression is the same as given by eq. (3) p. 583.

To determine the deflection at the center of a beam uniformly loaded, place the 1-lb. load as shown in Fig. 8. Substituting in eq. (6) values of M and m as shown on Fig. 8, we have

\[ y = \int_{x}^{l} \frac{1}{E} \left[ \frac{wz}{2} (l - z) \right] \left[ \frac{l}{2} \frac{l - z}{l} \right] dz + \int_{x}^{l} \frac{1}{E} \left[ \frac{wz}{2} (l - z) \right] \left[ \frac{l - z}{l} \right] dz \]

Integrating,

\[ y = \frac{5wl^4}{384EI} \]  
(10)
Since the \( m \) diagram is symmetrical about the beam center, as shown in Fig. 8, and since \( M \) for the entire beam is given by the same equation, the substitution in eq. (6) might have been written

\[
y = 2 \int_0^1 \frac{1}{E I} \left[ \frac{w x}{2} (l - x) \right] \frac{x}{2} dx
\]

On integrating this expression the result will be the same as given above.

**Illustrative Problem.**—A simple beam 16 ft. long supports a uniform load of 600 lb. per ft. Determine the maximum deflection of this beam in inches. Assume that the moment of inertia of the beam is 100 in.\(^4\), and that the material is steel for which \( E = 30,000,000 \) lb. per sq. in.

From conditions of symmetry, it is evident that the maximum deflection occurs at the center of the beam. The loading conditions are as shown in Fig. 8. Substituting given values in the formulas for \( M \) and \( m \), using foot units, we have

\[
M = 300x(16 - x) \text{ ft.-lb., and } m = \frac{x}{2} \text{ ft.-lb.}
\]

The equation for \( m \) holds for the left side of the beam. Noting from Fig. 8 that the \( M \) and \( m \) diagrams are symmetrical about the center of the beam, we have

\[
y = 2 \int_0^{16} \frac{M m dx}{E I} = \frac{(2)(1,728)}{30,000,000(100)} \int_0^{16} 300x(16 - x) \frac{x}{2} dx
\]

The multiplier 1,728 in the right-hand member of the above equation is made necessary by the fact that the moment equations are expressed in foot-pound units while the deflection is desired in inch units. Now, as stated in Art. 3, p. 591, \( M \) is expressed in linear-force units, \( m \) is expressed in linear units, and \( dx \) is also in linear units. The quantity \( \int M m dx \) therefore has the dimensions of force times linear units to the third power. Hence to reduce \( \int M m dx \) to inch units when the moments are expressed in foot units, we must multiply by \((12)^3 = 1,728\).

Performing the operations indicated by the above equation, we have

\[
y = 0.295 \text{ in.}
\]

**Beam with Uniform Load (Moment of Inertia Not Constant).**—Assume that the beam is a built-up girder, such as a plate girder with cover plates, as shown in Fig. 9. Let the moment of inertia of the end-quarters be \( I_1 \) and the moment of inertia of the middle half be \( I_2 \). Required the deflection of the center point. The \( M \) and \( m \) diagrams are the same as given in Fig. 8.

Since the moment of inertia is not constant for the entire girder, substitution in eq. (6) must be made for the sections for which the moment of inertia differs as well as for the sections for which the law of variation of \( M \) and \( m \) change.

Substitution in eq. (6) gives

\[
y = 2 \left[ \frac{1}{E I_1} \int_0^l \frac{wx}{2} (l - x) \frac{x}{2} dx + \frac{1}{I_2} \int_0^l \frac{wx}{2} (l - x) \frac{x}{2} dx \right]
\]

Integrating and reducing, we have finally

\[
y = \frac{w l^4}{E} \left[ \frac{5}{384} I_2 + \frac{13}{6,144} I_2 - I_1 \right]
\]

(11)

When the moment of inertia is constant, or when \( I_1 = I_3 \), this expression reduces to

\[
y = \frac{5}{384} \frac{w l^4}{E I} \text{ as on p. 584.}
\]

The above solution for deflection of a girder with varying moment of inertia is very convenient when there are a limited number of changes of moment of inertia and when the loading is comparatively simple. However, since a substitution must be made in eq. (6) for each change in moment of inertia and for each change in the law of variation of \( M \) and \( m \), the determination of the deflection of a long plate girder due to a set of concentrated loads becomes a long and tedious process. For such cases the Area Moment Method or the Elastic Weight Method are more convenient.

In some cases, the moment of inertia varies from section to section. When the moment of inertia can be expressed as a function of \( x \), it may be placed in the general equation and
the integration performed. Such integrations are generally very complicated. The above case represents the problem in the form usually encountered in practice.

**Illustrative Problem.**—Assume the following data for the beam of Fig. 9 and calculate the deflection in inches at the center of the beam. \( w = 1,200 \text{ lb. per ft.} \), \( l = 10 \text{ ft.} \), \( I_i = 36 \text{ in.}^4 \), \( I_1 = 48 \text{ in.}^4 \), and \( E = 30,000,000 \text{ lb. per sq. in.} \)

For the conditions shown, \( M = 600 x (10 - x) \text{ ft.-lb.} \), and \( m = \frac{x}{2} \text{ ft.-lb.} \). Noting that values of \( M \) and \( m \) are symmetrical about the beam center, and the deflection in inches is required, we have from eq. (6)

\[
y = \frac{(2)(1,728)}{30,000,000} \left[ \int_0^{10} 600x(10 - x) \frac{x}{2} dx + \int_{10}^8 600x(10 - x) \frac{x}{2} dx \right]
\]

from which

\[
y = 0.197 \text{ in.}
\]

Direct substitution in eq. (11) gives the same result.

**Simple Beam with Single Concentrated Load**

**Illustrative Problem.**—A 2- \( \times \) 1-in. piece of wood laid flatwise spans a 24-in. opening. The beam carries a 60-lb. load at a distance of 18 in. from the left end of the beam. Determine the deflection under the load and the maximum deflection of the beam in inches. Assume \( E = 1,500,000 \text{ lb. per sq. in.} \).

Figure 10a shows the given beam and Fig. 10b shows the \( M \) diagram. The moments are given in inch-pound units.

To determine the deflection under the load apply a 1-lb. load downward at point \( C \) of Fig. 10a. The \( m \) diagram is shown in Fig. 10c. From eq. (6)

\[
y_e = \frac{1}{EI} \left[ \int_0^{18} (15x) \left( \frac{1}{4} \right) dx + \int_{18}^{24} [45(24 - x)] \left( \frac{3}{4} \right) (24 - x) dx \right]
\]

For the given conditions, \( I = \frac{3}{2} \text{ in.}^4 \), \( M = (\frac{3}{2}) \text{ in.}^3 \), \( (\frac{3}{2}) \text{ in.}^3 \), and hence \( EI = 230,000 \).

Performing the integrations indicated above, we have

\[
y_e = 0.0389 \text{ in.}
\]

The maximum deflection occurs at the point where the tangent to the elastic curve is horizontal. Assume this point to be located at a distance \( x_0 \) from the left end of the beam. The value of \( x_0 \) may be determined by placing a unit couple at this point, as shown in Fig. 10d, substituting values of \( M \) and \( m \) in eq. (7) and solving for \( x_0 \) subject to the condition that \( \alpha \) = slope of tangent to elastic curve = 0.

Figure 10d shows the values of \( m \) in inch-pound units. Substituting in eq. (7) we have

\[
\alpha = \frac{1}{EI} \left[ \int_0^{x_0} (15x) \left( - \frac{x}{24} \right) dx + \int_{x_0}^{18} (15x) \left( - \frac{x}{24} \right) dx + \int_{x_0}^{24} [45(24 - x) \left( 1 - \frac{x}{24} \right) dx \right] = 0
\]

Integrating and solving for \( x_0 \), we have

\[
x_0 = 13.42 \text{ in.}
\]

That is, the maximum deflection occurs at a point 13.42 in. from the left end of the beam. Figure 10e shows the unit load in position at the point of maximum deflection and the resulting \( m \) diagram.
Substituting in eq. (6), we have

\[
y_{max} = \frac{1}{EI} \left( \int_{0}^{18} (15x)(0.441x)dx + \int_{18}^{19} (15x)(0.559(24 - x))dx \right) + \int_{18}^{24} 45(24 - x)(0.559(24 - x))dx
\]

from which

\[
y_{max} = 0.0484 \text{ in.}
\]

The above results check those given on p. 71.

**Cantilever Beams.**—Figure 11 shows a cantilever beam supporting a uniform load. Let it be required to determine the general formula for deflection of the free end. Assuming the deflection of the free end to be downward, the unit load acts as shown in Fig. 11. The \( M \) and \( m \) diagrams for the applied and unit load are as shown on Fig. 11. Substituting in eq. (6), we have

\[
y_{max} = \frac{1}{EI} \int_{0}^{l} \left( \frac{-wx^2}{2} \right) (-x)dx = \frac{1}{EI} \int_{0}^{l} \frac{wx^2}{2} dx
\]

Integrating,

\[
y_{max} = \frac{wL^3}{8EI}
\]

Illustrative Problem.—A 6-ft. cantilever beam supports a 1,000-lb. load placed as shown in Fig. 12. The beam consists of a 4- X 8-in. member with the 8-in. side vertical. Determine the deflection of the free end in inches. Assume \( E \) for timber as 1,500,000 lb. per sq. in.

Assume the deflection to be downward. For an origin at the free end, the \( M \) and \( m \) diagrams are as shown in Fig. 12. The moments are expressed in foot-pounds. Substituting in eq. (6), we have

\[
y = \frac{1}{EI} \int_{0}^{6} \left[-1,000(x^2 - 2)(-x)\right]dx = \frac{1}{EI} \int_{0}^{6} 1,000(x^2 - 2x)dx = \frac{1,000(37\frac{1}{2})}{EI}
\]

For a 4- X 8-in. rectangular section

\[
I = \frac{bh^3}{12} = \frac{(4)(8)^3}{12} = 170\frac{3}{4}
\]

Substituting values of \( E \) and \( I \) in the above equation, remembering that the deflection in inches is desired, we have

\[
y = \frac{1,000(37\frac{1}{2})(1,728)}{(1,500,000)(170\frac{3}{4})} = 0.252 \text{ in.}
\]

**Beam with Overhanging End.**

Illustrative Problem. — A 10-in. 25.4-lb. steel I-beam is used to form a beam of the type shown in Fig. 13. For the dimensions and loadings shown on Fig. 13 calculate the deflection at the free end of the cantilever arm and the maximum deflection in the span \( BC \). The moment of inertia of the given beam section is 122.1 in.\(^4\) and \( E = 30,000,000 \text{ lb. per sq. in.} \)

The moment diagram for the given loading is shown in Fig. 13a. The values shown on the diagram are expressed in foot-pound units.

To determine the deflection of the free end of the cantilever arm, apply a 1-lb. load acting downward
at point A, of Fig. 136. The resulting $m$ diagram, expressed in foot-pound units, is as shown in Fig. 13b. Substituting in eq. (6), we have

$$\frac{1.728}{30,000,000(122.1)} \left[ \int_0^5 [-6,000(5 - x)][-(5 - x)]\,dx + \int_5^8 [1,000(11x - 0.6x^2 - 30)][-(5 - x)]\,dx \right]$$

from which finally

$$y = +0.0738 \text{ in. downward deflection}$$

The maximum deflection in the span $BC$ occurs at the point where the slope of the tangent to the elastic curve is horizontal. Assume this point to be located at a distance $15k$ ft. from the left support, where $k$ is a fraction which is less than unity. Apply at this point a unit couple, as shown in Fig. 13c. The $m$ diagram for this couple is shown on Fig. 13c.

$$\text{Fig. 13.}$$

Substituting in eq. (7) we have

$$a = \frac{1}{EI} \int_0^5 \left[ 1,000(11x - 0.6x^2 - 30) \right] (1 - \frac{x}{15})\,dx + \int_5^8 \left[ 1,000(11x - 0.6x^2 - 30) \right] (1 - \frac{x}{15})\,dx$$

Integrating this expression, placing the result equal to zero, and solving for $k$, we have the cubic equation

$$k^3 - 1.833k^2 + 0.666k + 0.0278 = 0$$

Solving this equation by the method given in Art. 706, p. 537, we find $k = 0.565$. Hence the distance from the left support to the point of maximum deflection is $(15)(0.565) = 8.47$ ft. To determine the maximum deflection, apply a 1-lb. load as shown in Fig. 13d and determine the corresponding $m$ values. Substituting in eq. (6), we have, noting that the deflection in inches is desired,

$$u_{max} = \frac{1.728}{30,000,000(122.1)} \left[ \int_0^5 [1,000(11x - 0.6x^2 - 30)](0.455x)\,dx + \int_5^8 [1,000(11x - 0.6x^2 - 30)](0.565(15 - x))\,dx \right]$$

from which

$$u_{max} = 0.180 \text{ in.}$$

These values check the results given on p. 589.

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APPENDIX D

WORKING UNIT STRESSES FOR STRUCTURAL TIMBER

Adopted by the American Railway Engineering Association

The working unit stresses given in the table are intended for railroad bridges and trestles. For highway bridges and trestles, the unit stresses may be increased 25 per cent. For buildings and similar structures, in which the timber is protected from the weather and practically free from impact, the unit stresses may be decreased 50 per cent. To compute the deflection of a beam under long continued loading instead of that when the load is first applied, only 50 per cent of the corresponding modulus of elasticity given in the table is to be employed.

UNIT STRESSES IN POUNDS PER SQUARE INCH

<table>
<thead>
<tr>
<th>Kind of timber</th>
<th>Bending</th>
<th>Shearing</th>
<th>Compression</th>
<th>Working stresses for columns</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Extreme fiber stress</td>
<td>Modulus of elasticity</td>
<td>Parallel to to the grain</td>
<td>Longitudinal shear in beams</td>
</tr>
<tr>
<td>Douglas fir</td>
<td>6.100</td>
<td>1,200</td>
<td>1,510,000</td>
<td>690</td>
</tr>
<tr>
<td>Longleaf pine</td>
<td>6.500</td>
<td>1,300</td>
<td>1,610,000</td>
<td>720</td>
</tr>
<tr>
<td>Shortleaf pine</td>
<td>5.900</td>
<td>1,100</td>
<td>1,480,000</td>
<td>710</td>
</tr>
<tr>
<td>White pine</td>
<td>4.400</td>
<td>900</td>
<td>1,130,000</td>
<td>400</td>
</tr>
<tr>
<td>Spruce</td>
<td>4.800</td>
<td>1,000</td>
<td>1,310,000</td>
<td>600</td>
</tr>
<tr>
<td>Norway pine</td>
<td>4.200</td>
<td>800</td>
<td>1,190,000</td>
<td>590</td>
</tr>
<tr>
<td>Tamarack</td>
<td>4.400</td>
<td>900</td>
<td>1,220,000</td>
<td>670</td>
</tr>
<tr>
<td>Western hemlock</td>
<td>5.800</td>
<td>1,100</td>
<td>1,480,000</td>
<td>630</td>
</tr>
<tr>
<td>Redwood</td>
<td>5.000</td>
<td>900</td>
<td>800,000</td>
<td>300</td>
</tr>
<tr>
<td>Bald cypress</td>
<td>4.800</td>
<td>900</td>
<td>1,150,000</td>
<td>500</td>
</tr>
<tr>
<td>Red cedar</td>
<td>4.200</td>
<td>800</td>
<td>800,000</td>
<td>470</td>
</tr>
<tr>
<td>White oak</td>
<td>5.700</td>
<td>1,100</td>
<td>1,150,000</td>
<td>840</td>
</tr>
</tbody>
</table>

Unit stresses are for green timber and are to be used without increasing the live load stresses for impact. Values noted,* are for partially air dry timbers. In the formulas given for columns, l = length of column, and d = least side or diameter, in inches.
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